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ROTHE'S METHOD FOR AN ISOTHERMAL PHASE-FIELD MODEL OF A BINARY ALLOY WITH CONVECTION

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Abstract

A mathematical analysis of a phase-field model for solidification non-isothermal of a binary alloy with convection is presented. Convergence of the solutions of discretized scheme is proved and existence result for original problem are derived.

1 Introduction

Phase-field models provide an example of a diffuse interface model in which an order parameter, φ, is postulated whose value indicates the phase of the system at a particular point in space and time. When $\varphi = 0$ the phase is considered to be liquid, $\varphi = 1$ the phase is solid. The region when $0 < \varphi < 1$ corresponds to the solid-liquid interface (it is sometimes called the mushy region).

Concerning the possibility of motion of molten material during solidification/melting processes, an early attempt to include fluid motion within a phase-field model is due Caginalp and Jones [6]. Some other authors considering this subject are, for instance, Beckermann *et al.* [2], who employed the methodology of two-phase fluid flow, where φ is interpreted as a solid fraction; Anderson *et al.* [1] that also developed an approximate model which allows for convection, by treating both phases as fluids and assuming that the viscosity is dependent on the phase-field variable, becaming very large when the phase-field value corresponds to solid material.

Keywords: phase-field, semidiscretization, Navier-Stokes, alloy, solidification.

Concerning the rigorous mathematical analysis of phase-field models a pure material without convection, there are several articles examining questions about existence, uniqueness, regularity and large time behavior of solutions. Some of them are, for instance, [3, 5, 8, 13, 14, 17].

In this paper we are interested in the mathematical analysis of a solutal phase-field model for the isothermal solidification of a binary alloy with convection. We will consider a rather simple situation of this sort in the hope to obtain a better understanding of the mathematical difficulties brought by the coupling of terms describing phase change and the terms describing convection. Such model is related, although not exactly the same as, the one Anderson *et al.* [1] and Rappaz [18]. In fact, the phase-field equation is basically the one derived in [18], but with an advection term in the equation. The other equations are obtained by the usual physical balance laws; in our case, balace of mass and momentum, since we assumed an isothermal process. They can be viewed as generalizations of the corresponding ones of the other cited papers.

The problem involves a system of coupled nonlinear partial differential equations: a phase-field equation for the order parameter φ, a concentration equation for the relative solute concentration c, that is, the proportion of solute in the solvent, and a modified incompressible Navier-Stokes equation for the velocity field $\mathbf{u}$.

Given a bounded domain Ω and a constant $0 < T < +\infty$, define $Q = \Omega \times (0,T)$, $S = \partial\Omega \times (0,T)$. Then, we are interested in finding the fields φ, c, $\mathbf{u}$ and p, such that:

$$\partial_t \varphi - \xi^2 \Delta\varphi + \mathbf{u}\cdot\nabla\varphi = F_1(\varphi) + c\, F_2(\varphi) \qquad \text{in } Q, \tag{1.1}$$

$$\partial_t c - \operatorname{div}\Big(D_1(\varphi)\nabla c + D_2(c,\varphi)\nabla\varphi\Big) + \mathbf{u}\cdot\nabla c = 0 \qquad \text{in } Q, \tag{1.2}$$

$$\partial_t \mathbf{u} - \operatorname{div}\Big(\nu(\varphi)\nabla\mathbf{u}\Big) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \vec{\sigma}\, c, \ \ \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad Q, \tag{1.3}$$

$$\partial_\eta \varphi = 0, \quad \partial_\eta c = 0, \quad \mathbf{u} = 0 \qquad\qquad \text{on} \quad S, \tag{1.4}$$

$$\varphi(x,0) = \varphi_0(x), \quad c(x,0) = c_0(x), \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \qquad \text{in} \quad \Omega, \tag{1.5}$$

where ξ is positive constant, η is the unit normal in $\partial\Omega$, ∂_η and ∂_t are the derivate with respect to the normal η and the time t, respectively, p is the pressure, $\nu(\varphi)$ is the viscosity and $\vec{\sigma}$ is a constant vector associated to the Boussinesq approximation for the buoyancy forces appearing due to difference in solute concentration; again for simplicity of exposition we took the reference concentration to be zero. The initial data φ_0, c_0 and $\mathbf{u}_0$ are suitable given functions.

We remark that the phase field model that we consider here is rather simple and does not take care of phase-field model in the context of a free-boundary problem that was treated in [4]. We assume that the viscosity $\nu(\varphi)$ sufficiently smooth such that the Navier-Stokes equations hold in whole domain. For more details of the physics of the above model, we refer to [1, 18].

In this paper the problem above is analyzed by the discretization in time (method of semi-discretization, Rothe's method). With this, we will be able to prove existence of solution. We remark that this kind of technique was also employed in previous papers (e.g. [9, 10, 19]) addressing other types of phase-field problems.

The paper is organized as follows. The next section is dedicated to formulate our general assumptions and fix both the notations and the basic functional spaces to be used. In this sectiom, we also introduce some auxiliary results, define the time-discretization scheme and state the the main result of the paper. In Section 3, we consider the existence and uniqueness of solutions for stationary auxiliary problem which is directly related to the discrete problem. For this, we will use a suitable regularization and Leray-Schauder degree theory. Existence of a weak solution of the discretized scheme is proved in Section 4 by using of a Galerkin method. Section 5 contains a collection of estimates that will allow us to pass to the limit in semidiscretization scheme and obtain our main result.

Throughout the article, we will denote by M the constants depending only on known quantities.

2 Preliminary and Main result

2.1 Notation, assumptions and auxiliary results

We denote by $H^m(\Omega)$ the standard Sobolev space, with norm denoted as $||v||_{m,2,\Omega}$. The space $H_0^m(\Omega)$ is the closure with the norm $||\ ||_{m,2,\Omega}$ of the set of all infinitely differentiable functions with compact support in Ω.

Being X a Banach space, with norm $||\ ||_X$, we denote by $L^r(0,T;X)$ the Banach space of of all measurable functions $v : [0,T] \to X$ with the following finite norm $||v||_{L^r(0,T;X)} = \left(\int_0^T ||v(t)||_X^r \, dt\right)^{1/r}$, when $1 \leq r < \infty$ or $||v||_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{0\leq t\leq T} ||v(t)||_X$. In particular, we denote the norm of $L^r(0,T;L^q(\Omega))$ by $||v||_{q,r,Q} = \left(\int_0^T ||v(t)||_{q,\Omega}^r \, dt\right)^{1/r}$, and observe that $L^r(0,T;L^r(\Omega)) = L^r(Q)$.

For the treatment of the equations related to the Navier-Stokes type equations, we will need other functional spaces. For this, we denote the closure of $\mathcal{V} = \{\mathbf{v} \in C_0^\infty(\Omega)^3 : \operatorname{div} \mathbf{v} = 0\}$ in $L^2(\Omega)^3$ and $H_0^1(\Omega)^3$ by H and V, respectively. General properties for these spaces can be found for instance in Temam

[20].

Throughout the paper we shall make the following assumptions:

$(\mathbf{H}_0)$ $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^2-boundary.

$(\mathbf{H}_1)$ $\nu(\cdot) \in C^0(\mathbb{R})$ with $0 < \nu_0 \leq \nu(\cdot) \leq \nu_1,\ \nu_0, \nu_1 \in \mathbb{R}^+$.

$(\mathbf{H}_2)$ $D_1(\cdot) \in C^0(\mathbb{R})$ with $0 < \rho_0 \leq D_1(\cdot) \leq \rho_1,\ \rho_0, \rho_1 \in \mathbb{R}^+$.

$(\mathbf{H}_3)$ $D_2(\cdot,\cdot) \in C^0(\mathbb{R}^2)$ with $|D_2(\cdot,\cdot)| \leq \alpha_0,\ \alpha_0 \in \mathbb{R}^+$.

$(\mathbf{H}_4)$ $F_l(\cdot) \in C^0(\mathbb{R})$ with $|F_l(\cdot)| \leq \alpha_l,\ \alpha_l \in \mathbb{R}^+$, $l = 1, 2$.

In this paper we shall frequently make use of some auxiliary results, which we list below for further reference.

Frequently we shall use the Gronwall's Lemma in the discrete form.

Lemma 2.1 *Let $0 < k < 1$ and $(a_i)_{i\geq 1}$ and $(A_i)_{i\geq 1}$ be sequences of real, nonnegative numbers. Assume that $(A_i)_{i\geq 1}$ is nondecreasing and that*

$$a_i \leq A_i + k\sum_{j=0}^{i} a_j, \qquad \text{for} \quad i = 0, 1, 2, ...$$

then

$$a_i \leq \frac{1}{1-k} A_i \exp\left((i-1)\frac{k}{1-k}\right), \qquad \text{for} \quad i = 0, 1, 2, ...$$

At various places we shall use the following relation

$$2\int_\Omega a(a-b)\, dx = ||a||^2_{2,\Omega} - ||b||^2_{2,\Omega} + ||a-b||^2_{2,\Omega}. \tag{2.6}$$

Our next lemma provides a compactness criterion which shall be used to derive convergence properties of the Rothe approximations (see [16, Theorem 16.3 p. 580]):

Lemma 2.2 *Let X and Y be two (not necessarity reflexive) Banach spaces, such that $Y \subset X$, the injection being compact.*

Assume that $\mathcal{G}$ is a family of functions in $L^1(0,T;Y) \cap L^p(0,T;X)$ for some $T > 0$ and $p > 1$, such that

$$\mathcal{G} \text{ is bounded in } L^1(0,T;Y) \text{ and } L^p(0,T;X); \tag{2.7}$$

$$\sup_{g \in \mathcal{G}} \int_0^{T-a} ||g(a+s) - g(s)||_X^p \, ds \to 0 \quad as \quad a \to 0, \quad a > 0. \tag{2.8}$$

Then the family $\mathcal{G}$ is relatively compact in $L^p(0,T;X)$.

