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ON THE NUMERICAL APPROXIMATION OF THE HELMHOLTZ EQUATION *

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Abstract

In this paper we discuss the convergence of numerical algorithms for computing the solutions of the Helmholtz equation. At the continuous level two different methods have been proposed inspired by the idea of reducing the search of the desired solutions to a least squares problem for the periodically forced wave equation. The functional to be minimized proposed in [5] turns out be coercive only for non-trapping obstacles while the variant introduced in [2] is coercive, independently of the geometry of the domain under consideration. This paper is devoted to analyze the behavior of each of these approaches under numerical discretizations. We prove that, in order for the functional in [5] to be uniformly coercive at the numerical level with respect to the mesh-size parameter one needs to introduce numerical schemes with artificial numerical viscosity, an impose a non-trapping condition to the obstacle. In contrast, the method proposed in [2] is more robust since convergence of numerical solutions is guaranteed for standard conservative numerical schemes and without any geometric condition on the obstacle.

1 Introduction

This article is devoted to revisit, from a numerical analysis point of view, two methods, inspired by control theoretical ideas, for the computation of the outgoing solution of the Helmholtz equation in an exterior domain (the complement of an obstacle) that yields a harmonic, in time, solution at a single

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frequency, representing wave scattering about a body at the given frequency. The applications include acoustic, electromagnetic and geophysical wave propagation.

These methods were introduced, respectively, by Bristeau, Glowinski and Periaux in [5] and Bardos and Rauch in [2]. Both of them use a least square approach and design quadratic functionals for the periodically forced wave equation whose minimizers yield the solution of the Helmholtz equation one is looking for.

In this article we analyze the same issue but considering also the effect of introducing a numerical scheme for approximating the wave equation, something that is necessary to conduct numerical experiments in practice.

The analysis we develop in this paper is intimately related to previous work on the control and stabilization of numerical approximation schemes for the wave equation. We refer to [28] for a survey on this topic. To be more precise, our analysis follows very closely previous work on stabilization that we summarize now.

When considering numerical discretization schemes for wave equations, it is well known that most of them do not preserve the property of uniform (with respect to the mesh-size) decay property of the solutions of the continuous wave equation with damping as $t \to \infty$. This is due to spurious high frequency numerical solutions that generate wave packets that propagate at a vanishing group velocity, that take too long to reach the boundary where the dissipative mechanism is localized and which, consequently, is not efficient enough to guarantee the uniform exponential decay. In view of this in [25] and [26] we introduced a dissipative numerical scheme with artificial numerical viscosity that preserves the decay properties of the continuous wave equation. That scheme succeeds in yielding the uniform exponential decay since the numerical viscosity term damps out the high frequencies, while the low frequency components are damped by the dissipative properties inherited by the numerical scheme out of the continuous wave model.

Our analysis confirms that, in what concerns numerical approximations too, the functional introduced in [2] is more robust than the originally introduced one [5]. More precisely, the discrete functionals corresponding to [2] are uniformly coercive without any geometric condition on the obstacle and for conservative numerical schemes while the functional in [5] needs of viscous numerical schemes and non-trapping conditions on the obstacle.

In this paper we address the problem for a monocromatic forcing term. But the methods we develop, based on the time-dependent wave equation may be also appropriate for forcing terms involving many frequencies.

We refer to [27] for a survey of the state of the art on the numerical methods for the Helmholtz equations, in which special attention is paid to the appropriate choice of the absorbing boundary conditions. This is often done also by the method of Perfectly Matching Layers (PML), see [3], [4], [7] and [16]. We also refer to [8] for a rigorous analysis of the PML method both for the continuous and semi-discrete wave equation in $1-d$ along the lines of the methods developed in the present article.

The analysis of this paper could be extended in various directions:

- *Other numerical methods:* Here we deal with finite-difference space semi-discretizations but the same issues arise in what concerns full discretization schemes, finite elements, etc. In fact, the original numerical approach proposed in [5] was based on the finite element method for discretizing the space laplacian and a fully discrete scheme in time. We refer to [28] for a survey article on the numerical aspects of control of wave equations, a topic which is very closely related to the one addressed here.

At this respect the work [11] is worth metionning. There, the methods in [5] are adapted to a mixed finite element formulation of the wave equation.

- *Two-grid filtering:* Here we have used numerical viscosity to damp out the high frequency spurious numerical components. But, at the level of the control of the wave equation ([9], [21]), and also for what concerns the numerical approximation of nonlinear Schrödinger equations ([14]), the same goal can be achieved by making a two-grid preconditionning of the initial data, without modifying the numerical scheme that can in this way be kept to be conservative. The efficiency of this two-grid preconditionning method is also to be investigated at the level of the numerical approximation of the Helmholtz equation.

2 Problem formulation and preliminaries

We address the problem of the numerical computation of the outgoing periodic solutions of a periodically driven wave equation. To be more precise, let Ω be a connected open subset of $\mathbb{R}^d$, $d \geqslant 1$, whose complement is bounded. Denote by Γ_{int} the boundary of $\Omega : \Gamma_{\text{int}} = \partial\Omega$. As we shall see, this notation makes sense since, for computational purposes, the domain Ω has to be cut-off, i.e. it has to be replaced by $\Omega_R = \Omega \cap B_R$ (where B_R stands for the ball of radius R), for some $R >> 1$, in which case one generates an exterior boundary $\Gamma_{\text{ext}} = \partial B_R$ as well.

We consider the wave equation

$$\begin{cases} u_{tt} - \Delta u = e^{i\omega t} f(x) & \text{in} \quad \Omega,\ t > 0 \\ u = 0 & \text{on} \quad \partial\Omega,\ t > 0. \end{cases} \tag{2.1}$$

We are interested on the time-periodic solution of (2.1) of the form

$$u = e^{i\omega t} w(x) \tag{2.2}$$

where $w = w(x)$ solves the Helmholtz equation

$$\begin{cases} \big(\Delta + \omega^2\big) w = -f(x) & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

This paper is devoted to revisit the methods introduced by [5] and [2] for the computation of (2.2) from a numerical analysis point of view. These methods are based on a least square formulation of the problem and use the existing results on the control and stabilization of the wave equation.

The first step in the application of these methods is, as we mentioned above, cutting-off the domain Ω to transform it into a bounded domain Ω_R. The previous wave (2.1) and Helmholtz (2.3) equations are then considered in Ω_R. But to make the system complete, they have to be complemented with suitable boundary conditions on the external boundary $\Gamma_{\text{ext}} = \{x : |x| = R\}$. The most natural ones are the so-called absorbing boundary conditions.

The wave equation obtained in this way then reads:

$$\begin{cases} u_{tt} - \Delta u = e^{i\omega t} f(x) & \text{in} \quad \Omega_R,\ t > 0 \\ u = 0 & \text{on} \quad \Gamma_{\text{int}},\ t > 0 \\ \dfrac{\partial u}{\partial \nu} + \gamma(x)\partial_t u = 0 & \text{on} \quad \Gamma_{\text{ext}},\ t > 0, \end{cases} \tag{2.4}$$

with γ a smooth and strictly positive function defined on Γ_{ext}. Here and in the sequel ν stands for the unit normal vector to the boundary.

