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MULTIPLICITY RESULTS FOR EQUATIONS WITH SUBCRITICAL HARDY-SOBOLEV EXPONENT AND SINGULARITIES ON A HALF-SPACE

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Dedicated to Prof. Luiz Adauto Medeiros on the occasion of his 80th birthday

Abstract

We prove some multiplicity results for a class of singular quasilinear elliptic problems involving the critical Hardy-Sobolev exponent and singularities on a half-space.

1 Introduction

In this work we present some multiplicity results for a class of singular quasilinear elliptic problems of the form

$$\begin{cases} -\operatorname{div}\left[|x_N|^{-ap}|\nabla u|^{p-2}\nabla u\right] + \lambda|x_N|^{-(a+1)p}|u|^{p-2}u & \\ \qquad = \alpha|x_N|^{-bq}|u|^{q-2}u + \beta k(x)|x_N|^{-cr}|u|^{r-2}u & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{P}$$

involving the critical Hardy-Sobolev exponent and singularities on a half-space. We use notation $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ $(N \geqslant 3)$, $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$ and $\partial\mathbb{R}^N_+ = \{x \in \mathbb{R}^N_+ : x_N = 0\}$, we consider $1 < p < N$, $0 \leqslant a < (N-p)/p$, $a < b < a+1$, $c \in \mathbb{R}$, $d \equiv a + 1 - b$, $q = q(a,b) \equiv Np/[N-dp]$, which is the critical Hardy-Sobolev exponent, $1 < r < q(0,0) = p^* \equiv Np/(N-p)$,

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α, $\beta \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$ are parameters and $0 \leqslant k(x) \in L^{q/(q-r)}_{r(c-b)}(\mathbb{R}^N_+)$, where the notation $L^q_b(\mathbb{R}^N_+)$ stands for the weighted space defined by

$$L^q_b(\mathbb{R}^N_+) \equiv \Big\{ u : \mathbb{R}^N_+ \to \mathbb{R} \ : \ |u|_{q,b} \equiv \big| |x_N|^{-b} u \big|_q = \Big[\int |x_N|^{-bq} |u|^q \, \mathrm{d}x \Big]^{1/q} \Big\}.$$

To simplify the notation, hereinafter the symbol $\int$ denotes $\int_{\mathbb{R}^N_+}$.

We look for solutions of problem (P) in Sobolev space $\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ defined as the completion of space $C^\infty_0(\mathbb{R}^N_+)$ with respect to the norm given by

$$\|u\| \equiv \Big[\int |x_N|^{-ap} |\nabla u|^p \, \mathrm{d}x \Big]^{1/p}.$$

The pioneering work treating this kind of problem is [6], by Brézis and Nirenberg, in which the case $a = b = 0$ and $k(x) \equiv 1$ is studied in a bounded domain. This paper opened up a new stream to treat some types of semilinear as well as quasilinear elliptic problems, previously avoided because of the famous nonexistence result by Pohozaev. See [1, 4, 5, 9, 11, 12, 13, 14, 15, 19, 20]. We recall that all these problems consider the case in which the singular coefficient of the nonlinearity involves only the origin.

For problem (P) with the singular coefficient of the nonlinearity involving a half-space we cite [3, 10, 18, 17].

Our results generalize the work [10] and also some results in [13].

The starting point to treat problem (P) is a Caffarelli, Kohn, and Nirenberg type inequality stablished in [7]. See also [8, 16]. Let

$$S_\lambda = S(a,b,\lambda) \equiv \inf_{\substack{u \in \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+) \\ u \not\equiv 0}} \frac{\displaystyle\int |x_N|^{-ap} |\nabla u|^p \, \mathrm{d}x + \lambda \int |x_N|^{-(a+1)p} |u|^p \, \mathrm{d}x}{\displaystyle\Big[\int |x_N|^{-bq} |u|^q \, \mathrm{d}x \Big]^{p/q}} \tag{1}$$

be the Lagrange's multiplier. Then the best constant S_λ is positive and is attained by a function $u \in \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$. (See Theorem 1 below.) One of the main difficulties of our work is to prove this claim, because problem (P) involves unbounded domain and the usual methods of the calculus of variations do not apply immediately.