2.2 Semi-discretization. Main result

The Rothe method is based on a semidiscretization of problem (1.1), (1.2), (1.3), (1.4), (1.5) with respect to the time variable. For the purpose, we subdivide $[0,T]$ into N subintervals $[t_{m-1}, t_m]$, $t_m = k\,m$, $k = T/N$, and for $m = 1, 2, \cdots, N$ we consider the following system of elliptic problems

$$\delta_t \varphi^m - \xi^2 \Delta \varphi^m + \mathbf{u}^m \cdot \nabla \varphi^m = F_1(\varphi^m) + c^m\, F_2(\varphi^m) \qquad \text{a.e. in} \quad \Omega, \tag{2.9}$$

$$\delta_t c^m - \operatorname{div}\Big(D_1(\varphi^m) \nabla c^m + D_2(c^m, \varphi^m) \nabla \varphi^m \Big) + \mathbf{u}^m \cdot \nabla c^m = 0 \quad \text{a.e. in} \quad \Omega, \tag{2.10}$$

$$\delta_t \mathbf{u}^m - \operatorname{div}\Big(\nu(\varphi^m) \nabla \mathbf{u}^m \Big) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^m + \nabla p^m = \vec{\sigma}\; c^m \qquad \text{a.e. in} \quad \Omega, \tag{2.11}$$

$$\operatorname{div} \mathbf{u}^m = 0 \qquad \text{a.e. in} \quad \Omega, \tag{2.12}$$

$$\partial_\eta \varphi^m = 0, \quad \partial_\eta c^m = 0, \quad \mathbf{u}^m = 0 \qquad \text{a.e. on} \quad \partial\Omega, \tag{2.13}$$

assuming

$$\varphi^0 = \varphi_0(x), \quad c^0 = c_0(x), \quad \mathbf{u}^0 = \mathbf{u}_0(x).$$

Here, we used the notation

$$\delta_t \varphi^m = (\varphi^m - \varphi^{m-1})/k, \quad \delta_t c^m = (c^m - c^{m-1})/k, \quad \delta_t \mathbf{u}^m = (\mathbf{u}^m - \mathbf{u}^{m-1})/k,$$

and φ^m, c^m and $\mathbf{u}^m$, $m = 1, ..., N$, are expected to be approximations of $\varphi(x, t_m)$, $c(x, t_m)$ and $\mathbf{u}(x, t_m)$, respectively.

The following existence result for discrete scheme (2.9), (2.10), (2.11), (2.12), (2.13) will be proved by Galerkin method.

Theorem 2.1 *For time-step k small enough there exists a solution $(\varphi^m, c^m, \mathbf{u}^m)$ of the approximate problem* (2.9), (2.10), (2.11), (2.12), (2.13).

With this result we may introduce the Rothe's function given by

$$\widetilde{\varphi}_k(t) = \varphi^{m-1} + (t - t_{m-1})\delta_t\varphi^m, \quad \widetilde{c}_k(t) = c^{m-1} + (t - t_{m-1})\delta_t c^m,$$

$$\widetilde{\mathbf{u}}_k(t) = \mathbf{u}^{m-1} + (t - t_{m-1})\delta_t\mathbf{u}^m \quad \text{for} \quad t_{m-1} \le t \le t_m, \ 1 \le m \le N. \tag{2.14}$$

The corresponding step function is given by $(\varphi_k(t), c_k(t), \mathbf{u}_k(t)) = (\varphi^m, c^m, \mathbf{u}^m)$ for $t_{m-1} \le t \le t_m$, $1 \le m \le N$ and $(\varphi_k(0), c_k(0), \mathbf{u}_k(0)) = (\varphi_0, c_0, \mathbf{u}_0)$.

The sequence of Rothe's function $(\widetilde{\varphi}_k, \widetilde{c}_k, \widetilde{\mathbf{u}}_k)$ can be expected to converge for $k \to 0$, in suitable function spaces. The limit function $(\varphi, c, \mathbf{u})$ is expected to be a solution to the problem (1.1), (1.2), (1.3), (1.4), (1.5). Actually, the main result of this paper is

Theorem 2.2 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$ *holds.*

Suppose $(\varphi_0, c_0, \mathbf{u}_0) \in (L^2(\Omega))^2 \times H$. *Then there exists functions* $(\varphi, c, \mathbf{u})$ *satisfying*

$$\begin{aligned} &(\varphi, c) \in (L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))^2, \\ &\mathbf{u} \in L^\infty(0, T; H) \cap L^2(0, T; V), \end{aligned} \tag{2.15}$$

such that $(\varphi(0), c(0), \mathbf{u}(0)) = (\varphi_0, c_0, \mathbf{u}_0)$ *and*

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega \varphi(t)\,\phi\,dx\right) + \xi^2 \int_\Omega \nabla\varphi(t)\,\nabla\phi\,dx + \int_\Omega \mathbf{u}(t)\cdot\nabla\varphi(t)\,\phi\,dx = \\ &\qquad \int_\Omega (F_1(\varphi(t)) + c(t)F_2(\varphi(t)))\phi\,dx, \end{aligned} \tag{2.16}$$

$$\frac{d}{dt}\left(\int_\Omega c(t)\, z\, dx\right) + \int_\Omega D_1(\varphi(t))\nabla c(t)\nabla z\, dx + \int_\Omega \mathbf{u}(t)\cdot\nabla c(t)\, z\, dx = \\ -\int_\Omega D_2(\varphi(t), c(t))\nabla\varphi(t) z\, dx, \tag{2.17}$$

$$\frac{d}{dt}\left(\int_\Omega \mathbf{u}(t)\,\mathbf{v}\, dx\right) + \int_\Omega \nu(\varphi(t))\nabla\mathbf{u}(t)\,\nabla\mathbf{v}\, dx + \int_\Omega (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\,\mathbf{v}\, dx = \\ \int_\Omega \vec{\sigma}\; c(t)\mathbf{v}\, dx \tag{2.18}$$

for all $(\phi, z, \mathbf{v}) \in (H^1(\Omega))^2 \times V$.

3 Auxiliary Problem

In subsequent proofs we will utilize the following auxiliary problem:

$$\begin{aligned} &-k\,\xi^2\Delta\varphi + k\,\mathbf{u}\cdot\nabla\varphi + \varphi = k\,F_1(\varphi) + k\,c\,F_2(\varphi) + f \quad && \text{in } \Omega,\\ &-k\,\mathrm{div}(D_1(\varphi)\nabla c + D_2(c,\varphi)\nabla\varphi) + k\,\mathbf{u}\cdot\nabla c + c = g \quad && \text{in } \Omega,\\ &\partial_\eta\varphi = 0, \qquad \partial_\eta c = 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.19}$$

Proposition 3.1 *Assume that hypotheses* $(\mathbf{H}_0)$*,* $(\mathbf{H}_2)$*,* $(\mathbf{H}_3)$*,*$(\mathbf{H}_4)$ *hold. For* k *small enough, and* $\mathbf{u} \in V$*, and* $(f, g) \in (L^2(\Omega))^2$ *there exists a solution* $(\varphi, c) \in (H^1(\Omega))^2$ *of problem* (3.19) *satisfying*

$$||\varphi||_{1,2,\Omega} + ||c||_{1,2,\Omega} \leq M(1 + ||\,f\,||_{2,\Omega} + ||\,g\,||_{2,\Omega}). \tag{3.20}$$

where the constant M depends on Ω, ξ, and α_l with $l = 1, 2$.

3.1 Proof of Proposition 3.1

We prove the existence and uniqueness of problem (3.19) using a regularization technique: an auxiliary positive parameter will be introduced in the equations in such way that the original problem will be transformed in regularized problem. By solving this, one hopes to recover the solution of the original problem as the parameter approaches zero. To accomplish such program, we will firstly solve the regularized problem by using the Leray-Schauder degree theory (see Section 8.3, p. 56 in Deimling [7]).

Regularized Problem

We regularize problem (3.19) by changing the functions $u(x,t)$ in equations. For this, we wish to smooth $\mathbf{u}$, while keeping the Dirichlet conditions and the divergence-free property. One possible way is the following: let $\mathbf{u} \in V$, we set $\overline{\mathbf{u}}_\varepsilon$ to be the truncation in Ω_ε of $\mathbf{u}$ (extended by 0 to Ω) as defined in [15, Appendix A] and we defined ζ_ε by $\overline{\mathbf{u}}_\varepsilon * \eta_{\varepsilon/2}$. Then ζ_ε, which vanishes near $\partial\Omega$, is smooth in x and satisfies $\operatorname{div} \zeta_\varepsilon = 0$ in $\mathbb{R}^3$.

We show the following result of the existence for the associated regularized problem:

Lemma 3.1 *Assume that hypotheses* $(\mathbf{H}_0)$-$(\mathbf{H}_4)$ *hold. Fix* $\varepsilon \in (0,1]$ *and assume that* $(f,g) \in (L^2(\Omega))^2$ *and* $\mathbf{u} \in V$*. For* k *small enough there exists a solution* $(\varphi_\varepsilon, c_\varepsilon) \in (H^1(\Omega))^2$ *of the following problem:*

$$-k\,\xi^2 \Delta\varphi_\varepsilon + k\,\zeta_\varepsilon \cdot \nabla\varphi_\varepsilon + \varphi_\varepsilon = k\,F_1(\varphi_\varepsilon) + k\,c_\varepsilon\,F_2(\varphi_\varepsilon) + f \quad \text{in } \Omega, \tag{3.21}$$

$$-k\operatorname{div}(D_1(\varphi_\varepsilon)\nabla c_\varepsilon + D_2(c_\varepsilon,\varphi_\varepsilon)\nabla\varphi_\varepsilon) + k\,\zeta_\varepsilon \cdot \nabla c_\varepsilon + c_\varepsilon = g \quad \text{in } \Omega, \tag{3.22}$$

$$\partial_\eta \varphi_\varepsilon = 0, \qquad \partial_\eta c_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{3.23}$$

We will use the Leray-Schauder degree theory to prove existence of the solution to problem (3.21), (3.22), (3.23). For this, we will reformulate the problem as $\Phi(1,\varphi_\varepsilon,c_\varepsilon) = (\varphi_\varepsilon,c_\varepsilon)$, where $\Phi(\lambda,\cdot)$ is a compact homotopy depending on a parameter $\lambda \in [0,1]$ to be described shortly. Basic tool in our argument is L_p-theory of elliptic linear equations.