Similarly, the Helmholtz equation reads

$$\begin{cases} \big(\Delta + \omega^2\big) w = -f & \text{in} \quad \Omega_R \\ w = 0 & \text{on} \quad \Gamma_{\text{int}} \\ \dfrac{\partial w}{\partial \nu} + i\omega\gamma(x) w = 0 & \text{on} \quad \Gamma_{\text{ext}}. \end{cases} \tag{2.5}$$

Before going further we recall some basic properties of the solutions of the wave equation (2.4) in the absence of external forces, i. e. $f \equiv 0$. Let us introduce the energy of solutions of (2.4):

$$E_R(t) = \frac{1}{2} \int_\Omega \left[|u_t|^2 + |\nabla u|^2 \right] dx. \tag{2.6}$$

When $f \equiv 0$, we have the energy dissipation law

$$\frac{dE_R(t)}{dt} = - \int_{\Gamma_{\text{ext}}} \gamma(x) |\partial_t u|^2 d\sigma. \tag{2.7}$$

Assuming that $\gamma \geq \gamma_0 > 0$ a.e. on Γ_{ext}, it is easy to see that

$$E_R(t) \to 0 \text{ as } t \to \infty \tag{2.8}$$

for every solution of (2.4) with $f \equiv 0$. But the uniform exponential decay of the semigroup concerns the stronger condition about the existence of positive constants $C > 0$ and $\alpha > 0$ such that

$$E_R(t) \leq Ce^{-\alpha t} E_R(0), \quad \forall t > 0 \tag{2.9}$$

for all solution of (2.4) with $f \equiv 0$.

In [2] it was proved that the coercivity of the functional introduced in [5] (see (2.11) below) requires the uniform decay property (2.9), but they also indicated that such a property does not hold when the boundary Γ_{int} is trapping. This is due to the fact that, when Γ_{int} is trapping, there are Gaussian beam solutions (see [22] and [23]) that never reach the exterior boundary and, therefore, are very weakly dissipated.

In [5] an alternate functional was proposed whose coercivity only requires the strong stability property (2.8), which holds without any further geometric conditions.

In the sequel, to make the problem under consideration precise, we briefly recall the definition of these two functionals.

We consider the initial value problem associated to (2.4). Taking into account that we look for solutions of (2.4) of the form $u = e^{i\omega t} w(x)$ it is sufficient to take initial data of the form $u(0) = \psi$ and $u_t(0) = i\omega\psi$. We thus consider the system

$$\begin{cases} u_{tt} - \Delta u = e^{i\omega t} f(x) & \text{in} \quad \Omega_R,\ t > 0 \\ u = 0 & \text{on} \quad \Gamma_{\text{int}},\ t > 0 \\ \dfrac{\partial u}{\partial \nu} + \gamma(x)\partial_t u = 0 & \text{on} \quad \Gamma_{\text{ext}},\ t > 0 \\ u(x,0) = \psi(x),\ u_t(x,0) = i\omega\psi(x) & \text{in} \quad \Omega_R. \end{cases} \tag{2.10}$$

The functional proposed in [5] is as follows. Take $T = 2\pi/\omega$ and set

$$J(\psi) = \frac{1}{2}\int_{\Omega_R} \Big[\big|\nabla\big(u(x,T) - \psi(x)\big)\big|^2 + \big|u_t(x,T) - i\omega\psi(x)\big|^2\Big] dx. \tag{2.11}$$

Obviously, whenever ψ determines the initial datum of the $2\pi/\omega$-periodic solution of (2.10) (or, in other words, the solution of the Helmholtz equation (2.5)) this immediately produces a minimizer of J whose value is then zero. Minimizing J over $\psi \in V_R$ where

$$V_R = \Big\{\psi \in H^1(\Omega_R) : \psi\Big|_{\Gamma_{\text{int}}} = 0\Big\} \tag{2.12}$$

is therefore a natural way of looking for the $2\pi/\omega$-periodic solution of (2.10) and, consequently, for the solution of the Helmholtz equation (2.5).

However, as proved in [2], for the functional J to be coercive, i.e. to guarantee that

$$J(\psi) \geq c \parallel \psi - w \parallel^2_{H^1(\Omega_R)}, \quad \forall \psi \in V_R, \tag{2.13}$$

for some $c > 0$, for all $\psi \in V_R$, w being the solution of (2.5), one needs the uniform exponential decay property (2.9) of the semigroup generated by (2.10).

But, as we mentioned above, this is related to the geometry of $\Gamma_{\rm int}$ and, more precisely, to the condition of being non-trapping.

In [2], to avoid these geometric restrictions, the authors introduced a variant of J which has the advantage of being coercive under the strong stability condition (2.8), which holds regardless what the geometry of $\Gamma_{\rm int}$ is. The functional $\widetilde{J}$ in [2] is as follows:

$$\widetilde{J}(\psi) = \frac{1}{2}\int_0^T \int_{\Omega_R} \left[\left| e^{i\omega t}\nabla\psi(x) - \nabla u(x,\, t) \right|^2 + \left| i\omega e^{i\omega t}\psi(x) - u_t(x,\, t)\right|^2 \right] dxdt. \tag{2.14}$$

Once more, the minimum value of this functional is achieved when $\psi \equiv w$, w being the solution of (2.5).

In [2], without any geometric condition on $\Gamma_{\rm int}$, it was proved that

$$\widetilde{J}(\psi) \geq c \parallel \psi - w \parallel^2_{H^1(\Omega_R)}, \quad \forall \psi \in V_R, \tag{2.15}$$

for some $c > 0$. Obviously, this coercivity property is essential to guarantee the convergence of any descent algorithm for the minimization of $\widetilde{J}(\psi)$.

In practice, these functionals have to be minimized not on the basis of the continuous wave equation (2.4) whose solutions can not be computed, but rather on the solutions of a numerical approximation scheme.

This paper is devoted to discuss how the nature of the discretization schemes we employ can affect the efficiency of the minimization method, distinguishing the functionals J and $\widetilde{J}$. As in the continuous case, the uniform (on the mesh-size parameter) coercivity property of the corresponding discrete functionals will strongly influence the efficiency of the minimization method. As we shall see, this topic is closely related to the existing literature on the stabilization of the numerical approximation schemes for the wave equation that states that: For most numerical schemes, the strong stability property (2.8) is kept under numerical discretizations, but the exponential decay rate fails to be uniform on the mesh-size parameter. As we shall see, this suffices to prove the uniform (with respect to the mesh-size parameter h) coercivity of a discrete version of the functional $\widetilde{J}$ on the discretized wave equation that we shall denote by $\widetilde{J}_h$. But the situation changes when considering the functional J that requires the use of viscous numerical schemes (see [25], [26]) and non-trapping conditions on Γ_{int}.