Using the claim we can show that the functional $I : \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+) \to \mathbb{R}$ given by

$$
\begin{aligned}
I(u) \quad \equiv \quad & \frac{1}{p}\int |x_N|^{-ap}|\nabla u|^p \,\mathrm{d}x + \frac{\lambda}{p}\int |x_N|^{-(a+1)p}|u|^p \,\mathrm{d}x \\
& -\frac{\alpha}{q}\int |x_N|^{-bq}|u|^q \,\mathrm{d}x - \frac{\beta}{r}\int k(x)|x_N|^{-cr}|u|^r \,\mathrm{d}x
\end{aligned}
$$

is well defined. Moreover, $I \in C^1(\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+), \mathbb{R})$ (see Lemma 1) and a weak solution of problem (P) is a critical point of the functional I. This means that $I'(u) = 0$, where

$$
\begin{aligned}
\langle I'(u), v\rangle \quad = \quad & \int |x_N|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla v \,\mathrm{d}x + \lambda\int |x_N|^{-(a+1)p}|u|^{p-2}uv \,\mathrm{d}x \\
& -\alpha\int |x_N|^{-bq}|u|^{q-2}uv \,\mathrm{d}x - \beta\int k(x)|x_N|^{-cr}|u|^{r-2}uv \,\mathrm{d}x
\end{aligned}
$$

for every $v \in \mathcal{D}^{1,p}_{a}(\mathbb{R}^N_+)$.

Our main results are the following.

Theorem 1 (Extremal problem) *Let $\alpha = 1$ and $\beta = 0$. The best constant S_λ is attained by a function $u \in \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ provided that one of the conditions below is verified.*

1. *$a < b < a+1$ and $-S(a, a+1) < \lambda \leqslant 0$.*
2. *$a < b < a+1$ and $0 < \lambda$ small.*

Suppose that the Lebesgue measure of the set $M \equiv \{x \in \mathbb{R}^N_+ \,:\, k(x) > 0\}$ is positive.

Theorem 2 (Hardy-Sobolev critical case) *Let $1 < r < p < N$ be given. Then the following claims hold.*

1. *For every* $\alpha \in \mathbb{R}_+$ *there exists* $B \in \mathbb{R}_+$ *such that if* $0 < \beta < B$, *then problem* (P) *has a sequence of solutions* $(u_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ *with* $I(u_n) < 0$ *and* $\lim_{n\to\infty} I(u_n) = 0$.
2. *For every* $\beta \in \mathbb{R}_+$ *there exists* $A \in \mathbb{R}_+$ *such that if* $0 < \alpha < A$, *then problem* (P) *has a sequence of solutions* $(u_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ *with* $I(u_n) < 0$ *and* $\lim_{n\to\infty} I(u_n) = 0$.

2 Auxiliary results

We begin by stating the following lemma, whose proof can be adapted from [14].

Lemma 1 *The functional* $F : \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+) \to \mathbb{R}$ *given by*

$$F(u) \equiv \int k(x)|x_N|^{-cr}|u|^r \,\mathrm{d}x$$

is well defined, is weakly continuous, and is continuously differentiable.

We say that a functional $\phi : X \to \mathbb{R}$ defined in a Banach X verifies the Palais-Smale condition at the level c, in short ϕ verifies $(PS)_c$, if every sequence $(u_n) \subset X$ such that $\phi(u_n) \to c$ and $\phi'(u_n) \to 0$ has a convergent subsequence. To find the critical points of functional I we have to establish conditions under which I verifies $(PS)_c$. Therefore we need a concentration-compactness lemma. Next we state a useful lemma.