Proof of Lemma 3.1: For simplicity of notation, in this proof we will omit the subscript ε and consider the homotopy

$$\begin{gathered}\Phi : [0,1] \times H^1(\Omega) \times L^4(\Omega) \to H^1(\Omega) \times L^4(\Omega) \quad \text{as} \\ \Phi(\lambda, \psi, z) = (\varphi, c),\end{gathered} \tag{3.24}$$

where (φ, c) is the unique solution of the following problem:

$$-k\,\xi^2 \Delta\varphi + k\,\zeta \cdot \nabla\varphi + \varphi = \lambda\, k\, F_1(\psi) + \lambda\, k\, z\, F_2(\psi) + \lambda\, f \quad \text{in } \Omega, \tag{3.25}$$

$$-k\,\mathrm{div}(D_1(\varphi)\nabla c) + k\,\zeta \cdot \nabla c + c = \lambda\, k\,\mathrm{div}(D_2(z,\varphi)\nabla\varphi) + \lambda\, g \quad \text{in } \Omega, \tag{3.26}$$

$$\partial_\eta \varphi = 0, \qquad \partial_\eta c = 0 \quad \text{on } \partial\Omega. \tag{3.27}$$

We have to check that $\Phi(\lambda, \cdot)$ is well defined. In fact, observe that $\zeta \in L^4(\Omega)$, $z \in L^4(\Omega)$ $f \in L^2(\Omega)$ and, using $(\mathbf{H}_4)$ and L_p-theory elliptic linear equation (see [11] chapter 3), we conclude that the equation (3.25) has a unique solution $\varphi \in H^2(\Omega)/\mathbb{R}$. In addition, $(\mathbf{H}_2)$-$(\mathbf{H}_3)$, $\zeta \in L^4(\Omega)$ and $g \in L^2(\Omega)$ imply again by L_p-theory elliptic linear equation that there is a unique solution $c \in H^1(\Omega)/\mathbb{R}$ for the equation (3.22).

To check the continuity of $\Phi(\lambda, \cdot)$, let $(\lambda_n, \psi_n, z_n) \to (\lambda, \psi, z)$ in $[0,1] \times H^1(\Omega) \times L^4(\Omega)$. Denoting $\Phi(\lambda_n, \psi_n, z_n) = (\varphi_n, c_n)$, from (3.24), we write

$$\begin{aligned}-k\,\xi^2 \Delta\varphi_n + k\,\zeta \cdot \nabla\varphi_n + \varphi_n = \lambda_n\, k\, F_1(\psi_n) + \lambda_n\, k\, z_n\, F_2(\psi_n) \\ +\lambda_n\, f \qquad\qquad\qquad\qquad \text{in } \Omega,\end{aligned} \tag{3.28}$$

$$\begin{aligned}-k\,\mathrm{div}(D_1(\varphi_n)\nabla c_n) + k\,\zeta \cdot \nabla c_n + c_n = \lambda_n\, k\,\mathrm{div}(D_2(z_n,\varphi_n)\nabla\varphi_n) \\ +\lambda_n\, g \qquad\qquad\qquad\qquad \text{in } \Omega,\end{aligned} \tag{3.29}$$

$$\partial_\eta \varphi = 0, \qquad \partial_\eta c = 0 \qquad\qquad\qquad \text{on } \partial\Omega. \tag{3.30}$$

Applying the L_p-theory elliptic linear equations of the equation (3.28) with $\zeta \in L^4(\Omega)$, $f \in L^2(\Omega)$ together $(\mathbf{H}_0)$ and $(\mathbf{H}_4)$, we obtain that $\varphi_n \in H^1(\Omega)$ and

$$||\varphi_n||_{1,2,\Omega} \le M|\lambda_n|\Big(1 + \,||z_n||_{2,\Omega} + ||\, f\,||_{2,\Omega}\Big). \tag{3.31}$$

Now, applying the L_p-theory elliptic linear equations of the equation (3.29) with $\zeta \in L^4(\Omega)$, $g \in L^2(\Omega)$ together $(\mathbf{H}_0)$,$(\mathbf{H}_2)$, $(\mathbf{H}_3)$, we obtain that $c_n \in H^1(\Omega)$ and

$$||c_n||_{1,2,\Omega} \leq M|\lambda_n|\Big(||\nabla\varphi_n||_{2,\Omega)} + ||\, g\,||_{2,\Omega}\Big). \tag{3.32}$$

and using (3.31) in (3.32), we have

$$||c_n||_{1,2,\Omega} \leq M|\lambda_n|^2\Big(1 + \,||z_n||_{2,\Omega} + ||\, f\,||_{2,\Omega}\Big) + M\,|\lambda_n|\,||\, g\,||_{2,\Omega}. \tag{3.33}$$

Since the sequence (λ_n, ψ_n, z_n) is bounded (uniformly with respect to n) in $[0,1] \times H^1(\Omega) \times L^4(\Omega)$, we obtain

$$||\varphi_n||_{1,2,\Omega} \leq M \quad \text{ and } \quad ||c_n|_{1,2,\Omega} \leq M,$$

consequently (φ_n, c_n) is bounded (uniformly with respect to n) in $H^1(\Omega) \times L^4(\Omega)$.

Moreover, applying the L_p-theory elliptic linear equations of the equation (3.28), we obtain

$$||\varphi_n||_{2,2,\Omega} \leq M|\lambda_n|\Big(1 + \,||z_n||_{2,\Omega)} + ||\, f\,||_{2,\Omega}\Big) + ||\varphi_n||_{2,\Omega}, \tag{3.34}$$

and using (3.31) in (3.34), we get

$$||\varphi_n||_{2,2,\Omega} \leq M|\lambda_n|\Big(1 + \,||z_n||_{2,\Omega)} + ||\, f\,||_{2,\Omega}\Big), \tag{3.35}$$

consequently φ_n is bounded (uniformly with respect to n) in $H^2(\Omega)$.

Since $H^2(\Omega)\times H^1(\Omega)$ is compactly embedding into $H^1(\Omega)\times L^q(\Omega)$, $2 \leq q < 6$. Then, there exist $(\varphi, c) \in H^2(\Omega) \times H^1(\Omega)$ and subsequences of (φ_n, c_n) (we still denote (φ_n, c_n)) such that, as $n \to +\infty$,

$$(\varphi_n, c_n) \rightharpoonup (\varphi, c) \quad \text{weakly in} \quad H^2(\Omega) \times H^1(\Omega), \tag{3.36}$$

$$(\varphi_n, c_n) \to (\varphi, c) \quad \text{strongly in} \quad H^1(\Omega) \times L^q(\Omega). \tag{3.37}$$

where $2 \leq q < 6$.

Now, let us verify that $\Phi(\lambda, \psi, z) = (\varphi, c)$, in other words, that (φ, c) is solution of (3.25), (3.26), (3.27). For this, we are going to pass to the limit with respect to the above subsequence of problem (3.28), (3.29), (3.30).

Next, we infer from (3.37) and the convergence $(\psi_n, z_n) \to (\psi, z)$ in $H^1(\Omega)\times L^4(\Omega)$ that

$$(\varphi_n, \psi_n, z_n) \to (\varphi, \psi, z) \quad \text{a.e. in } \Omega. \tag{3.38}$$

Combining (3.38) with the assumptions $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ of $D_1(\cdot)$, $D_2(\cdot,\cdot)$ and $F_l(\cdot)$ with $l = 1, 2$, we deduce

$$\begin{aligned} D_1(\varphi_n) &\to D_1(\varphi) && \text{a.e. in } \Omega, \\ F_l(\psi_n) &\to F_l(\psi) && \text{a.e. in } \Omega, \\ D_2(z_n, \varphi_n) &\to D_2(z_n, \varphi) && \text{a.e. in } \Omega. \end{aligned} \tag{3.39}$$

Now, let $2 \leq p < \infty$ and $\beta_n = |D_1(\varphi_n) - D_1(\varphi)|^p$, $\delta_n = |F_l(\psi_n) - F_l(\psi)|^p$, $\gamma_n = |D_2(z_n, \varphi_n) - D_2(z, \varphi)|^p$. Using $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ and (3.39), we obtain

$$|\beta_n| \leq (2\rho_1)^p, \quad |\gamma_n| \leq (2\alpha_0)^p, \quad |\delta_n| \leq (2\alpha_l)^p, \quad \beta_n \to 0, \quad \delta_n \to 0, \quad \gamma_n \to 0.$$

Then, by Lebesgue's Dominated Convergence Theorem, we conclude

$$\begin{aligned} D_1(\varphi_n) &\to D_1(\varphi) && \text{in } L^p(\Omega) \text{ strongly}, \\ F_l(\psi_n) &\to F_l(\psi) && \text{in } L^p(\Omega) \text{ strongly}, \\ D_2(z_n, \varphi_n) &\to D_2(z, \varphi) && \text{in } L^p(\Omega) \text{ strongly}. \end{aligned} \tag{3.40}$$

with $2 \leq p < \infty$ and $l = 1, 2$.

By (3.36) we have that $(\nabla\varphi_n, \nabla c_n) \rightharpoonup (\nabla\varphi, \nabla c)$ in $(L^2(\Omega))^2$ weakly. Moreover, $(\psi_n, z_n) \to (\psi, z)$ in $H^1(\Omega) \times L^4(\Omega)$. Combining this results with (3.40), we obtain

$$\begin{aligned} D_1(\varphi_n)\,\nabla c_n &\rightharpoonup D_1(\varphi)\nabla c && \text{in } L^2(\Omega) \text{ weakly}, \\ z_n\,F_2(\psi_n) &\rightharpoonup z\,F_2(\psi) && \text{in } L^2(\Omega) \text{ weakly}, \\ D_2(z_n, \varphi_n)\nabla\varphi_n &\rightharpoonup D_2(z, \varphi)\nabla\varphi && \text{in } L^2(\Omega) \text{ weakly}. \end{aligned} \tag{3.41}$$

Thus, passing to the limit in the system of equations (3.28) and (3.29), using the convergence (3.36), (3.37) and (3.41), we obtain the equations (3.25) and (3.26).

Moreover, the uniqueness can be obtained in a standard way and the required boundary conditions are included in the definitions of the functional spaces where (φ, c) is in.

On the other hand, note that for any given subsequence of $\{\Phi(\lambda_n, \psi_n, z_n)\}$, the above arguments can be applied to conclude that this subsequence admits another subsequence converging to a solution of (3.25), (3.26), (3.27). Since (ψ, z) is also fixed and the solution of this last problem is unique, we conclude that $\{\Phi(\lambda_n, \psi_n, z_n)\}$ is a sequence with the property that any one of its subsequences has by its turn a subsequence converging to a limit that is independent of the chosen subsequence. Hence, $\{\Phi(\lambda_n, \psi_n, z_n)\}$ converges to this limit, and the continuity of Φ is proved.