The rest of this article is organized as follows. Section 3 is devoted to analyze the functional J introduced in [5]. In a first subsection we discuss the strong stability and the lack of uniform exponential decay of the conservative semi-discrete wave equation. We conclude the impossibility of using a discrete version of the functional J to obtain a numerical approximation of the solution of the Helmholtz equation. We then consider a viscous numerical scheme, show its uniform exponential decay under a star-shaped geometric condition on Γ_{int}, and derive the convergence of the minimizers of the corresponding discrete versions of the functional J for T large enough, i. e. for ω small enough. The case of large ω remains open. In Section 4 we address the functional $\widetilde{J}$ introduced in [2]. We derive the convergence as $h \to 0$ of the minimizers of the corresponding discrete functionals $\widetilde{J}_h$ for the standard conservative scheme and without any geometric assumption on the domain.

3 Analysis of the functional J

3.1 Decay for the conservative scheme with boundary dissipation

We consider a finite-difference approximation of the wave equation (2.10). To simplify the presentation we consider the case of two space dimensions ($d = 2$). For, given $h > 0$, we replace the domain Ω_R by a finite-difference approximation

Ω_R^h whose boundary Γ_R^h is constituted by mesh-points of the form

$$x_{j,\,k} = (jh,\, kh), \quad j,\, k \in \mathbb{N}. \tag{3.1}$$

We denote by $u_{j,\,k}$, $\gamma_{j,\,k}$, $f_{j,\,k}$, $\psi_{j,\,k}$ the approximation of the functions u, γ, f and ψ on these mesh points. We decompose the boundary of Ω_R^h into two subsets $\Gamma_{\rm int}^h$ and $\Gamma_{\rm ext}^h$ corresponding to the approximation of $\Gamma_{\rm int}$ and $\Gamma_{\rm ext}$, respectively. By Δ_h we denote the 5-point approximation scheme for the Laplacian so that

$$-\Delta_h u_{j,\,k} = \frac{1}{h^2}\Big[4u_{j,\,k} - u_{j+1,\,k} - u_{j-1,\,k} - u_{j,\,k+1} - u_{j,\,k-1}\Big]. \tag{3.2}$$

Then, we consider the following numerical approximation scheme of (2.10):

$$\begin{cases} u''_{j,\,k} - \Delta_h u_{j,\,k} = e^{i\omega t} f_{j,\,k} & \text{in} \quad \Omega_R^h, \quad t > 0 \\ u_{j,\,k} = 0 & \text{on} \quad \Gamma_{\rm int}^h, \quad t > 0 \\ \dfrac{\partial u_{j,\,k}}{\partial \nu^h} + \gamma_{j,\,k} u'_{j,\,k} = 0 & \text{on} \quad \Gamma_{\rm ext}^h, \quad t > 0 \\ u_{j,\,k}(0) = \psi_{j,\,k};\; u'_{j,\,k}(0) = i\omega\psi_{j,\,k} & \text{in} \quad \Omega_R^h. \end{cases} \tag{3.3}$$

Here and in the sequel when saying that an identity holds in a set we refer to the fact that it holds for all indexes $(j,\, k)$ such that $x_{j,\,k}$ is in that set. The time derivative is denoted by a prime, i. e. $u'_{j,\,k}$ stands for the time-derivative of $u_{j,\,k}$. In the sequel, in order to simplify the notation, we will avoid the index R for denoting the domain and its boundaries, both the interior and the exterior one.

By $\partial u_{j,\,k}/\partial \nu^h$ we denote the discrete normal derivative. For most boundary points we have

$$\frac{\partial u_{j,\,k}}{\partial \nu^h} = \frac{1}{h}\big[u_{j,\,k} - u_{j',\,k'}\big] \tag{3.4}$$

where $\big(j',\, k'\big)$ stands for the neighboring index in the direction perpendicular to the boundary. For instance when the index $(j,\, k)$ corresponds to a point of the boundary $x_{j,\,k}$ that lies on an horizontal subset of the boundary, then $\big(j',\,k'\big) = \big(j,\,k+1\big)$ when the domain is in the upper side of the boundary and $\big(j',\,k'\big) = \big(j,\,k-1\big)$ if it is in the lower one. Note also that the boundary points that lie on a convex corner of the domain do not enter on the five-point scheme when written on the interior nodes. Thus, no boundary conditions are needed on those points. However, the interpretation of the dissipative boundary condition on the points of $\Gamma^h_{\rm ext}$ corresponding to concave corners needs to be slightly modified since, in this case, there are two interior neighbors and the value of the solution on that node enters on the equation satisfied on the two neighboring ones (see Figure 1).

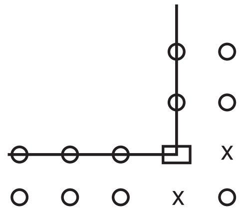

Figure 1: *Node* $\square$ *at a concave corner. It enters on the discrete scheme on the two neighboring nodes denoted by* $\times$.

In that case one sees that the right definition of the normal derivative to guarantee the dissipativity of the system (3.3) is of the form

$$\frac{\partial u_{j,\,k}}{\partial \nu^h} = \frac{1}{h}\big[(u_{j,\,k} - u_{j',\,k'}) + (u_{j,\,k} - u_{\tilde{j},\,\tilde{k}})\big] \tag{3.5}$$

where (j', k') and $(\tilde{j},\,\tilde{k})$ stand for the two neighboring nodes of (j, k).

The energy for the semi-discrete system (3.3) with the definition (3.4) and (3.5) of the normal derivatives is given by

$$E^h(t) = \frac{h^2}{2}\left[\sum_{(j,\,k)\in\Omega^h_{\rm int}} \left|u'_{j,\,k}\right|^2 + \sum_{(j,\,k)\in\Omega^h_x} \left|\frac{u_{j+1,\,k} - u_{j,\,k}}{h}\right|^2 + \sum_{(j,\,k)\in\Omega^h_y} \left|\frac{u_{j,\,k+1} - u_{j,\,k}}{h}\right|^2\right]. \tag{3.6}$$

The first sum runs over the set of interior indexes. The other two run over the sets Ω_x^h and Ω_y^h and involve the discrete derivatives in the variables x and y respectively. The sets Ω_x^h and Ω_y^h are constituted by the indexes such that the segments $[j,\, j+1] \times \{h\}$ and $\{j\} \times [k,\, k+1]$ respectively cover the whole discrete domain Ω^h.

It is easy to see that the energy $E^h(t)$ satisfies the law

$$\frac{dE^h(t)}{dt} = h^2 \sum_{(j,\,k)\in\Omega^h_{\text{int}}} f_{j,\,k} e^{i\omega t} u'_{j,\,k} - h \sum_{(j,\,k)\in\Gamma^h_{\text{ext}}} \gamma_{j,\,k} \left|u'_{j,\,k}\right|^2. \tag{3.7}$$

In particular, in the absence of external source, i.e. when $f_{j,\,k} \equiv 0$, we have

$$\frac{dE^h(t)}{dt} = -h \sum_{(j,\,k)\in\Gamma^h_{\text{ext}}} \gamma_{j,\,k} \left|u'_{j,\,k}\right|^2, \tag{3.8}$$

which is the discrete version of the energy dissipation law (2.7).