Lemma 2 *Let* $K \subset \mathbb{R}^{N-1}$ *be a compact subset and* $R \in \mathbb{R}_+$. *If* $u_n \rightharpoonup u$ *weakly in* $\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$, *then the following claims hold.*

1. $|x_N|^{-a}u_n \to |x_N|^{-a}u$ *in* $L^p_{\mathrm{loc}}(\mathbb{R}^N_+)$.
2. $|x_N|^{-a}u_n \to |x_N|^{-a}u$ *in* $L^p(K \times [0, R])$.

Proof. The proof of item 1 is similar to the corresponding one in [20]. To prove item 2 we consider $\delta \in \mathbb{R}_+$ and divide the argument in the cases $K \times [\delta, R]$ (whose proof is similar to that of claim 1) and $K \times [0, \delta]$. In the last case we have

$$\int_{K\times[0,\delta]} |x_N|^{-ap}|u|^p \,\mathrm{d}x \leqslant \delta^p \int_{K\times[0,\delta]} |x_N|^{-(a+1)p}|u|^p \,\mathrm{d}x$$

and this concludes the proof of claim 2. The lemma is proved.

□

Thanks to Lemma 2, the proof of the following concentration-compactness lemma is similar to that of Lemma 4.1 in [2]. See also [20].

Lemma 3 *Let $\mathcal{M}(\mathbb{R}^N_+)$ be the space of positive, bounded Radon measures in $\mathbb{R}^N_+$. Let $(u_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ be a sequence such that the following convergences hold.*

1. *$u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$.*
2. *$\big||x_N|^{-a}\nabla(u_n - u)\big|^p + \lambda\big||x_N|^{-(a+1)}(u_n - u)\big| \rightharpoonup \gamma$ weakly in $\mathcal{M}(\mathbb{R}^N_+)$.*
3. *$\big||x_N|^{-bq}(u_n - u)\big|^q \rightharpoonup \nu$ weakly in $\mathcal{M}(\mathbb{R}^N_+)$.*
4. *$u_n \to u$ a.e. in $\mathbb{R}^N_+$.*

We define the measures of concentration at infinity (where the integrals are taken over the set $\mathbb{R}^N_+ \cap \{|x| \geqslant R\}$),

$$\begin{aligned}
\gamma_\infty &\equiv \lim_{R\to\infty}\varlimsup_{n\to\infty}\Big[\int |x_N|^{-ap}|\nabla u_n|^p \,\mathrm{d}x + \lambda\int |x_N|^{-(a+1)p}|u_n|^p \,\mathrm{d}x\Big],\\
\nu_\infty &\equiv \lim_{R\to\infty}\varlimsup_{n\to\infty}\int |x_N|^{-bq}|u_n|^q \,\mathrm{d}x.
\end{aligned}$$

Then there exist a set of indexes J (at most denumerable), a corresponding set *of points* $(z_j)_{j\in J} \subset \mathbb{R}^{N-m}$*, and two sets of measures* $(\mu_j)_{j\in J}$*,* $(\nu_j)_{j\in J}$ *such that*

$$
\begin{aligned}
\gamma &\geqslant |x_N|^{-ap}|\nabla u|^p + \lambda |x_N|^{-(a+1)p}|u|^p \\
&\quad + \sum_{j\in J} S_\lambda \big[\gamma_j(\{x_j\})\big]^{p/q} + \gamma_\infty, && (2)\\
\nu &= |x_N|^{-bq}|u|^q + \sum_{j\in J} \nu_j(\{x_j\}) + \nu_\infty, && (3)\\
\big[\nu_j(\{x_j\})\big]^{p/q} &\leqslant S_\lambda^{-1}\gamma_j(\{x_j\}), && (4)\\
\nu_\infty^{p/q} &\leqslant S_\lambda^{-1}\gamma_\infty. && (5)
\end{aligned}
$$

Moreover, for $u(x) \equiv 0$*, if* $b < a+1$ *and* $\|\nu\|^{p/q} = S_\lambda^{-1}\|\gamma\|$*, then the measures* ν *and* γ *concentrate at a single point.*