The mapping Φ given by (3.24) is also compact. In fact, if (ψ_n, z_n) is any bounded sequence in $H^1(\Omega) \times L^4(\Omega)$, the above arguments can be applied to obtain exactly the same sort of estimates for $\Phi(\lambda, \psi_n, z_n)$. These imply that (φ_n, c_n) is bounded (uniformly with respect to n) in $H^2(\Omega) \times H^1(\Omega)$. Since that embedding of $H^2(\Omega) \times H^1(\Omega)$ into $H^1(\Omega) \times L^q(\Omega)$, $2 \leq q < 6$, is compact, we have that (φ_n, c_n) is relatively compact in $H^1(\Omega) \times L^4(\Omega)$. Therefore, the compactness is proved.

In the following, we will show that any possible fixed point of $\Phi(\lambda, \cdot)$ can be estimated independently of $\lambda \in [0, 1]$, that is, we will show that if $(\varphi, c) \in H^1(\Omega) \times L^4(\Omega)$ is such that $\Phi(\lambda, \varphi, c) = (\varphi, c)$, for some $\lambda \in [0, 1]$, then there exists a constant $\beta > 0$ such that

$$||(\varphi, c)||_{H^1(\Omega)\times L^4(\Omega)} < \beta \tag{3.42}$$

For this, we recall that such fixed point $(\varphi, c) \in H^1(\Omega) \times L^4(\Omega)$ solves the problem

$$-k\,\xi^2\Delta\varphi + k\,\zeta\cdot\nabla\varphi + \varphi = \lambda\,k\,F_1(\psi) + \lambda\,k\,z\,F_2(\psi) + \lambda\,f \quad \text{in } \Omega, \tag{3.43}$$

$$-k\,\mathrm{div}(D_1(\varphi)\nabla c) + k\,\zeta\cdot\nabla c + c = \lambda\,k\,\mathrm{div}(D_2(z,\varphi)\nabla\varphi) + \lambda\,g \quad \text{in } \Omega, \tag{3.44}$$

$$\partial_\eta\varphi = 0, \qquad \partial_\eta c = 0 \quad \text{on } \partial\Omega. \tag{3.45}$$

We start by multiplying (3.43) by φ_ϵ and (3.44) by c_ϵ. After integrating of the result over Ω, we use $(\mathbf{H}_2)$ $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ together $\mathrm{div}\,\zeta = 0$, Green's formula, Hölder's and Young's inequalities to obtain

$$\xi^2||\nabla\varphi||^2_{2,\Omega} + \frac{1}{8\,k}||\varphi||^2_{2,\Omega} \le k\,M\,\alpha_1^2 + k\,\alpha_2^2||c||^2_{2,\Omega} + \frac{2}{k}||f||^2_{2,\Omega},$$

$$k\,\rho_0||\nabla c||^2_{2,\Omega} + ||c||^2_{2,\Omega} \le \frac{k\alpha_0^2}{\rho_0}||\nabla\varphi||^2_{2,\Omega} + ||g||^2_{2,\Omega}.$$

Thus, by taking $k \le \min\left\{\dfrac{1}{2\,\alpha_2^2}, \dfrac{\rho_0\,\xi^2}{2\,\alpha_0^2}\right\}$, we conclude

$$\xi^2||\nabla\varphi||^2_{2,\Omega} + \frac{1}{8\,k}||\varphi||^2_{2,\Omega} \le M + \frac{1}{2}||c||^2_{2,\Omega} + \frac{2}{k}||f||^2_{2,\Omega}, \tag{3.46}$$

$$k\,\rho_0||\nabla c||^2_{2,\Omega} + ||c||^2_{2,\Omega} \le \frac{\xi^2}{2}||\nabla\varphi||^2_{2,\Omega} + ||g||^2_{2,\Omega}. \tag{3.47}$$

Next, adding (3.46) and (3.47), we get

$$||\varphi||_{1,2,\Omega} + ||c||_{1,2,\Omega} \le M\Big(1 + ||f||_{2,\Omega} + ||g||_{2,\Omega}\Big). \tag{3.48}$$

where the constant M depends on Ω, ξ, ρ_0 and α_l, $l = 1, 2$.

Thus, it is enough to take any constant $\beta > M\Big(1 + ||f||_{2,\Omega} + ||g||_{2,\Omega}\Big)$ to obtain the stated result. By denoting

$$B_\beta = \Big\{(\varphi, c) \in H^1(\Omega) \times L^4(\Omega)\ ;\ ||(\varphi, c)||_{H^1(\Omega)\times L^4(\Omega)} < \beta\Big\}$$

(3.42) ensures in particular that

$$\Phi(\lambda,\varphi,c) \neq (\varphi,c) \qquad \forall(\varphi,c)\in\partial B_\beta,\ \ \forall\lambda\in[0,1] \tag{3.49}$$

According to property (3.49) and the compactness of $\Phi(\lambda,\cdot)$, we may consider the Leray-Schauder degree $D(Id-\Phi(\lambda,\cdot),B_\beta,0)$, $\forall\lambda\in[0,1]$ (see [7]). The homotopy invariance of the degree implies

$$D(Id-\Phi(0,\cdot),B_\beta,0) = D(Id-\Phi(1,\cdot),B_\beta,0) \tag{3.50}$$

Now, we consider $\Phi(0,\varphi,c)=(\varphi,c)$ given by

$$\begin{array}{ll} -k\,\xi^2\Delta\varphi + k\,\zeta\cdot\nabla\varphi + \varphi = 0 & \text{in } \Omega, \\ -k\,\mathrm{div}(D_1(\varphi)\nabla c) + k\,\zeta\cdot\nabla c + c = 0 & \text{in } \Omega, \\ \partial_\eta\varphi = 0, \qquad \partial_\eta c = 0 & \text{on } \partial\Omega. \end{array} \tag{3.51}$$

Applying L_p-theory of elliptic linear equations together the assumption $(\mathbf{H}_2)$, we obtain that exists an unique solution $(\varphi,c)\in(H^1(\Omega)/\mathbb{R})^2$ of the problem (3.51).

Now, if we choose $\beta>0$ large enough so that the ball B_β contains the unique solution of problem $\Phi(0,\varphi,c)=(\varphi,c)$, then $D(Id-\Phi(0,\cdot),B_\beta,0)=1$, and from (3.50) we conclude that problem (3.21), (3.22), (3.23) has a solution $(\varphi_\varepsilon,c_\varepsilon)\in H^1(\Omega)\times L^4(\Omega)$.

Again, applying L_p-theory of elliptic linear equations together the assumptions $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$, it is easy to conclude that $(\varphi_\varepsilon,c_\varepsilon)\in H^2(\Omega)\times H^1(\Omega)$.

Uniform Estimates

Next we obtain some uniform estimates. We start by multiplying (3.21) by φ and (3.22) by c. After integrating of the result over Ω, we use $(\mathbf{H}_2)$ $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ together $\mathrm{div}\,\zeta=0$, Green's formula, Hölder's and Young's inequalities to obtain

$$\xi^2||\nabla\varphi_\varepsilon||^2_{2,\Omega} + \frac{1}{8\,k}||\varphi_\varepsilon||^2_{2,\Omega} \le k\,M\,\alpha_1^2 + k\,\alpha_2^2||c_\varepsilon||^2_{2,\Omega} + \frac{2}{k}||f||^2_{2,\Omega},$$

$$k\,\rho_0||\nabla c_\varepsilon||^2_{2,\Omega} + ||c_\varepsilon||^2_{2,\Omega} \le \frac{k\alpha_0^2}{\rho_0}||\nabla\varphi_\varepsilon||^2_{2,\Omega} + ||g||^2_{2,\Omega}.$$

In an analogous way as in (3.46) and (3.47), we obtain the following estimates for k small enough:

$$||\varphi_\varepsilon||_{1,2,\Omega} + ||c_\varepsilon||_{1,2,\Omega} \le M\Big(1 + ||f||_{2,\Omega} + ||g||_{2,\Omega}\Big). \tag{3.52}$$

where the constant M depends on Ω, ξ, ρ_0 and α_l, $l = 1, 2$ but is independent of ε.

This completes the proof of Lemma 3.1.

□

Taking the limit

Next we pass to limits as $\varepsilon \to 0^+$. For this we deduce from (3.52) that $(\varphi_\varepsilon, c_\varepsilon)$ is bounded (uniformly with respect to ε) in $(H^1(\Omega))^2$. Moreover, $H^1(\Omega)$ is compactly embedding into $L^q(\Omega)$ with $2 \le q < 6$. Hence, there exist $(\varphi, c) \in (H^1(\Omega))^2$ and subsequences of $(\varphi_\varepsilon, c_\varepsilon)$ (we still denote $(\varphi_\varepsilon, c_\varepsilon)$) such that, as $\varepsilon \to 0^+$,

$$(\varphi_\varepsilon, c_\varepsilon) \rightharpoonup (\varphi, c) \quad \text{weakly} \quad \text{in} \quad (H^1(\Omega))^2, \tag{3.53}$$

$$(\varphi_\varepsilon, c_\varepsilon) \to (\varphi, c) \quad \text{strongly in} \quad L^q(\Omega). \tag{3.54}$$

Using the estimate (3.53) and that $\zeta_\varepsilon \to u$ in $L^2(Q)^3$, we obtain

$$\begin{aligned} \zeta_\varepsilon \cdot \nabla\varphi_\varepsilon &\rightharpoonup \mathbf{u}\cdot\nabla\varphi \quad \text{weakly in} \quad L^2(\Omega), \\ \zeta_\varepsilon \cdot \nabla c_\varepsilon &\rightharpoonup \mathbf{u}\cdot\nabla c \quad \text{weakly in} \quad L^2(\Omega). \end{aligned} \tag{3.55}$$

In an analogous way as in (3.41), we obtain that

$$\begin{aligned}
&D_1(\varphi_\varepsilon)\,\nabla c_\varepsilon \rightharpoonup D_1(\varphi)\nabla c && \text{in} \quad L^2(\Omega) \quad \text{weakly},\\
&c_\varepsilon\, F_2(\varphi_\varepsilon) \rightharpoonup c\, F_2(\varphi) && \text{in} \quad L^2(\Omega) \quad \text{weakly},\\
&D_2(c_\varepsilon,\varphi_\varepsilon)\nabla\varphi_\varepsilon \rightharpoonup D_2(c,\varphi)\nabla\varphi && \text{in} \quad L^2(\Omega) \quad \text{weakly}.
\end{aligned} \tag{3.56}$$

We now take the limit in problem (3.21), (3.22), (3.23) and find that (φ, c) is solution of problem (3.19).