The following results hold:

Theorem 3.1. *In the absence of external source, i.e. when $f_{j,\,k} \equiv 0$, the following asymptotic properties hold true as $t \to +\infty$:*

(a) *For every initial datum and $h > 0$:*

$$E_h(t) \to 0, \quad \text{as } t \to \infty, \tag{3.9}$$

(b) *The decay rate is not uniform on the parameter h: In other words, there are no positive constants $C > 0$ and $\alpha > 0$ such that*

$$E_h(t) \le Ce^{-\alpha t} E_h(0), \tag{3.10}$$

for all $t > 0$, all solution of (3.3) and all $h > 0$.

We shall come back to the proof of this Theorem later on but it is first convenient to comment on its important consequences on the solutions of (3.3):

(a) In view of (3.9), for all $h > 0$, because the dynamical system (3.3) is finite-dimensional, there exist $C_h > 0$ and $\alpha_h > 0$ such that

$$E^h(t) \leq C_h e^{-\alpha_h t} E_h(0),\ \forall t > 0 \tag{3.11}$$

for all solution of (3.3) with $f_{j,\,k} \equiv 0$. Obviously this does not imply that the constants C_h and α_h are uniform on h. In fact, as indicated in the second statement of Theorem 3.1, they are not uniform on h.

(b) According to (3.11), for all $f_{j,\,k}$ and $h > 0$ there exists a $2\pi/\omega$-periodic solution of (3.3). However, in view of the fact that the exponential stability properties (3.11) are not independent of h, one can not derive uniform bounds on the energy of these periodic solutions.

The time-periodic solution of (3.3) can be found by minimizing a discrete version of the functional J in (2.11). It can be written in the form

$$\begin{aligned} J_h(\psi) = {} & \frac{h^2}{2} \sum_{(j,\,k)\in\Omega^h_{\mathrm{int}}} \left| u'_{j,\,k}(T) - i\omega\psi_{j,\,k} \right|^2 \\ & + \frac{h^2}{2} \sum_{(j,\,k)\in\Omega^h_x} \left| \frac{\big(u_{j+1,\,k}(T) - u_{j,\,k}(T)\big) - \big(\psi_{j+1,\,k} - \psi_{j,\,k}\big)}{h} \right|^2 \\ & + \frac{h^2}{2} \sum_{(j,\,k)\in\Omega^h_y} \left| \frac{\big(u_{j,\,k+1}(T) - u_{j,\,k}(T)\big) - \big(\psi_{j,\,k+1} - \psi_{j,\,k}\big)}{h} \right|^2, \end{aligned} \tag{3.12}$$

with $T = 2\pi/\omega$.

Following the arguments in [2], it can be seen that J_h is coercive because of the exponential decay property (3.11). But, the functionals J_h may not be proved to be uniformly coercive because of the lack of uniform exponential decay stated in (b) of Theorem 3.1. Consequently, when the uniform coercivity property fails, minimizing the functionals J_h might not be an efficient algorithm for approximating the minimizer of J, since the Γ-convergence of J_h towards J can not be guaranteed.

Proof of Theorem 3.1. We sketch it briefly.

Proof of (a). Using La Salle's invariance principle the decay property (3.9) is equivalent to showing that the only solution of (3.3) with $f_{j,\,k} \equiv 0$ for which the dissipated quantity vanishes, i.e.

$$u'_{j,\,k} \equiv 0, \quad \forall (j,\,k) \in \Gamma^h_{\text{ext}}, \tag{3.13}$$

is the trivial one $u \equiv 0$.

The problem is then reduced to proving the following discrete unique-continuation property: Let $u = u_{j,\,k}$ be a discrete solution of

$$\begin{cases} u''_{j,\,k} - \Delta_h u_{j,\,k} = 0 & \text{in} \quad \Omega^h_R, \quad t > 0 \\ u_{j,\,k} = 0 & \text{on} \quad \Gamma^h_{\text{int}}, \quad t > 0 \\ \dfrac{\partial u_{j,\,k}}{\partial \nu^h} = u'_{j,\,k} = 0 & \text{on} \quad \Gamma^h_{\text{ext}}, \quad t > 0, \end{cases} \tag{3.14}$$

then $u_{j,\,k} \equiv 0$.

This result can be proved arguing as in [6]. Indeed, the fact $u'_{j,\,k} = 0$ on Γ^h_{ext} and $\partial u_{j,\,k}/\partial \nu^h = 0$, implies that $u'_{j,\,k}$ vanishes on all the interior nodes which are neighbors of Γ^h_{ext}. For doing this we observe that the fact that $\partial u_{j,\,k}/\partial \nu^h = 0$ on the boundary implies that $\partial u'_{j,\,k}/\partial \nu^h = 0$ as well. Taking the time-derivative of the semi-discrete wave equation satisfied on all those nodes one can show that $u'_{j,\,k}$ vanishes on the next layer of interior nodes too. Iterating this argument one can prove that $u'_{j,\,k} = 0$ for all node $(j,\,k)$. Consequently $u_{j,\,k}(t) = v_{j,\,k}$ is time-independent. It is then easy to see that $v_{j,\,k} = 0$ for all node $(j,\,k)$ since $v_{j,\,k}$ satisfies the following discrete elliptic equation with Dirichlet boundary conditions both on the interior and the exterior boundaries:

$$\begin{cases} \Delta_h v_{j,\,k} = 0, & \text{in} \quad \Omega^h_{\text{int}} \\ v_{j,\,k} = 0, & \text{on} \quad \Gamma^h_R \cup \Gamma^h_{\text{ext}}. \end{cases}$$

Obviously, the unique solution of this discrete elliptic equation is the trivial one. Consequently, $u_{j,\,k} = 0$, as we wanted to prove.

Of course, this proof yields the strong stability property (3.9) but it does not yield any estimate on the exponential decay rate.

Proof of (b). The fact that (3.10) does not hold was already pointed out in [26]. In fact, the uniform exponential decay property (3.10) fails even in $1-d$. This is due, not to the dissipative boundary conditions we have considered, but rather to the structure of the numerical scheme we have employed to discretize the wave equation. Indeed, this scheme generates spurious high frequency solutions that travel at a group velocity of the order of h. Therefore there is no uniform time $T > 0$ and constant $C > 0$ such that the following boundary observability property holds:

$$E_0^h \leq C \int_0^T \Big[h \sum_{(j,\,k) \in \Gamma^h_{\mathrm{ext}}} \Big| \frac{\partial u_{j,\,k}}{\partial \nu^h} \Big|^2 \Big] dt. \tag{3.15}$$

It turns out that an uniform observability inequality of the form (3.15) with $T > 0$ and $C > 0$ independent of h is equivalent to a uniform exponential decay property (3.10). Accordingly the later fails too.

□

3.2 The functional J for the conservative scheme

As mentioned above, the uniform coercivity of J_h is related to the uniform exponential decay property (3.10). For the sake of completeness we briefly sketch the argument.