Lemma 4 *Suppose that* $1 < r < p$ *and let* $(u_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ *be a Palais-Smale sequence for the functional* I *at the level* $c \in \mathbb{R}$*. Then we have:*

1. *For every* $\alpha > 0$*, there exists* $B > 0$ *such that if* $0 < \beta < B$*, then the sequence* (u_n) *has a convergent subsequence in* $\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$*.*
2. *For every* $\beta > 0$*, there exists* $A > 0$ *such that if* $0 < \alpha < A$*, then the sequence* (u_n) *has a convergent subsequence in* $\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$*.*

Proof. It is standard to prove that the sequence $(u_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ is bounded. Passing to a subsequence (still denoted in the same way), we can suppose that the hypotheses of Lemma 3 hold. Let $x_j \in \mathbb{R}^N_+$ be a singular point for the measures μ and ν; then following the ideas of Chen and Li in [9] we can prove that $\nu(\{x_j\}) = 0$ and $\nu_\infty = 0$. Using Claim 1 in [1] and substituting the singular term for $|x_N|^{-ap}$, we can prove that the sequence (u_n) converges strongly to $u \in \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$.

$\square$

3 Proofs of the theorems

Proof of Theorem 1. Following Maz'ja [16] and adapting a result from [10], we have

$$S_\lambda \Big[\int_{\mathbb{R}^N_+} |x_N|^{-bq} u^q \,\mathrm{d}x \Big]^{p/q} \leqslant \int |x_N|^{-ap} |\nabla u|^p \,\mathrm{d}x + \lambda \int |x_N|^{-(a+1)p} |u|^p \,\mathrm{d}x \quad (6)$$

for every $u \in \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$. Let $(u_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ be a minimizing sequence for S_λ, i. e.,

$$\int |x_N|^{-bq} |u_n|^q \,\mathrm{d}x = 1$$

and

$$\int |x_N|^{-ap} |\nabla u_n|^p \,\mathrm{d}x + \lambda \int |x_N|^{-(a+1)p} |u_n|^p \,\mathrm{d}x \to S_\lambda$$

as $n \to \infty$. We define

$$Q_n(\lambda) \equiv \sup_{y \in \mathbb{R}^N_+} \int_{B(y,\lambda)} |x_N|^{-bq} |u_n|^q \,\mathrm{d}x.$$

Then there exist sequences $(\tau_n) \subset \mathbb{R}_+$ and $(y_n) \subset \mathbb{R}^N_+$ (where $y_n = (y_n', (y_n)_N)$) such that $Q_n(\tau_n) = 1/2$, because

$$\lim_{\lambda \to 0^+} Q_n(\lambda) = 0 \qquad \text{and} \qquad \lim_{\lambda \to \infty} Q_n(\lambda) = 1.$$

Let $v_n(x) \equiv \tau_n^{[N-p(a+1)]/p} u_n(\tau_n x + z_n)$, where $z_n = (y_n', 0)$. Using $x_n = (0, y_n^N/\lambda_n)$, we have

$$\frac{1}{2} = \int_{B(x_n,1)} |x_N|^{-bq} |v_n|^q \,\mathrm{d}x = \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |x_N|^{-bq} |v_n|^q \,\mathrm{d}x. \qquad (7)$$

A simple computation shows that $(v_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ is also a minimizing sequence; in other words, the problem is invariant under the given transformation.

Now we divide the argument in two cases.

Case 1. The sequence $(x_n) \subset \mathbb{R}^N_+$ is bounded. In this case we can suppose (passing to a subsequence if necessary, still denoted in the same way) that the sequence $(v_n) \subset \mathcal{D}^{1,p}_a(\mathbb{R}^N_+)$ verifies the hypotheses of Lemma 3. Using it, we have