This completes the proof of Proposition 3.1.

□

4 Proof of Theorema 2.1

We employ a semi-Galerkin method. For a fixed m, assuming that φ^{m-1}, c^{m-1} and $\mathbf{u}^{m-1}$ are already known and we consider the following nonlinear system:

$$-k\xi^2\Delta\varphi + k\mathbf{u}\cdot\nabla\varphi + \varphi = k\,F_1(\varphi) + k\,c\,F(\varphi) + f \qquad \text{in} \quad \Omega, \tag{4.57}$$

$$-k\,\mathrm{div}(D_1(\varphi)\nabla c + D_2(c,\varphi)\nabla\varphi) + k\,\mathbf{u}\cdot\nabla c + c = g \qquad \text{in}\ \Omega, \tag{4.58}$$

$$-k\mathrm{div}\Big(\nu(\varphi)\nabla\mathbf{u}\Big) + k(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u} + k\nabla p = k\,\vec{\sigma}\,c + \mathbf{h} \qquad \text{in} \quad \Omega, \tag{4.59}$$

$$\begin{aligned}
&\mathrm{div}\,\mathbf{u} = 0 && \text{in} \quad \Omega,\\
&\partial_\eta\varphi = 0, \quad \partial_\eta c = 0, \quad \mathbf{u} = 0 && \text{on} \quad \partial\Omega,
\end{aligned} \tag{4.60}$$

where $(\varphi, c, \mathbf{u}) = (\varphi^m, c^m, \mathbf{u}^m)$ and $(f, g, \mathbf{h}) = (\varphi^{m-1}, c^{m-1}, \mathbf{u}^{m-1})$.

We employ a semi-Galerkin method to prove the Theorem 2.1. We consider a complete orthonormal basis of H and orthogonal basis of V consisting of eigenfunctions of the Stokes operator $\mathbf{A}$ denoted as $\{\mathbf{w}_i\}_{i\geq 1}$. Denote by $\mathbf{V}_j$ the finite vector space spanned by $\{\mathbf{w}_i\}_{1\leq i\leq j}$.

For each $j \geq 1$, we then consider the approximate problem of finding $(\varphi_j, c_j, \mathbf{u}_j) \in (H^1(\Omega))^2 \times \mathbf{V}_j$ satisfying:

$$
\begin{aligned}
& k \int_\Omega \nu(\varphi_j)\, \nabla \mathbf{u}_j\, \nabla \mathbf{v}\, dx + k \int_\Omega (\mathbf{u}_j \cdot \nabla)\mathbf{u}_j\, \mathbf{v}\, dx + \int_\Omega \mathbf{u}_j\, \mathbf{v}\, dx = \\
& \qquad\qquad k \int_\Omega \vec{\sigma}\; c_j\, \mathbf{v}\, dx + \int_\Omega \mathbf{h}\, \mathbf{v}\, dx, \qquad \forall\, \mathbf{v} \in \mathbf{V}_j, \qquad (4.61) \\
& -k\,\xi^2 \Delta\varphi_j + k\, \mathbf{u}_j \cdot \nabla\varphi_j + \varphi_j = k\, F_1(\varphi_j) + k\, c_j\, F_2(\varphi_j) + f \quad \text{in } \Omega, \\
& -k\, \mathrm{div}(D_1(\varphi_j)\nabla c_j + D_2(c_j, \varphi_j)\nabla\varphi_j) + k\, \mathbf{u}_j \cdot \nabla c_j + c_j = g \quad \text{in } \Omega, \quad (4.62) \\
& \qquad\qquad \partial_\eta \varphi_j = 0, \qquad \partial_\eta c_j = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Existence of approximate solutions

The next lemma guarantees the existence of a solution of the previous problem.

Lemma 4.1 *Assume that hypotheses* $(\mathbf{H}_0)$-$(\mathbf{H}_4)$ *hold. Let* $(f, g) \in (L^2(\Omega))^2$. *For k small enough and for each integer* $j \geq 1$ *there exists a solution* $(\varphi_j, c_j, \mathbf{u}_j) \in (H^1(\Omega))^2 \times \mathbf{V}_j$ *of problem* (4.61), (4.62).

Proof: We infer from Proposition 3.1 that for k small enough there exists a solution $(\varphi_j, c_j) \in (H^1(\Omega))^2$ of problem (4.62) satisfying

$$
||\varphi_j||_{1,2,\Omega} + ||c_j||_{1,2,\Omega} \leq M(1 + ||\, f\,||_{2,\Omega} + ||\, g\,||_{2,\Omega}). \qquad (4.63)
$$

where the constant M depends on Ω, ξ, ρ_0 and α_l, $l = 1, 2$.

To prove the existence of $\mathbf{u}_j$ we using the analogous arguments as in [20, p. 164]. We define Ψ_j by

$$
\begin{aligned}
\Big\langle \Psi_j(\mathbf{u}_j), \mathbf{v} \Big\rangle = {} & k \int_\Omega \nu(\varphi_j)\, \nabla \mathbf{u}_j\, \nabla \mathbf{v}\, dx + k \int_\Omega (\mathbf{u}_j \cdot \nabla)\mathbf{u}_j\, \mathbf{v}\, dx + \int_\Omega \mathbf{u}_j\, \mathbf{v}\, dx \\
& -k \int_\Omega \vec{\sigma}\; c_j\, \mathbf{v}\, dx - \int_\Omega \mathbf{h}\, \mathbf{v}\, dx, \quad \forall\, \mathbf{u}_j, \mathbf{v} \in \mathbf{V}_j.
\end{aligned}
$$

The continuity the mapping Ψ_j is obvious; moreover

$$\Big\langle \Psi_j(\mathbf{u}_j), \mathbf{u}_j \Big\rangle > k\,\nu_0 ||\mathbf{u}_j||_V^2 + ||\mathbf{u}_j||_{2,\Omega}^2 - (k\,M||c_j||_{2,\Omega} + ||\mathbf{h}||_{2,\Omega})||\mathbf{u}_j||_V.$$

Note that we use $(\mathbf{H}_1)$ in above inequality. Now, from (4.63), we obtain

$$\Big\langle \Psi_j(\mathbf{u}_j), \mathbf{u}_j \Big\rangle > \Big(k\,\nu_0 ||\mathbf{u}_j||_V - k\,M(1 + ||f||_{2,\Omega} + ||g||_{2,\Omega}) - ||\mathbf{h}||_{2,\Omega}\Big)||\mathbf{u}_j||_V,$$

It follows that $\Big\langle \Psi_j(\mathbf{u}_j), \mathbf{u}_j \Big\rangle > 0$ for $||\mathbf{u}_j||_V > \dfrac{1}{k\nu_0}\Big(k\,M(1 + ||f||_{2,\Omega} + ||g||_{2,\Omega}) + ||\mathbf{h}||_{2,\Omega}\Big)$.

Thus, the hypotheses of the Lemma 1.4, p. 164, of Temam [20] are satisfied for our case and therefore there exists a solution $\mathbf{u}_j$ of (4.61).

$\square$

Passage to the limit

Then setting $\mathbf{v} = \mathbf{u}_j$ in (4.61) we see that the sequence is bounded in V, independently of j, thanks to $(\mathbf{H}_1)$:

$$k\,\nu_0 ||\mathbf{u}_j||_V^2 + \frac{1}{4}||\mathbf{u}_j||_{2,\Omega}^2 \leq k^2\,M||c_j||_{2,\Omega}^2 + \frac{1}{2}||\mathbf{h}||_{2,\Omega}^2.$$

From (4.63), we get

$$k\,\nu_0 ||\mathbf{u}_j||_V^2 + \frac{1}{4}||\mathbf{u}_j||_{2,\Omega}^2 \leq k^2\,M(1 + ||f||_{2,\Omega}^2 + ||g||_{2,\Omega}^2) + \frac{1}{2}||\mathbf{h}||_{2,\Omega}^2 \qquad (4.64)$$

Hence we deduce from (4.63) and (4.64) that $(\varphi_j, c_j, \mathbf{u}_j)$ is bounded (uniformly with respect to j) in $(H^1(\Omega))^2 \times V$. Since $(H^1(\Omega))^2 \times V$ is compactly embedding into $(L^q(\Omega))^2 \times H$, for any $2 \leq q < 6$. Hence, there exists $(\varphi, c, \mathbf{u})$ in $(H^1(\Omega))^2 \times V$ and subsequences of $(\varphi_j, c_j, \mathbf{u}_j)$, still denoted $(\varphi_j, c_j, \mathbf{u}_j)$, such that, as $j \to +\infty$,

$$\begin{array}{lll} (\varphi_j, c_j) \rightharpoonup (\varphi, c) & \text{weakly in} & (H^1(\Omega))^2, \\ (\varphi_j, c_j) \to (\varphi, c) & \text{strongly in} & (L^q(\Omega))^2, \\ \mathbf{u}_j \rightharpoonup \mathbf{u} & \text{weakly in} & V, \\ \mathbf{u}_j \to \mathbf{u} & \text{strongly in} & H. \end{array} \qquad (4.65)$$

Due to the estimates (4.65), the convergence of almost all the terms in the equation (4.61) will be standard ones, except the firts term that follows from the Lebesgue dominated convergence theorem as in the case of estimates (3.40) and (3.41).

Moreover, the convergence of all the terms in the equations in the problem (4.62) will be analogous as in the proof of Proposition 3.1.

□

5 Proof of Theorem 2.2

5.1 A priori estimates

In this subsection we will be interested in obtaining some *a priori* estimates for the functions φ^m, c^m and $\mathbf{u}^m$. Throughout the analysis k is taken to be sufficiently small.