As observed in [2], the functional J_h can be rewritten in the form

$$J_h(\psi) = \frac{1}{2} \parallel [e^{i\omega T} I - S_h(T)]v \parallel_h^2 \tag{3.16}$$

where $v=\psi-w_h$. Here I stands for the identity operator, S_h for the semigroup generated by the semi-discrete system with $f\equiv 0$ and $\|\cdot\|_h$ for the norm associated with the discrete energy E_h. Moreover, here, when writing Lv (with $L=e^{i\omega T}I-S_h(T)$) we are in fact refering to $L\tilde{v}$, where $\tilde{v}$ is the vector valued initial datum $\tilde{v}=(v,\,i\omega v)=\big(\psi-w_h, i\omega(\psi-w_h)\big)$. Similarly, Lv is vector valued, its two components being the two components of the state. Finally, w_h stands for the solution of the discrete version of (2.5), i.e.

$$\begin{cases} \big(\Delta_h+\omega^2\big)w_{j,k}=-f_{j,k} & \text{in} \quad \Omega_R^h \\ w_{j,k}=0 & \text{on} \quad \Gamma_{\text{int}}^h \\ \dfrac{\partial w_{j,k}}{\partial \nu^h}+i\gamma w_{j,k}=0 & \text{on} \quad \Gamma_{\text{ext}}^h. \end{cases} \tag{3.17}$$

As pointed out in [2], it follows that there exists $c_h>0$ such that

$$J_h(\psi)\geq c_h \parallel \psi-w_h \parallel_h^2, \tag{3.18}$$

if and only if $e^{i\omega T}$ does not belong to the spectrum of $S_h(T)$. Obviously a sufficient condition for this to hold for all ω, T and f is that $\parallel S_h(T)\parallel_h\to 0$ as $T\to\infty$, which is equivalent to (3.11). But in order for the coercivity inequality (3.18) to be uniform, i.e. for it to hold with a uniform coercivity constant $c>0$ independent of h, further properties are required. A natural condition for proving the uniform coervity would be the uniform (with respect to h) exponential decay property (3.10). We know however that such a uniform decay property is not true, even if the solutions of the continuous wave equation decay exponentially to zero under the non-trapping geometric condition on Γ_{int}^h. This is due to the existence of high frequency wave packets propagating with group velocity of the order of the mesh size, on which the boundary damping term has a very weak effect (see [28], [8]).

Consequently, minimizing J_h on the solutions of the conservative semi-discrete scheme is not an efficient way of approximating the solution of the Helmholtz equation.

3.3 The functional J for the viscous scheme

Following [25] and [26], in order to overcome the lack of uniform exponential stability properties of the scheme (3.3) it is natural to introduce a viscous variant of it. Of course, this method may only work under a non-trapping condition on Γ. It reads as follows

$$\begin{cases} u''_{j,k} - \Delta_h u_{j,k} - h^2 \Delta_h u'_{j,k} = e^{i\omega t} f_{j,k} & \text{in} \quad \Omega^h, \quad t>0 \\ u_{j,k} = 0 & \text{on} \quad \Gamma^h_{\text{int}}, \quad t>0 \\ \dfrac{\partial u_{j,k}}{\partial \nu^h} + \gamma_{j,k} u'_{j,k} = 0 & \text{on} \quad \Gamma^h_{\text{ext}}, \quad t>0 \\ u_{j,k}(0) = \psi_{j,k},\ u'_{j,k}(0) = i\omega\psi_{j,k} & \text{on} \quad \Omega^h_{\text{int}}. \end{cases} \tag{3.19}$$

In fact the only difference between the previous scheme (3.3) and the new one (3.19) is the addition of the numerical viscosity term in the interior. Indeed, the discretization scheme $\partial_t^2 - \Delta_h$ of the wave operator $\partial_t^2 - \Delta$ has been replaced in (3.16) by the viscous version $\partial_t^2 - \Delta_h - h^2\Delta_h\partial_t$.

For the solutions of this viscous semi-discrete scheme the energy dissipation law reads:

$$\begin{aligned} \frac{d\tilde{E}^h(t)}{dt} &= h^2 \sum_{(j,k)\in\Omega^h_{\text{int}}} f_{j,k} e^{i\omega t} u'_{j,k} - h \sum_{(j,k)\in\Gamma^h_{\text{ext}}} \gamma_{j,k} \left|u'_{j,k}\right|^2 \\ &\quad - h^4 \Big[\sum_{(j,k)\in\Omega^h_x} \left| \frac{u'_{j+1,k} - u'_{j,k}}{h} \right|^2 + \sum_{(j,k)\in\Omega^h_y} \left| \frac{u'_{j,k+1} - u'_{j,k}}{h} \right|^2 \Big], \end{aligned} \tag{3.20}$$

the new energy being

$$\tilde{E}^h(t) = \frac{h^2}{2} \Bigg[\sum_{(j,k)\in\Omega^h_{\text{int}}} \left|u'_{j,k}\right|^2 + \sum_{(j,k)\in\Omega^h_x} \left| \frac{u_{j+1,k} - u_{j,k}}{h} \right|^2$$

$$+ \sum_{(j,\,k)\in\Omega_y^h} \Big|\frac{u_{j,\,k+1}-u_{j,\,k}}{h}\Big|^2\Bigg] + \frac{h^2}{2}\Bigg[\sum_{(j,\,k)\in\Gamma^h_{\rm ext}} \gamma_{j,\,k}\big|u'_{j,\,k}\big|^2\Bigg]. \quad (3.21)$$

In particular, when the forcing term vanishes, i.e. $f_{j,\,k}=0$, we have

$$\begin{aligned}\frac{d\tilde{E}^h(t)}{dt} &= -h\sum_{(j,\,k)\in\Gamma^h_{\rm ext}} \gamma_{j,\,k}\big|u'_{j,\,k}\big|^2 \\ &\quad -h^4\Big[\sum_{(j,\,k)\in\Omega_x^h}\Big|\frac{u'_{j+1,\,k}-u'_{j,\,k}}{h}\Big|^2+\sum_{(j,\,k)\in\Omega_y^h}\Big|\frac{u'_{j,\,k+1}-u'_{j,\,k}}{h}\Big|^2\Big]. \quad (3.22)\end{aligned}$$

This energy dissipation law clearly reflects the effect of the added numerical viscosity term that provides a second dissipative mechanism which is effective everywhere in the domain $\Omega^h_{\rm int}$.

Remark: We could consider the same viscous numerical approximation scheme but with different dissipative boundary conditions. More precisely, in (3.19) we could take the boundary condition

$$\frac{\partial u_{j,\,k}}{\partial\nu^h} + h^2\frac{\partial u'_{j,\,k}}{\partial\nu^h} + \gamma_{j,\,k}u'_{j,\,k} = 0 \text{ on } \Gamma^h_{\rm ext}, t>0$$

on the exterior boundary instead of the one in (3.19).

In that case the energy to be considered is E_h as in (3.6) and the energy dissipation law (3.22) remains unchanged. Very likely the decay property of the energy E_h for the viscous scheme with these new boundary conditions are similar to those of $\tilde{E}_h$. But this issue is to be further investigated.