$$\begin{aligned}
\|\nu\|^{p/q} &\leqslant S_\lambda^{-1}\|\gamma\| && (8)\\
\nu_\infty^{p/q} &\leqslant S_\lambda^{-1}\gamma_\infty && (9)\\
S_\lambda &= \overline{\lim_{n\to\infty}} \int |x_N|^{-ap}|\nabla v_n|^p \,\mathrm{d}x + \lambda \int |x_N|^{-(a+1)p}|v_n|^p \,\mathrm{d}x \\
&= \int |x_N|^{-ap}|\nabla v|^p \,\mathrm{d}x + \lambda \int |x_N|^{-(a+1)p}|v|^p \,\mathrm{d}x + \|\gamma\| + \gamma_\infty && (10)\\
1 &= \overline{\lim_{n\to\infty}} \int |x_N|^{-bq}|v_n|^q \,\mathrm{d}x = \int |x_N|^{-bq}|v|^q \,\mathrm{d}x + \|\nu\| + \nu_\infty. && (11)
\end{aligned}$$

By definition of S_λ and equation (8), using inequality (10) we obtain

$$S_\lambda \geqslant S_\lambda \Big\{ \Big[\int |x_N|^{-bq}|v|^q \Big]^{p/q} + \|\nu\|^{p/q} + \nu_\infty^{p/q} \Big\}. \tag{12}$$

It follows from equation (11) and inequality (12) that $\displaystyle\int |x_N|^{-bq}|v|^q \,\mathrm{d}x$, $\|\nu\|$, and ν_∞ are equal either to 0 or to 1.

Since we are supposing that $(x_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ is bounded, from equation (7) we have $\nu_\infty \leqslant 1/2$; hence, $\nu_\infty = 0$.

Now we suppose that $\|\nu\| = 1$; therefore $v = 0$ and we will get a contradiction. From inequality (10) it follows that

$$1 = \|\nu\|^{p/q} \leqslant S_\lambda^{-1}\|\gamma\| = \frac{\|\gamma\|}{\|\gamma\| + \gamma_\infty} \leqslant 1$$

so that $\|\nu\|^{p/q} = S_\lambda^{-1}\|\gamma\|$. By the concentration-compactness principle, the measure ν concentrates at a single point z. Using equation (11) we get

$$\frac{1}{2} = \sup_{y\in\mathbb{R}^N_+} \int_{B(y,1)} |x_N|^{-bq}|v_n|^q \,\mathrm{d}x \geqslant \int_{B(z,1)} \mathrm{d}\nu_n \to \|\nu\| = 1,$$

because $\nu_n \rightharpoonup \nu$ and $\displaystyle\int_{B(z,1)} \mathrm{d}\nu_n \to \delta_z = 1$, which is a contradiction. Thus $\displaystyle\int |x_N|^{-bq}|v|^q = 1$ and S_λ is attained.

Case 2. The sequence $(x_n) \subset \mathbb{R}^N_+$ **is unbounded.** In this case we can suppose that $x_n \to \infty$ as $n \to \infty$. We define the function

$$G = G\big(\tau^{[N-p(a+1)]/p} v_n(\tau x)\big) \equiv \int \frac{|x|}{1+|x|} |x_N|^{-bq} \big[\tau^{[N-p(a+1)]/p} v_n(\tau x)\big]^q \, \mathrm{d}x.$$

We have

$$\lim_{\tau\to 0^+} G = \lim_{\tau\to 0^+} \int \frac{|z|}{\tau+|z|} |z_N|^{-bq} |v_n(z)|^q \, \mathrm{d}z = \int |z_N|^{-bq}|v_n(z)|^q \, \mathrm{d}z = 1$$

and

$$\lim_{\tau\to\infty} G = \lim_{\tau\to\infty} \int \frac{|z|}{\tau+|z|} |z_N|^{-bq}|v_n(z)|^q \, \mathrm{d}z = 0.$$

Then there exists a sequence $(\tau_n) \subset \mathbb{R}_+$ such that $G\big(\tau_n^{[N-p(a+1)]/p} v_n(\tau_n x)\big) = 1/2$ for every $n \in \mathbb{N}$. Now let $w_n(x) \equiv \tau_n^{[N-p(a+1)]/p} v_n(\tau_n x)$; then

$$\frac{1}{2} = \int \frac{|x|}{1+|x|} |x_N|^{-bq}|w_n(x)|^q \, \mathrm{d}x \qquad \text{and} \qquad \int |x_N|^{-bq}|w_n|^q \, \mathrm{d}x = 1. \quad (13)$$

The sequence $(w_n) \subset \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N_+)$ is minimizing and verifies the hypotheses of Lemma 3.