First, we multiplying the equation (2.9) by φ^m, the equation (2.10) by c^m and the equation (2.11) by $\mathbf{u}^m$, integrating over Ω and using Green's formula and (2.12), we obtain

$$\frac{1}{k}\int_\Omega \varphi^m(\varphi^m - \varphi^{m-1})\,dx + \xi^2\int_\Omega |\nabla\varphi^m|^2\,dx = \int_\Omega F_1(\varphi^m)\varphi^m\,dx + \int_\Omega c^m\,F_2(\varphi^m)\varphi^m\,dx,$$

$$\frac{1}{k}\int_\Omega c^m(c^m - c^{m-1})\,dx + \int_\Omega D_1(\varphi^m)|\nabla c^m|^2\,dx = - \int_\Omega D_2(c^m,\varphi^m)\nabla\varphi^m\nabla c^m\,dx,$$

$$\frac{1}{k}\int_\Omega \mathbf{u}^m(\mathbf{u}^m - \mathbf{u}^{m-1})\,dx + \int_\Omega \nu(\varphi^m)|\nabla\mathbf{u}^m|^2\,dx = \int_\Omega \vec{\sigma}\,c^m\,\mathbf{u}^m\,dx.$$

Using assumptions $(\mathbf{H}_0)$-$(\mathbf{H}_4)$, Hölder's and Young's inequalities and the relation (2.6), we obtain

$$\begin{aligned}\frac{1}{2k}\Big(||\varphi^m||^2_{2,\Omega}-||\varphi^{m-1}||^2_{2,\Omega}+||\varphi^m-\varphi^{m-1}||^2_{2,\Omega}\Big)+\xi^2||\nabla\varphi^m||^2_{2,\Omega}\\ \leq M\Big(1+||c^m||^2_{2,\Omega}+||\varphi^m||^2_{2,\Omega}\Big),\end{aligned}\tag{5.66}$$

$$\begin{aligned}\frac{1}{2k}\Big(||c^m||^2_{2,\Omega}-||c^{m-1}||^2_{2,\Omega}+||c^m-c^{m-1}||^2_{2,\Omega}\Big)+\frac{\rho_0}{2}||\nabla c^m||^2_{2,\Omega}\\ \leq\frac{\alpha_0^2}{2\rho_0}||\nabla\varphi^m||^2_{2,\Omega},\end{aligned}\tag{5.67}$$

$$\begin{aligned}\frac{1}{2k}\Big(||\mathbf{u}^m||^2_{2,\Omega}-||\mathbf{u}^{m-1}||^2_{2,\Omega}+||\mathbf{u}^m-\mathbf{u}^{m-1}||^2_{2,\Omega}\Big)+\nu_0||\nabla\mathbf{u}^m||^2_{2,\Omega}\\ \leq M\Big(||c^m||^2_{2,\Omega}+||\mathbf{u}^m||^2_{2,\Omega}\Big).\end{aligned}\tag{5.68}$$

Multiplying (5.66) by α_0^2/ρ_0 and (5.67) by ξ^2, and adding the resulting, we obtain

$$\begin{aligned}&||\varphi^m||^2_{2,\Omega}-||\varphi^{m-1}||^2_{2,\Omega}+||c^m||^2_{2,\Omega}-||c^{m-1}||^2_{2,\Omega}+||\varphi^m-\varphi^{m-1}||^2_{2,\Omega}\\ &\quad+||c^m-c^{m-1}||^2_{2,\Omega}+k||\nabla\varphi^m||^2_{2,\Omega}+k||\nabla c^m||^2_{2,\Omega}\leq\\ &\qquad M\Big(1+||c^m||^2_{2,\Omega}+||\varphi^m||^2_{2,\Omega}\Big).\end{aligned}$$

Summing these relations for $m=1,2,...,r$, with $1\leq r\leq N$, we find

$$\begin{aligned}&||\varphi^r||^2_{2,\Omega}+||c^r||^2_{2,\Omega}+\sum_{m=1}^{r}\Big(||\varphi^m-\varphi^{m-1}||^2_{2,\Omega}+||c^m-c^{m-1}||^2_{2,\Omega}\Big)+\\ &k\sum_{m=1}^{r}\Big(||\nabla\varphi^m||^2_{2,\Omega}+||\nabla c^m||^2_{2,\Omega}\Big)\leq M\Big(1+||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}+\\ &\qquad k\sum_{m=1}^{r}||c^m||^2_{2,\Omega}+||\varphi^m||^2_{2,\Omega}\Big),\end{aligned}\tag{5.69}$$

where the constant M depends on Ω, ξ, ρ_0, ν_0 and α_l, $l=0,1,2$.

Applying discrete Gronwall's Lemma (5.69), we find

$$\max_{1\leq r\leq N}||\varphi^r||_{2,\Omega}+\max_{1\leq r\leq N}||c^r||_{2,\Omega}\leq M\Big(||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}\Big),\tag{5.70}$$

$$k\sum_{m=1}^{N}\Big(||\nabla\varphi^m||^2_{2,\Omega}+||\nabla c^m||^2_{2,\Omega}\Big)\leq M\Big(||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}\Big)\tag{5.71}$$

and consequently

$$\sum_{m=1}^{N}\Big(||\varphi^m-\varphi^{m-1}||^2_{2,\Omega}+||c^m-c^{m-1}||^2_{2,\Omega}\Big)\le M\Big(||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}\Big). \quad (5.72)$$

Now, summing (5.68) for $m=1,2,...,r$, with $1\le r\le N$ and using (5.70), we get

$$\begin{aligned}||\mathbf{u}^r||^2_{2,\Omega}+\sum_{m=1}^{r}||\mathbf{u}^m-\mathbf{u}^{m-1}||^2_{2,\Omega}+k\nu_0\sum_{m=1}^{r}||\nabla\mathbf{u}^m||^2_{2,\Omega}\le\\ M\Big(||\mathbf{u}_0||^2_H+||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}+k\sum_{m=1}^{r}||\mathbf{u}^m||^2_{2,\Omega}\Big).\end{aligned} \quad (5.73)$$

Therefore, the discrete Gronwall's Lemma (2.1) implies that

$$\max_{1\le r\le N}||\mathbf{u}^r||^2_H\le M\Big(||\mathbf{u}_0||^2_H+||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}\Big) \quad (5.74)$$

and consequently

$$k\sum_{m=1}^{N}||\nabla\mathbf{u}^m||^2_{2,\Omega}\le M\Big(||\mathbf{u}_0||^2_H+||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}\Big), \quad (5.75)$$

$$\sum_{m=1}^{N}||\mathbf{u}^m-\mathbf{u}^{m-1}||^2_{2,\Omega}\le M\Big(||\mathbf{u}_0||^2_H+||\varphi_0||^2_{2,\Omega}+||c_0||^2_{2,\Omega}\Big). \quad (5.76)$$

By rewriting the above estimates in terms of the Rothe function $(\widetilde{\varphi}_k,\widetilde{c}_k,\widetilde{\mathbf{u}}_k)$, and the associated step function $(\varphi_k,c_k,\mathbf{u}_k)$, we obtain

Lemma 5.1 *For k sufficiently small the following estimates are satisfied:*

$$||(\varphi_k,c_k)||_{(L^\infty(0,T;L^2(\Omega)))^2}\le M,\quad ||(\varphi_k,c_k)||_{L^2(0,T;H^1(\Omega)))^2}\le M,$$

$$||\mathbf{u}_k||_{L^\infty(0,T;H)}\le M,\quad ||\mathbf{u}_k||_{L^2(0,T;V)}\le M,$$

$$||(\widetilde{\varphi}_k,\widetilde{c}_k)||_{(L^\infty(0,T;L^2(\Omega)))^2}\le M,\quad ||(\widetilde{\varphi}_k,\widetilde{c}_k)||_{L^2(0,T;H^1(\Omega)))^2}\le M,$$

$$||\widetilde{\mathbf{u}}_k||_{L^\infty(0,T;H)}\le M,\quad ||\widetilde{\mathbf{u}}_k||_{L^2(0,T;V)}\le M.$$

Moreover, $||(\widetilde{\varphi}_k-\varphi_k,\widetilde{c}_k-c_k)||_{(L^2(Q))^2}\le M\,k$, $||\widetilde{\mathbf{u}}_k-\mathbf{u}_k||_{L^2(0,T;H)}\le M\,k$.

Proof: Using estimates (5.70), (5.71) and (5.72), we get

$$||\varphi_k||_{L^\infty(0,T;L^2(\Omega))} = \max_{1\le r\le N} ||\varphi^r||_{2,\Omega} \le M,$$

$$||\nabla\varphi_k||^2_{2,Q} = \sum_{m=1}^{N} \int_{(m-1)k}^{mk} ||\nabla\varphi^m||^2_{2,\Omega}\, dt \ \le k \sum_{m=1}^{N} ||\nabla\varphi^m||^2_{2,\Omega} \ \le \ M,$$

$$\begin{aligned}||\widetilde{\varphi}_k||^2_{2,Q} &= \sum_{m=1}^{N} \int_{(m-1)k}^{mk} ||\varphi^m||^2_{2,\Omega} + ((t-t_m)/k)^2 ||\varphi^m - \varphi^{m-1}||^2_{2,\Omega}\, dt\\ &\le k \sum_{m=1}^{N} ||\varphi^m||^2_{2,\Omega} + \frac{k}{3} \sum_{m=1}^{N} ||\varphi^m - \varphi^{m-1}||^2_{2,\Omega} \le M.\end{aligned}$$

Moreover,

$$\begin{aligned}||\widetilde{\varphi}_k - \varphi_k||^2_{2,Q} &= \sum_{m=1}^{N} \int_{t_{m-1}}^{t_m} ((t-t_m)/k)^2 ||\varphi^m - \varphi^{m-1}||^2_{2,\Omega}\, dt\\ &= \frac{k}{3} \sum_{m=1}^{N} ||\varphi^m - \varphi^{m-1}||^2_{2,\Omega} \le M\, k.\end{aligned}$$

By a similar argumentation using the estimates (5.70), (5.71), (5.72), (5.74), (5.75) and (5.76), we obtain the other estimates of the statement and the proof of Lemma 5.1 is complete.