Let us now discuss the decay properties of system (3.19), and more precisely of the energy $\tilde{E}_h$. Before doing that it is convenient to recall the main features of the decay property of the continuous dissipative wave equation (2.10). The energy of the solutions of the wave equation decays exponentially in a bounded domain Ω provided the damping mechanism is effective on a subset of the boundary of the form

$$\Gamma(x^0) = \{x \in \partial\Omega : (x - x^0) \cdot \nu(x) > 0\},$$

x^0 being any point of $\mathbb{R}^d$ and $\nu = \nu(x)$ the unit outward normal to the domain Ω. The same holds when the damping term is effective in a neighborhood in Ω of a subset of the boundary of the form $\Gamma(x^0)$ (see [15] and [29]).

In the present setting, when the domain limited by Γ_{int} is star shaped with respect to a point x^0, for the domain Ω_R the subset $\Gamma(x^0)$ of its boundary coincides with Γ_{ext}. Accordingly the multiplier method yields the exponential decay for (2.10) with damping on Γ_{ext} only.

Note that the condition on the domain limited by Γ_{int} to be star-shaped is natural since it excludes the existence of trapped rays that never reach the exterior boundary Γ_{ext}, in which case the exponential decay properties would fail. Note however that the sharp necessary and sufficient condition for the exponential decay property to hold is the so-called Geometric Control Condition (GCC) (see [1]). The subset $\Gamma(x^0)$ as above is a particular case in which this condition holds and can be handled by multipliers. However, multipliers do not allow proving the decay under the most general GCC.

In what concerns the viscous semi-discrete scheme (3.19), the uniform exponential decay of its solutions was proved in [26] in the $1 - d$ case. In [20] the problem of stabilization was considered for a similar model in which the boundary damping term was replaced by an interior one localized in a neighborhood of the boundary, i.e. the model in which the Dirichlet boundary conditions are considered everywhere on the boundary of the domain and the equation is replaced by

$$u''_{j,\,k} - \Delta_h u_{j,\,k} - h^2 \Delta u'_{j,\,k} + 1_{\omega^h} u'_{j,\,k} = 0 \tag{3.23}$$

where 1_{ω^h} stands for the characteristic function of a subset ω^h of Ω^h constituting a neighborhood of Γ^h_{ext}. The results in [20], derived using discrete multiplier techniques, allow showing that the solutions of (3.23) satisfy the uniform, with respect to h, exponential decay property (3.10) provided the bounded domain limited by Γ_{int} is star-shaped. However, as in the continuous case, the result may not be achieved under the most general sharp condition of the obstacle being non-trapped. This would require the use of microlocal tools that so far have not been sufficiently developed at the discrete level to yield such an optimal result.

It is very likely that, combining the techniques in [20] and [26], the uniform, with respect to $h > 0$, exponential decay property for (3.19) with $f_{j,\,k} = 0$, will hold under the assumption on the set limited by Γ_{int} being star-shaped. But the details of the proof are still to be completed. Once the uniform exponential property (3.10) has been proved, one can immediately prove the following results, for ω small enough, i. e. for $T > 0$ large enough such that

$$||S_h(T)|| \leq \rho < 1, \tag{3.24}$$

for all $h > 0$, $S_h(\cdot)$ being the semigroup generated by the semi-discrete viscous scheme:

1. System (3.19) posesses a $2\pi/\omega$-periodic solution for all $h > 0$. Moreover, the energies of these periodic solutions are uniformly bounded with respect to h.

2. These discrete periodic solutions converge, as $h \to 0$, to the time-periodic solution of the continuous wave equation (2.4). Convergence holds in the energy spaces when considering the piecewise multi-linear and continuous extension of the discrete solutions to the continuous domain.

3. The time-periodic solutions of the discrete system (3.19) can be found by minimizing the functionals J_h in (3.12).

4. Consequently, the functionals J_h Γ-converge to the functional J in (2.11) associated to the wave equation (2.10).

Whether the same is true for all ω is an open problem. Indeed, according to (3.16), the problem is reduced to showing a uniform lower bound of the form

$$||(e^{i\omega T} I - S_h(T))v||_h \geq c||v||_h, \tag{3.25}$$

with $c > 0$ independent of h. Under the assumption that T is large enough, and in view of (3.24), this condition holds since

$$||(e^{i\omega T} I - S_h(T))v||_h \geq ||v||_h - ||S_h(T)v||_h \geq (1-\rho)||v||_h, \tag{3.26}$$

with $0 < \rho < 1$ as indicated above. But, obviously, this argument fails when T is small, i. e. when ω is large. The arguments in [2] (see Theorem 3) show that, for each $h > 0$, due to the exponential decay of the energy of solutions, there exists $c_h > 0$ such that

$$||(e^{i\omega T} I - S_h(T))v||_h \geq c_h||v||_h, \tag{3.27}$$

but a lower uniform bound on c_h can not be directly deduced. This is an interesting open problem.

3.4 Conclusion

As a consequence of the results of this section we conclude that, in order to guarantee the uniform coercivity of the functional J_h, one needs to employ the viscous scheme (3.16), under the assumption that the domain limited by Γ_{int} is star-shaped. Even in that case the problem is open for large values of ω.

Let us summarize some of the open problems that appeared along this section:

- To prove the uniform coercivity of the functionals J_h for the viscous numerical scheme under the sharp geomeric condition of the obstacle being non-trapping.
- To address the case where ω is large.

4 Analysis of the functional $\tilde{J}$

In the previous section we have seen that for using the discrete versions J_h of the functional J to efficiently solve the Helmholtz equation one needs to use a viscous numerical scheme and to assume that the region limited by Γ_{int} is star-shaped.

This section is devoted to analyze the possible use of the functional $\tilde{J}$ in (2.14).

As we shall see, from an analytical point of view this functional is more robust since its uniform coercivity is fulfilled not only for the viscous system but for the conservative one as well and without any geometric restrictions on the obstacle.

4.1 Uniform coercivity

As mentioned above, in [2] it was shown that the coercivity of this functional can be achieved more easily at the continuous level since it is related to the strong stability (2.8) of each individual trajectory and does not need the exponential decay of the norm of the semigroup (2.9). It is therefore natural to analyze whether its discrete versions $\tilde{J}_h$ are uniformly (with respect to h) coercive for the conservative scheme (3.3) and without any geometric assumption on Γ_{int} and any restriction on the frequency ω.

