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SUPERLINEAR ORDINARY ELLIPTIC SYSTEMS INVOLVING PARAMETERS

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Abstract

We use a unified approach to study certain classes of elliptic problems. More precisely, we consider a Dirichlet problem for a system of two ordinary differential equations which depends on two numerical parameters a and b, and with nonlinearities satisfying very general superlinear local growth conditions. Using the upper–lower solutions method, fixed point theorems of cone expansion/compression type and some degree–theoretic arguments, we prove that there exists a non–increasing function Γ of the parameter a such that the problem has (i) at least one positive solution for $0 \leq b \leq \Gamma(a)$, (ii) no positive solution for $b > \Gamma(a)$, and (iii) at least two positive solutions for $0 < b < \Gamma(a)$. We apply the main results to a class of semilinear elliptic systems in both bounded annular domains and exterior domains with non–homogeneous Dirichlet boundary conditions. In addition, we apply our results to fourth–order boundary value problems. The nonlinearities may have singularities, as well as may vanish in parts of the domain.

1 Introduction

In this paper we will outline some recent advances concerning existence, non-existence and multiplicity of positive solutions for a class of systems of two ordinary differential equations which depends on two numerical parameters with Dirichlet homogeneous boundary conditions. We will apply these results to study certain classes of semilinear elliptic systems in both bounded annular domains and exterior domains with non–homogeneous Dirichlet boundary

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conditions. In addition, we apply our results to fourth–order boundary value problems.

We will mainly deal with the class of ordinary differential equations

$$\left\{\begin{array}{rcll} -u'' & = & g_1(t,u,v,a,b) & \text{in} \quad (0,1), \\ -v'' & = & g_2(t,u,v,a,b) & \text{in} \quad (0,1) \end{array}\right. \tag{S_{ab}}$$

with Dirichlet boundary conditions

$$u(0) = v(0) = u(1) = v(1) = 0\,. \tag{BC}$$

Here the parameters a and b are non–negative, and the nonlinearities g_1, g_2 : $(0,1) \times [0,+\infty)^4 \longrightarrow [0,+\infty)$ are continuous and non–decreasing in the last four variables, satisfying the following hypotheses:

(H_1) Given $a, b \geq 0$, we have that, for all $M > 0$ and $i = 1, 2$, there exists $h_i \in \mathcal{C}((0,1),(0,+\infty))$ such that

$$\int_0^1 t(1-t)h_i(t)dt < +\infty \tag{1.1}$$

and

$$0 \leq g_i(t,u,v,a,b) \leq h_i(t), \quad \text{for all} \quad (t,u,v) \in (0,1) \times [0,M]^2.$$

(H_2) There exist a function $h \in \mathcal{C}((0,1),(0,+\infty))$ and an $M^* > 0$ such that

$$\int_0^1 t(1-t)h(t)dt < \frac{1}{M^*}$$

and

$$\lim_{|(u,v,a,b)| \to 0^+} \frac{g_1(t,u,v,a,b) + g_2(t,u,v,a,b)}{u+v+a+b} < h(t)M^*, \quad \text{for each} \;\; t \in (0,1).$$

(H_3) There exist $\alpha, \beta, \gamma, \delta, \eta, \xi \in (0,1)$, with $\alpha < \beta$, $\gamma < \delta$ and $\eta < \xi$, such that

$$\lim_{u\to+\infty}\frac{g_1(t,u,v,0,0)+g_2(t,u,v,0,0)}{u} = +\infty$$

uniformly for $v \geq 0$ and $t \in [\alpha,\beta]$,

$$\lim_{v\to+\infty}\frac{g_1(t,u,v,0,0)+g_2(t,u,v,0,0)}{v} = +\infty$$

uniformly for $u \geq 0$ and $t \in [\gamma,\delta]$, and

$$\lim_{|(a,b)|\to+\infty}(g_1(t,0,0,a,b)+g_2(t,0,0,a,b)) = +\infty$$

uniformly for $t \in [\eta,\xi]$.

1.1 Main Result

Theorem 1.1 *Suppose* g_1, $g_2 : (0,1)\times[0,+\infty)^4 \longrightarrow [0,+\infty)$ *are continuous, non–decreasing in the last four variables and satisfy assumptions* (H_1),(H_2) *and* (H_3). *Then there exist a constant* $\bar{a} > 0$ *and a non–increasing function* $\Gamma : [0,\bar{a}] \to [0,+\infty)$ *so that, for all* $a \in [0,\bar{a}]$, *the System* (S_{ab}) *has:*

(i) *at least one positive solution for* $0 \leq b \leq \Gamma(a)$*;*

(ii) *no positive solutions for* $b > \Gamma(a)$*;*

(iii) *at least two positive solutions for* $0 < b < \Gamma(a)$.

The approach taken to prove our main result is based on some well known fixed point theorem of expansion/compression type, the upper–lower solutions method, and some topological degree arguments.