□

5.2 Taking the limit

Using Rothe's function $(\widetilde{\varphi}_k, \widetilde{c}_k, \widetilde{\mathbf{u}}_k)$ (defined in (2.14)), and corresponding step function $(\varphi_k, c_k, \mathbf{u}_k)$ we rewrite (2.16), (2.17), (2.18), (2.12), (2.13) in the form

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \widetilde{\varphi}_k(t)\,\phi\, dx\right) + \xi^2 \int_\Omega \nabla\varphi_k(t)\,\nabla\phi\, dx +\\ &\int_\Omega \mathbf{u}_k(t)\cdot\nabla\varphi_k(t)\,\phi\, dx = \int_\Omega (F_1(\varphi_k(t)) + c_k(t) F_2(\varphi_k(t)))\phi\, dx,\end{aligned} \tag{5.77}$$

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \widetilde{c}_k(t)\, z\, dx\right) + \int_\Omega D_1(\varphi_k(t))\nabla c_k(t)\nabla z\, dx +\\ &\int_\Omega \mathbf{u}_k(t).\nabla c_k(t)\, z\, dx = -\int_\Omega D_2(\varphi_k(t), c_k(t))\nabla\varphi_k(t) z\, dx,\end{aligned} \tag{5.78}$$

$$\frac{d}{dt}\left(\int_\Omega \widetilde{\mathbf{u}}_k(t)\,\mathbf{v}\,dx\right)+\int_\Omega \nu(\varphi_k(t))\nabla\mathbf{u}_k(t)\,\nabla\mathbf{v}\,dx+ \\ \int_\Omega(\mathbf{u}_k(t)\cdot\nabla)\mathbf{u}_k(t)\,\mathbf{v}\,dx=\int_\Omega \vec{\sigma}\;c_k(t)\mathbf{v}\,dx \tag{5.79}$$

for all $(\phi, z, \mathbf{v})\in (H^1(\Omega))^2\times V$.

Due to Lemma 5.1, there exist subsequences still denote $(\varphi_k, c_k, \mathbf{u}_k)$, $(\widetilde{\varphi}_k, , \widetilde{c}_k, \widetilde{\mathbf{u}}_k)$, such that as $k\to 0$,

$$\begin{array}{llcc} (\varphi_k, c_k)\rightharpoonup(\varphi,c), & (\widetilde{\varphi}_k,\widetilde{c}_k)\rightharpoonup(\widetilde{\varphi},\widetilde{c}) & \text{in} & (L^2(0,T;H^1(\Omega)))^2, \\ (\varphi_k, c_k)\overset{*}{\rightharpoonup}(\varphi,c), & (\widetilde{\varphi}_k,\widetilde{c}_k)\overset{*}{\rightharpoonup}(\widetilde{\varphi},\widetilde{c}) & \text{in} & (L^2(Q))^2, \\ \mathbf{u}_k\rightharpoonup\mathbf{u}, & \widetilde{\mathbf{u}}_k\rightharpoonup\widetilde{\mathbf{u}} & \text{in} & L^2(0,T;V), \\ \mathbf{u}_k\overset{*}{\rightharpoonup}\mathbf{u}, & \widetilde{\mathbf{u}}_k\overset{*}{\rightharpoonup}\widetilde{\mathbf{u}} & \text{in} & L^2(0,T;H). \end{array} \tag{5.80}$$

Since $(\widetilde{\varphi}_k-\varphi_k, \widetilde{c}_k-c_k)$ converge to 0 in $(L^2(Q))^2$ and $\widetilde{\mathbf{u}}_k-\mathbf{u}_k$ converge to 0 in $L^2(0,T;H)$ as $k\to 0$, we conclude that $(\varphi, c, \mathbf{u})=(\widetilde{\varphi},\widetilde{c},\widetilde{\mathbf{u}})$.

Now, we are able to prove convergence we need the estimate of discrete time derivate, else. For this we now apply Lemma 2.2 (see p. 4).

We set $Y=(H^1(\Omega))^2\times V$ and $X=(L^2(\Omega))^2\times H$; the imbedding of $(H^1(\Omega))^2\times V$ in $(L^2(\Omega))^2\times H$ is compact by Rellich's theorem; we choose $p=2$ and $\mathcal{G}$ is the family of functions $(\widetilde{\varphi}_k, , \widetilde{c}_k, \widetilde{\mathbf{u}}_k)$.

We assume for simplicity that $(\varphi_0, c_0, \mathbf{u}_0)\in (H^1(\Omega))^2\times V$ (see [16, p. 585] for more details); we then know from Lemma 5.1 the $(\widetilde{\varphi}_k,\widetilde{c}_k,\widetilde{\mathbf{u}}_k)$ is bounded in $(L^2(0,T;H^1(\Omega)))^2\times L^2(0,T;V)$ and $(L^\infty(0,T;L^2(\Omega)))^2\times L^\infty(0,T;H)$. There remains to show (2.7). For this, we integrate, respectively, (5.77), (5.78), (5.79) between t and $t+a$, $t\in(0,T)$, $a>0$:

$$\int_\Omega(\widetilde{\varphi}_k(t+a)-\widetilde{\varphi}_k(t))\phi\,dx=-\xi^2\int_t^{t+a}\int_\Omega\nabla\varphi_k(s)\nabla\phi\,dx\,ds$$

$$-\int_t^{t+a}\int_\Omega \mathbf{u}_k(s)\nabla\varphi_k(s)\phi\,dx\,ds+\int_t^{t+a}\int_\Omega(F_1(\varphi_k(s))+c_k(s)\,F_2(\varphi_k(s))\,\phi\,dx\,ds,$$

$$\int_\Omega(\widetilde{c}_k(t+a)-\widetilde{c}_k(t))z\,dx=-\int_t^{t+a}\int_\Omega D_1(\varphi_k(s))\nabla c_k(s)\nabla z\,dx\,ds$$

$$-\int_t^{t+a}\int_\Omega \mathbf{u}_k(s)\nabla c_k(s)\, z\, dx\, ds - \int_t^{t+a}\int_\Omega D_2(\varphi_k(s), c_k(s))\nabla\varphi_k(s)\nabla z\, dx\, ds,$$

$$\int_\Omega (\widetilde{\mathbf{u}}_k(t+a) - \widetilde{\mathbf{u}}_k(t))\,\mathbf{v}\, dx = -\int_t^{t+a}\int_\Omega \nu(\varphi_k(s))\nabla\mathbf{u}_k(s)\nabla\mathbf{v}\, dx\, ds$$

$$-\int_t^{t+a}\int_\Omega (\mathbf{u}_k(s)\cdot\nabla)\mathbf{u}_k(s)\,\mathbf{v}\, dx\, ds + \int_t^{t+a}\int_\Omega \vec{\sigma}\; c_k(s)\,\mathbf{v}\, dx\, ds,$$

$\forall(\phi, z, \mathbf{v}) \in (H^1(\Omega))^2 \times V$.

We choose $\phi = \widetilde{\varphi}_k(t+a) - \widetilde{\varphi}_k(t)$, $z = \widetilde{c}_k(t+a) - \widetilde{c}_k(t)$, $\mathbf{v} = \widetilde{\mathbf{u}}_k(t+a) - \widetilde{\mathbf{u}}_k(t)$ and integrate these relations between 0 and $T-a$. We find

$$\int_0^{T-a} ||\widetilde{\varphi}_k(t+a) - \widetilde{\varphi}_k(t)||_{2,\Omega}^2 = I_1 + I_2 + I_3,$$
$$\int_0^{T-a} ||\widetilde{c}_k(t+a) - \widetilde{c}_k(t)||_{2,\Omega}^2 = J_1 + J_2 + J_3,$$
$$\int_0^{T-a} ||\widetilde{\mathbf{u}}_k(t+a) - \widetilde{\mathbf{u}}_k(t)||_{2,\Omega}^2 = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

Now, we consider the following integrals:

$$I_1 = -\xi^2\int_0^{T-a}\int_t^{t-a}\int_\Omega \nabla\varphi_k(s)(\nabla\widetilde{\varphi}_k(t+a) - \nabla\widetilde{\varphi}_k(t))\, dx\, ds\, dt,$$
$$J_1 = -\int_0^{T-a}\int_t^{t-a}\int_\Omega D_1(\varphi_k(s))\nabla c_k(s)(\nabla\widetilde{c}_k(t+a) - \nabla\widetilde{c}_k(t))\, dx\, ds\, dt,$$
$$\mathcal{I}_1 = -\int_0^{T-a}\int_t^{t-a}\int_\Omega \nu(\varphi_k(s))\nabla\mathbf{u}_k(s)(\nabla\widetilde{\mathbf{u}}_k(t+a) - \nabla\widetilde{\mathbf{u}}_k(t))\, dx\, ds\, dt.$$

Using assumptions $(\mathbf{H}_0)$-$(\mathbf{H}_4)$, Fubini's theorem and Schwarz inequality, we get

$$|I_1| \leq \xi^2 T^{1/2} a^{1/2} I_{11}\left(\int_0^{T-a} ||\nabla\widetilde{\varphi}_k(t+a) - \nabla\widetilde{\varphi}_k(t)||_{2,\Omega}^2 dt\right)^{1/2},$$
$$|J_1| \leq \rho_1 T^{1/2} a^{1/2} J_{11}\left(\int_0^{T-a} ||\nabla\widetilde{c}_k(t+a) - \nabla\widetilde{c}_k(t)||_{2,\Omega}^2 dt\right)^{1/2},$$
$$|\mathcal{I}_1| \leq \nu_1 T^{1/2} a^{1/2} \mathcal{I}_{11}\left(\int_0^{T-a} ||\nabla\widetilde{\mathbf{u}}_k(t+a) - \nabla\widetilde{\mathbf{u}}_k(t)||_{2,\Omega}^2 dt\right)^{1/2},$$

where $I_{11} = \left(\int_0^T ||\nabla\varphi_k(s)||^2_{2,\Omega} ds\right)^{1/2}$, $J_{11} = \left(\int_0^T ||\nabla c_k(s)||^2_{2,\Omega} ds\right)^{1/2}$ and $\mathcal{I}_{11} = \left(\int_0^T ||\nabla \mathbf{u}_k(s)||^2_{2,\Omega} ds\right)^{1/2}$.