The key observation in [2] was that

$$\tilde{J}(\psi) \geq \frac{T}{2} \parallel (I-K)(\psi - w) \parallel^2, \tag{4.1}$$

where K is the linear operator

$$Kv := \frac{1}{T}\int_0^T e^{-i\omega s} S(s) v ds, \tag{4.2}$$

S being the semigroup associated to (2.4) with $f \equiv 0$. Thus, for the coercivity of $\tilde{J}$ it suffices to prove that the norm of the operator K is strictly smaller than 1. The latter is a consequence of the compactness of the operator K and the fact that $\parallel K(\psi) \parallel < 1$ for all ψ such that $\parallel \psi \parallel = 1$. This holds since Kv is a convex linear combination of the elements $\{e^{-i\omega t} S(t)\psi,\, t \in [0,\, T]\}$ that belong to the unit ball of the energy space. By strict convexity $\parallel Kv \parallel < 1$ unless this set is reduced to a single vector. In that case $S(t)\psi = e^{i\omega t}\psi$ for $t \in (0,\, T)$ and this holds if and only if ψ is an eigenvector of the generator of the semigroup with eigenvalue $i\omega$. But, because of the decay of each individual solution as $t \to \infty$, this possibility can be immediately excluded.

The same argument can be applied to prove the coercivity of the corresponding discrete functionals $\tilde{J}_h$ and this may be done uniformly on h. These functionals read as follows:

$$\begin{aligned} J_h(\psi) = \; & \frac{h^2}{2} \sum_{(j,\,k)\in\Omega^h_{\text{int}}} \int_0^T \left| u'_{j,\,k}(t) - i\omega e^{i\omega t}\psi_{j,\,k} \right|^2 dt \\ & + \frac{h^2}{2} \sum_{(j,\,k)\in\Omega^h_x} \int_0^T \left| \frac{\Big(u_{j+1,\,k}(t) - u_{j,\,k}(t)\Big) - e^{i\omega t}\Big(\psi_{j+1,\,k} - \psi_{j,\,k}\Big)}{h} \right|^2 dt \\ & + \frac{h^2}{2} \sum_{(j,\,k)\in\Omega^h_y} \int_0^T \left| \frac{\Big(u_{j,\,k+1}(t) - u_{j,\,k}(t)\Big) - e^{i\omega t}\Big(\psi_{j,\,k+1} - \psi_{j,\,k}\Big)}{h} \right|^2 dt. \end{aligned} \tag{4.3}$$

This functional can be defined for both schemes and it is, indeed, a discrete version of the functional $\tilde{J}$. These functionals are coercive for both schemes since each individual trajectory tends to zero as $t \to \infty$. Moreover, they are uniformly coercive as stated in the following Theorem:

Theorem 4.1. *The functionals $\tilde{J}_h$ are uniformly (with respect to h) coercive both for the conservative scheme (3.3) and the viscous one (3.19), without any geometric assumption of the interior boundary $\Gamma_{\rm int}$.*

Remark. In view of this Theorem we see that the uniform coercivity of the functionals $\tilde{J}_h$ holds much more easily than that of J_h since the uniform (with respect to h) exponential decay of the semigroups S_h is not needed. This makes the functional J and its discrete version to be more robust because of their uniform (with respect to the mesh-size) coercivity.

Proof of Theorem 4.1. Let us now sketch the proof of the uniform, with respect to h, coercivity of $\tilde{J}_h$. Let us denote by S_h the semigroup generated by any of the two numerical schemes under consideration. Distinctions will be made later, when needed. We also denote by K_h the corresponding operators as in (4.2), associated to S_h.

We have to show that there exists $0 < \rho < 1$ such that

$$\| K_h v \|_h \leq \rho \| v \|_h, \tag{4.4}$$

for all $h > 0$ and v. We argue by contradiction. If (4.4) does not hold, there exists a sequence of mesh-sizes $h \to 0$ and a sequence of data v_h such that

$$\begin{cases} \| K_h v_h \|_h \to 1, \\ \| v_h \|_h = 1. \end{cases} \tag{4.5}$$

As we shall see the two facts in (4.5) are in contradiction. To see this we first show that $K_h v_h$ are relatively compact in the energy space $H^1(\Omega_R) \times L^2(\Omega_R)$. In fact, to make the meaning of this statement precise we consider the piecewise multi-linear and continuous extension of $K_h v_h$ to the domain Ω_R that we shall denote as $P_1 K_h v_h$. Note also that, since the systems under consideration are second order in time, the vectors $K_h v_h$ have two components, the first one corresponding to the state and the second one to its time derivative.

Assuming for the moment that this compactness result holds, we can extract a subsequence, still denoted by the index h, such that

$$(P_1(v_h), P_1(v_{h,t})) \rightharpoonup (v, v_t) \text{ weakly in } L^2(0,T; H^1(\Omega_R) \times L^2(\Omega_R)) \tag{4.6}$$

and

$$(P_1(K_h v_h), P_1(K_h v_{h,t})) \to (Kv, Kv_t) \text{ strongly in } L^2(0,T; H^1(\Omega_R) \times L^2(\Omega_R)), \tag{4.7}$$

where K stands for the operator associated to the continuous wave equation (2.4).

In view of (4.5) we deduce that

$$\begin{cases} \| Kv \| = 1, \\ \| v \| \leq 1 \end{cases} \tag{4.8}$$

and these two facts are in contradiction since the norm of the operator K associated with the continuous wave equation is strictly smaller than one, as shown in [2] and sketched above.

Let us now conclude by showing the compactness of $(P_1(K_h v_h), P_1(K_h v_{h,t}))$ in the energy space $H^1(\Omega_R) \times L^2(\Omega_R)$. By definition

$$K_h v_h = \frac{1}{T} \int_0^T e^{-i\omega t} S_h(t) v_h \, dt, \tag{4.9}$$

that can be characterized by the equation

$$A_h(K_h v_h) = \frac{i\omega}{T}\int_0^T e^{-i\omega t} S_h(t) v_h \, dt + \frac{1}{T}\Big(e^{-i\omega T} S_h(T) - I\Big) v_h, \qquad (4.10)$$

A_h being the generator of the semi-discrete semigroup.

The right hand side terms in (4.10) are uniformly bounded in the discrete energy space. Thus $K_h v_h$ are uniformly bounded in the domains of the operators A_h. It is then sufficient to show that this guarantees the compactness of the P_1 extensions in the continuous energy space.

As stated in the Theorem, this is true both for the dissipative and the conservative numerical schemes without any geometric restriction on Γ_{int}.

For the sake of completeness we now prove the compactness of $K_h v_h$ for the conservative semi-discrete scheme. To do it we first consider the continuous equation (2.4) and characterize Kv as the solution of an elliptic equation.

We have

$$\begin{cases} -\Delta(Kv) + \omega^2 Kv = -\dfrac{1}{T} e^{-i\omega t}[i\omega v + v_t]\Big|_0^T & \text{in} \quad \Omega \\ Kv = 0 & \text{on} \quad \Gamma_{\text{int}} \\ \partial_\nu(Kv) + i\omega\gamma Kv = -\dfrac{\gamma}{T} e^{-i\omega t} v\Big|_0^T & \text{on} \quad \Gamma_{\text{ext}}. \end{cases} \qquad (4.11)$$

To be more precise, this elliptic equation characterizes the first component of Kv. The second one, that we denote by $K^2 v$ can be characterized as

$$K^2 v = \frac{1}{T}\int_0^T e^{-i\omega s} v_t(s) ds = \frac{i\omega}{T}\int_0^T e^{-i\omega s} v(s) ds + \frac{e^{-i\omega t} v(t)}{T}\Big|_0^T. \qquad (4.12)$$

Note that the fact that the trajectory $t \to v(t)$ is bounded in $H^1(\Omega_R)$ implies that $K^2 v$ lies in $H^1(\Omega_R)$ as well. This guarantees the compactness with values in $L^2(\Omega_R)$.