By the concentration-compactness principle, inequalities (8), (9), (10), and (11) also hold for sequence (w_n). As before, $\displaystyle\int |x_N|^{-bq}|w_n|^q$, $\|\nu\|$, and ν_∞ are equal either to 0 or to 1. Since

$$\begin{aligned}\frac{1}{2} &= \int_{\mathbb{R}^N_+} \frac{|x|}{1+|x|} |x_N|^{-bq}|w_n|^q \, \mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^N_+ \cap \{|x|\geqslant R\}} \frac{|x|}{1+|x|} |x_N|^{-bq}|w_n|^q \, \mathrm{d}x \geqslant \frac{R}{1+R} \int_{\mathbb{R}^N_+ \cap \{|x|\geqslant R\}} |x_N|^{-bq}|w_n|^q \, \mathrm{d}x,\end{aligned}$$

it follows that $\tilde{\nu}_\infty \leqslant 1/2$; then, $\tilde{\nu}_\infty = 0$.

Suppose that $\|\tilde{\nu}\| = 1$; then $\|\tilde{\nu}\|^{p/q} = S^{-1}\|\gamma\|$. Hence the measure $\tilde{\nu}$ concentrates at a single point z. By equation (13), we have

$$\frac{1}{2} = \int \frac{|x|}{1+|x|}|x_N|^{-bq}|w_n|^q \,\mathrm{d}x \to \frac{|z|}{1+|z|},$$

which implies that $|z| = 1$. Here we used $\nu_n = |x_N|^{-bq}|w_n|^q \,\mathrm{d}x$ and $\nu_n \rightharpoonup \nu$ as $n \to \infty$, which means that

$$\int \frac{|x|}{1+|x|}\nu_n \to \int \frac{|x|}{1+|x|}\delta_z = \frac{|z|}{1+|z|}.$$

But since $q < p^* \equiv Np/(N-p)$, taking a ball $B \subset \mathbb{R}^N_+$ and using item 1 of Lemma 2, we have

$$|x_N|^{p/q} w_n \to 0 \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^N_+).$$

Therefore $z_N = 0$; but from equation (7) it follows that

$$\int_{B(x_n/\tau_n, 1/\tau_n)} |x_N|^{-bq}|w_n|^q \,\mathrm{d}x = \frac{1}{2}.$$

If the sequence $(\tau_n) \subset \mathbb{R}_+$ is bounded, then

$$\inf\{|x| : x \in B(x_n/\tau_n, 1/\tau_n)\} \geqslant \frac{|x_n|-1}{|\tau_n|} \to \infty$$

since we are supposing that the sequence (x_n) is unbounded. This contradicts the fact that $\tilde{\nu}_\infty = 0$.

If $\tau_n \to \infty$, using the facts that $x_n = (0, (y_n)_N/\tau_n)$ and $x_n \to z$ as $n \to \infty$ (with $|z| = 1$), we have to consider two possibilities. One possibility occurs if $(y_n)_N/\tau_n$ is unbounded, which is the case just treated. The other possibility occurs if $(y_n)_N/\tau_n$ is bounded. In this case we have $(y_n)_N/\tau_n \to 0$, which implies that $z = (0,0)$, a contradiction with $|z| = 1$.

Then we have $\|\nu\| = 0$, $\nu_\infty = 0$ and $\int_{\mathbb{R}^N_+} |x_N|^{-bq}|w(x)|^q\,\mathrm{d}x = 1$. This shows that S_λ is attained and concludes the proof of the theorem.

□

Proof of Theorem 2. To prove Theorem 2 we use Lemmata 1 and 4 and follows the same ideas of [12]. See also [9, 1].

□

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