From Lemma 5.1, we obtain

$$|I_1| \le M\, a^{1/2}, \quad |J_1| \le M\, a^{1/2}, \quad |\mathcal{I}_1| \le M\, a^{1/2}. \tag{5.81}$$

Now, we consider the following integrals:

$$\begin{aligned}
I_2 &= -\int_0^{T-a}\int_t^{t-a}\int_\Omega \mathbf{u}_k(s)\cdot\nabla\varphi_k(s)(\widetilde{\varphi}_k(t+a)-\widetilde{\varphi}_k(t))\,dx\,ds\,dt,\\
J_2 &= -\int_0^{T-a}\int_t^{t-a}\int_\Omega \mathbf{u}_k(s)\cdot\nabla c_k(s)(\widetilde{c}_k(t+a)-\widetilde{c}_k(t))\,dx\,ds\,dt,\\
\mathcal{I}_2 &= -\int_0^{T-a}\int_t^{t-a}\int_\Omega (\mathbf{u}_k(s)\cdot\nabla)\mathbf{u}_k(s)(\widetilde{\mathbf{u}}_k(t+a)-\widetilde{\mathbf{u}}_k(t))\,dx\,ds\,dt.
\end{aligned}$$

Using Sobolev imbedding, Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}
|I_2| &\le \int_0^T\int_{[s-a,s]\cap[0,T-a]} ||\mathbf{u}_k(s)||_V\,||\nabla\varphi_k(s)||_{2,\Omega}||\widetilde{\varphi}_k(t+a)-\widetilde{\varphi}_k(t)||_{1,2,\Omega}\,ds\,dt,\\
|J_2| &\le \int_0^T\int_{[s-a,s]\cap[0,T-a]} ||\mathbf{u}_k(s)||_V\,||\nabla c_k(s)||_{2,\Omega}||\widetilde{c}_k(t+a)-\widetilde{c}_k(t)||_{1,2,\Omega}\,ds\,dt,\\
|\mathcal{I}_2| &\le M\int_0^T\int_{[s-a,s]\cap[0,T-a]} ||\mathbf{u}_k(s)||^2_V\,||\widetilde{\mathbf{u}}_k(t+a)-\widetilde{\mathbf{u}}_k(t)||_V\,ds\,dt.
\end{aligned}$$

From Schwarz inequality, we get

$$\begin{aligned}
|I_2| &\le Ma^{1/2}\mathcal{I}_{11}I_{11}\left(\int_0^{T-a}||\widetilde{\varphi}_k(t+a)-\widetilde{\varphi}_k(t)||^2_{2,\Omega}dt\right)^{1/2},\\
|J_2| &\le Ma^{1/2}\mathcal{I}_{11}J_{11}\left(\int_0^{T-a}||\widetilde{c}_k(t+a)-\widetilde{c}_k(t)||^2_{2,\Omega}dt\right)^{1/2},\\
|\mathcal{I}_2| &\le Ma^{1/2}\mathcal{I}_{11}\left(\int_0^{T-a}||\widetilde{\mathbf{u}}_k(t+a)-\widetilde{\mathbf{u}}_k(t)||^2_V\,dt\right)^{1/2}.
\end{aligned}$$

From Lemma 5.1, we obtain

$$|I_2| \le M\, a^{1/2}, \quad |J_2| \le M\, a^{1/2}, \quad |\mathcal{I}_2| \le M\, a^{1/2}. \tag{5.82}$$

In an analogous way, we get

$$
\begin{aligned}
|I_3| &\le MT^{1/2}a^{1/2}I_{31}\left(\int_0^{T-a}||\widetilde{\varphi}_k(t+a)-\widetilde{\varphi}_k(t)||^2_{2,\Omega}dt\right)^{1/2},\\
|J_3| &\le MT^{1/2}a^{1/2}I_{11}\left(\int_0^{T-a}||\widetilde{c}_k(t+a)-\widetilde{c}_k(t)||^2_{2,\Omega}dt\right)^{1/2},\\
|\mathcal{I}_3| &\le MT^{1/2}a^{1/2}J_{11}\left(\int_0^{T-a}||\widetilde{\mathbf{u}}_k(t+a)-\widetilde{\mathbf{u}}_k(t)||^2_{2,\Omega}dt\right)^{1/2},
\end{aligned}
$$

where $I_{31} = \left(\int_0^T 1+||c_k(s)||^2_{2,\Omega}ds\right)^{1/2}$.

From Lemma 5.1, we obtain

$$
|I_3| \le M\,a^{1/2}, \quad |J_3| \le M\,a^{1/2}, \quad |\mathcal{I}_3| \le M\,a^{1/2}. \tag{5.83}
$$

Combining (5.81), (5.82) and (5.83), we obtain (2.8). Hence there exist subsequences still denote $(\varphi_k, c_k, \mathbf{u}_k)$, such that as $k \to 0$,

$$
\begin{array}{rcl}
(\varphi_k, c_k) \to (\varphi, c) & \text{in} & (L^2(Q))^2,\\
\mathbf{u}_k \to \mathbf{u} & \text{in} & L^2(0,T;H).
\end{array} \tag{5.84}
$$

Applying arguments that are similar to the ones used to proof of Proposition 3.1 (see for instance the estimates (3.39), (3.40), (3.41)), we can show the following convergences:

$$
\begin{array}{llll}
\nu(\varphi_k) \to \nu(\varphi), & F_l(\varphi_k) \to F_l(\varphi) & \text{in} \quad L^p(Q) & \text{strongly},\\
D_1(\varphi_k) \to D_1(\varphi), & D_2(\varphi_k, c_k) \to D_2(\varphi, c) & \text{in} \quad L^p(Q) & \text{strongly},
\end{array} \tag{5.85}
$$

with $2 \le p < \infty$ and $l = 1, 2$.

We now take the limit in problem (5.77), (5.78), (5.79). For this we consider $(\phi, z, \mathbf{v}) \in (C_0^\infty(\Omega))^2 \times \mathcal{V}$ and $(\psi, \sigma, \pi) \in (C^1([0,T];\mathbb{R}))^3$ with $\phi(T) = \sigma(T) = \pi(T) = 0$.

Next, we multiplying the equation (5.77) by $\psi(t)$, the equation (5.78) by $\sigma(t)$ and the equation (5.79) by $\pi(t)$, integrate over Q, integrate by parts, we find

$$-\int_0^T\int_\Omega \widetilde{\varphi}_k(t)\,\psi'(t)\phi\,dx\,dt+\xi^2\int_0^T\int_\Omega \nabla\varphi_k(t)\,\psi(t)\nabla\phi\,dx\,dt+$$
$$+\int_0^T\int_\Omega \mathbf{u}_k(t)\cdot\nabla\varphi_k(t)\,\psi(t)\phi\,dx\,dt=\int_\Omega \varphi_0\psi(0)\phi\,dx+$$
$$\int_0^T\int_\Omega F_1(\varphi_k(t))\psi(t)\phi\,dx\,dt+\int_0^T\int_\Omega c_k(t)F_2(\varphi_k(t))\psi(t)\phi\,dx\,dt,$$

$$-\int_0^T\int_\Omega \widetilde{c}_k(t)\,\sigma'(t)z\,dx\,dt+\int_0^T\int_\Omega D_1(\varphi_k(t))\nabla c_k(t)\,\sigma(t)\nabla z\,dx\,dt+$$
$$+\int_0^T\int_\Omega \mathbf{u}_k(t)\cdot\nabla c_k(t)\,\sigma(t)z\,dx\,dt=\int_\Omega c_0\sigma(0)z\,dx-$$
$$\int_0^T\int_\Omega D_2(\varphi_k(t),c_k(t))\nabla\varphi_k(t)\sigma(t)z\,dx\,dt,$$

$$-\int_0^T\int_\Omega \widetilde{\mathbf{u}}_k(t)\,\pi'(t)\mathbf{v}\,dx\,dt+\int_0^T\int_\Omega \nu(\varphi_k(t))\nabla\mathbf{u}_k(t)\,\pi(t)\nabla\mathbf{v}\,dx\,dt+$$
$$+\int_0^T\int_\Omega (\mathbf{u}_k(t)\cdot\nabla)\mathbf{u}_k(t)\,\pi(t)\mathbf{v}\,dx\,dt=\int_\Omega \mathbf{u}_0\pi(0)\mathbf{v}\,dx+\int_0^T\int_\Omega \vec{\sigma}\;c_k(t)\pi(t)\mathbf{v}\,dx\,dt.$$

Setting $k\to 0$ and using the convergences (5.80), (5.84), (5.85), we get

$$\begin{aligned}&-\int_0^T\int_\Omega \varphi(t)\,\psi'(t)\phi\,dx\,dt+\xi^2\int_0^T\int_\Omega \nabla\varphi(t)\,\psi(t)\nabla\phi\,dx\,dt+\\&+\int_0^T\int_\Omega \mathbf{u}(t)\cdot\nabla\varphi(t)\,\psi(t)\phi\,dx\,dt=\int_\Omega \varphi_0\psi(0)\phi\,dx+\\&\int_0^T\int_\Omega F_1(\varphi(t))\psi(t)\phi\,dx\,dt+\int_0^T\int_\Omega c(t)F_2(\varphi(t))\psi(t)\phi\,dx\,dt,\end{aligned}\tag{5.86}$$

$$\begin{aligned}&-\int_0^T\int_\Omega c(t)\,\sigma'(t)z\,dx\,dt+\int_0^T\int_\Omega D_1(\varphi(t))\nabla c(t)\,\sigma(t)\nabla z\,dx\,dt+\\&+\int_0^T\int_\Omega \mathbf{u}(t).\nabla c(t)\,\sigma(t)z\,dx\,dt=\int_\Omega c_0\sigma(0)z\,dx-\\&\int_0^T\int_\Omega D_2(\varphi(t),c(t))\nabla\varphi(t)\sigma(t)z\,dx\,dt,\end{aligned}\tag{5.87}$$

$$
\begin{aligned}
&-\int_0^T \int_\Omega \mathbf{u}(t)\,\pi'(t)\mathbf{v}\,dx\,dt + \int_0^T \int_\Omega \nu(\varphi(t))\nabla\mathbf{u}(t)\,\pi(t)\nabla\mathbf{v}\,dx\,dt+ \\
&\quad + \int_0^T \int_\Omega (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\,\pi(t)\mathbf{v}\,dx\,dt = \int_\Omega \mathbf{u}_0\pi(0)\mathbf{v}\,dx+ \\
&\qquad \int_0^T \int_\Omega \vec{\sigma}\; c(t)\pi(t)\mathbf{v}\,dx\,dt.
\end{aligned}
\tag{5.88}
$$

Relations (5.86), (5.87) and (5.88) holds by continuity for every $(\phi, z, \mathbf{v}) \in (H^1(\Omega))^2 \times V$.

By choosing $(\psi, \sigma, \pi) \in (C_0^\infty([0,T]))^3$, we obtain from (5.86), (5.87) and (5.88) the equations (2.16), (2.17) and (2.18), respectively, in the weak sense on $(0,T)$ for all $(\phi, z, \mathbf{v}) \in (H^1(\Omega))^2 \times V$. Moreover, by standard arguments we show that $(\varphi(0), c(0), \mathbf{u}(0)) = (\varphi_0, c_0, \mathbf{u}_0)$ (see [16, p. 576] for more datails).

This completes the proof of Theorem 2.2.

$\square$

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