The same can be said about the discrete versions $K_h^2 v_h$. The fact that the solutions $\big(v_h, v_{h,t}\big)$ are uniformly bounded in the discrete energy spaces implies immediately that $P_1(K_h^2 v_h)$ are uniformly bounded in $H^1(\Omega_R)$ and this guarantees the relative compactness in $L^2(\Omega_R)$.

The discrete functions $K_h v_h$ can be characterized as in (4.11) by replacing the continuous Laplacian Δ by its discrete counterpart and the Neumann boundary conditions accordingly:

$$\begin{cases} -\Delta_h K_h v_{j,k} - \omega^2 K_h v_{j,k} = -\dfrac{1}{T} e^{-i\omega t}[i\omega v_{j,k} + v'_{j,k}]\Big|_0^T & \text{in} \quad \Omega^h \\ K_h v_{j,k} = 0 & \text{on} \quad \Gamma^h_{\text{int}} \\ \dfrac{\partial K_h v_{j,k}}{\partial \nu^h} + i\omega\gamma_{j,k} K_h v_{j,k} = -\dfrac{\gamma_{j,k}}{T} e^{-i\omega t} v_{j,k}\Big|_0^T & \text{on} \quad \Gamma^h_{\text{ext}}. \end{cases} \tag{4.13}$$

It is then sufficient to check the relative compactness of $P_1(K_h v_h)$ in $H^1(\Omega_R)$, $K_h v_h$ being characterized as the solutions of (4.13).

At the continuous level such a result would be easy to prove since Kv, solution of (4.11), belongs to $H^2(\Omega_R)$ by elliptic regularity.

In what concerns the discrete problems, taking into account that $P_1(K_h v_h)$ is a piecewise multilinear extension of $K_h v_h$ its gradient may be discontinuous. Thus, it does not make sense to search for uniform estimates in H^2. We may however look for compactness in H^1. To do it we argue as follows. We first observe that, as a consequence of the uniform discrete energy estimates for v_h, $P_1(K_h v_h)$ is uniformly bounded in $H^1(\Omega_R)$. By extracting subsequences we pass weakly to the limit in $H^1(\Omega_R)$. It is then easy to identify the limit as Kv, v being the limit solution of the continuous wave equation and K the corresponding operator in (4.11). To conclude that the sequence actually converges strongly it is sufficient to check the convergence of the H^1-norms. At this point it is useful to use the fact that the continuous H^1-norm of $P_1(K_h v_h)$ coincides with the discrete H^1-norm of $K_h v_h$ that we simply denote as $||K_h v_h||_{1,h}$. In view of the variational formulation of (4.13) we have

$$||K_h v_h||_{1,h}^2 = \omega^2 ||K_h v_h||_{0,h}^2 - \frac{1}{T}(e^{-i\omega t}[i\omega v_{j,k} + v'_{j,k}]\Big|_0^T, K_h v_{j,k})_{0,h}$$
$$-(i\omega\gamma_{j,k} K_h v_{j,k} + \frac{\gamma_{j,k}}{T} e^{-i\omega t} v_{j,k}\Big|_0^T, K_h v_{j,k})_{\Gamma^h_{\mathrm{ext}}} \tag{4.14}$$

where $|| \cdot ||_{0,h}$ and $(\cdot,\cdot)_{0,h}$ stand, respectively, for the discrete L^2-norms and scalar products and $(\cdot,\cdot)_{\Gamma^h_{\mathrm{ext}}}$ for the L^2-scalar product on Γ_{ext}. It is not hard to see that the weak convergence in H^1 allows passing to the limit in the right hand side term of (4.14). In the limit we get the corresponding continuous term which, by the variational formulation of (4.11), yields

$$||Kv||_{H^1(\Omega_R)}^2 = \omega^2 ||Kv||_{L^2(\Omega_R)}^2 - \frac{1}{T}(e^{-i\omega t}[i\omega v + v_t]\Big|_0^T, Kv)_{L^2(\Omega_R)}$$
$$-(i\omega\gamma Kv + \frac{\gamma}{T} e^{-i\omega t} v\Big|_0^T, Kv)_{\Gamma_{\mathrm{ext}}}. \tag{4.15}$$

Consequently, we deduce that

$$||K_h v_h||_{1,h}^2 \to ||Kv||_{H^1(\Omega_R)}^2.$$

Taking into account that, as mentioned above, $||K_h v_h||_{1,h}^2 = ||P_1(K_h v_h)||_{H^1(\Omega_R)}^2$ this concludes the proof of the convergence of norms and, consequently, of the strong convergence of $P_1 K_h v_h$ to Kv in $H^1(\Omega)$. This completes the proof of Theorem 4.1.

The convergence of the terms on the right hand side of (4.14) towards those in (4.15) can be easily deduced from the weak convergence in $H^1(\Omega_R)$. The first two are consequence of the strong convergence of $K_h v_h$ and v_h towards Kv and v in $L^2(\Omega_R)$. This is due to the compact embedding from $H^1(\Omega_R)$ into $L^2(\Omega_R)$. The last one holds because of the compactness of traces in $L^2(\Gamma_{\mathrm{ext}})$.

The proof can be developed similarly for the viscous scheme as well.

4.2 Conclusion

As a consequence of the results of this section we conclude that the functional $\widetilde{J}_h$ can be used efficiently for numerical computations in a much more flexible setting. In particular one can use both the conservative and the viscous schemes (3.16), without any geometric assumption on the domain limited by $\Gamma_{\rm int}$ and any restriction on the size of ω.

This is in contrast with the results of the previous section on the functional J_h that needed both, the star-shaped condition on Γ_{int}, and the use of the scheme with the added numerical viscosity term.

5 Concluding remarks

The analysis in this paper shows that the functional $\widetilde{J}_h$ is more robust than J with respect to numerical discretizations since the later one requires specific viscous numerical schemes and non-trapping conditions on the obstacle, that the first one does not need.

It is however worth mentioning that the numerical experiments conducted in [5] using the functional J showed to be very efficient also when $\Gamma_{\rm int}$ was trapping. This could be related to the fact that the forcing terms under consideration are monocromatic and that, consequently, the high frequency spurious effects are not determinant.

In the literature on the exact controllability of the numerical approximation schemes for the wave equation there is an analytical result showing that this may happen. Indeed, according to the results in [18] about the controllability of the semi-discrete wave equation, despite the fact that boundary controls for the $1-d$ semi-discrete equations do not converge for all initial data, they do it when the Fourier expansion of the data to be controlled is truncated conveniently. But the results in [18] only concern the $1-d$ case and there is a lot to be done in this context to extend it to multi-dimensional problems and to analyze the possible connections with the good numerical performance of the functional J.

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