Remark 1.2 *The results in this paper were in part motivated by several recent papers on elliptic problems. Here we extend and complement some of those previous works, see Section 1.2 and compare with the results of [5], [6], [3], [4],* *[7], [8], [10], [12], [11], [14], [15] and [16] and references therein. We consider a more general class of superlinear nonlinearities. As a typical example of functions satisfying our assumptions above we have*

$$\begin{aligned} g_1(t,u,v,a,b) &= c_1(t)(e^u - 1 + a)^q (v + c_2(t)) \\ g_2(t,u,v,a,b) &= d_1(t)(e^{v^5} - 1 + b)^p (u + d_2(t)) \end{aligned}$$

where $p, q > 0$ and c_1, d_1 satisfy the integrability condition (1.1) and may have disjoint support, and c_2, d_2 are positive continuous function defined on $[0,1]$. These nonlinearities satisfy assumptions (H_1) through (H_3), but do not satisfy the particular classical superlinear assumptions

$$\lim_{u+v\to+\infty} \frac{g_1(t,u,v,a,b)}{u+v} = +\infty \quad \textit{or} \quad \lim_{u+v\to+\infty} \frac{g_2(t,u,v,a,b)}{u+v} = +\infty.$$

1.2 Applications

Next we give some examples of elliptic problems to which we can apply Theorem 1.1 to prove existence, multiplicity and nonexistence of positive solutions.

I. Elliptic Systems in Annular Domains.

$$\left\{ \begin{array}{rcll} -\Delta u &=& f(|x|,u,v) & \\ -\Delta v &=& g(|x|,u,v), & \text{for} \quad r_1 < |x| < r_2 \;\text{ and }\; x \in \mathbb{R}^N (N \geq 3), \\ (u(x),v(x)) &=& (a,b), & \text{for} \quad |x| = r_1, \\ (u(x),v(x)) &=& (0,0), & \text{for} \quad |x| = r_2. \end{array} \right. \tag{1.2}$$

By applying the change of variable $t = a(r)$, with

$$a(r) = -\frac{A}{r^{N-2}} + B$$

where

$$A = \frac{(r_1 r_2)^{N-2}}{r_2^{N-2} - r_1^{N-2}} \quad \text{and} \quad B = \frac{{r_2}^{N-2}}{r_2^{N-2} - r_1^{N-2}}$$

it easy to see that (1.2) is equivalent to a system like (S_{ab}) with boundary conditions (BC).

II. Elliptic Systems in Exterior Domains.

$$\begin{cases} -\Delta u &= f(|x|, u, v) \\ -\Delta v &= g(|x|, u, v), \quad \text{for} \quad |x| > 1 \text{ and } x \in \mathbb{R}^N (N \geq 3), \\ (u(x), v(x)) &= (a, b), \quad \text{for} \quad |x| = 1, \\ (u(x), v(x)) &\to (0, 0) \quad \text{as} \quad |x| \to +\infty. \end{cases} \tag{1.3}$$

We denote the radial solutions by

$$u, v : [1, +\infty) \to \mathbb{R},\ u(x) = u(|x|),\ \ v(x) = v(|x|).$$

By applying the changes of variables

$$z(t) = u\left((1-t)^{1/(2-N)}\right),\ \ w(t) = v\left((1-t)^{1/(2-N)}\right),$$

we may transform (1.3) into a system like (S_{ab}) with boundary conditions (BC).

III. Fourth–order Elastic Beam Equation.

$$\begin{cases} -u^{(4)} &= f(t, u, u'') \quad \text{in} \quad (0, 1), \\ u(0) &= a,\ u(1) = b, \\ u''(0) &= 0,\ u''(1) = 0. \end{cases} \tag{1.4}$$

In this case, it suffices to take $g_2(t, u, v, a, b) = f(t, (1-t)a + tb + u, -v),\ v = -u''$, and $g_1(t, u, v, a, b) = v$.

1.3 Comments

Second-order elliptic problems in symmetric euclidian domains have been received considerable attention in the recent years. We start from the study of simpler case. In [3], it was used fixed point theorems of cone expansion/ compression type, the upper–lower solutions method and degree arguments, in order to study the existence, non–existence and multiplicity of positive solutions for a class of second–order ordinary differential equations with multi–parameters. They applied their results to study a class of semilinear elliptic equations in bounded annular domains with non–homogeneous Dirichlet boundary conditions of the form

$$\begin{aligned} -\Delta u &= \lambda f(|x|, u) && \text{in} \quad r_1 < |x| < r_2\,, \\ u(x) &= a && \text{on} \quad |x| = r_1\,, \\ u(x) &= b && \text{on} \quad |x| = r_2\,, \end{aligned}$$

where a, b and λ are non–negative parameters. One feature of the hypotheses on the nonlinearities that they consider is that they have some sort of local character.

Several papers on existence and multiplicity of positive radial solutions of elliptic systems in annular bounded domains involving nonlinearities as in Problem (1.2) and imposing Dirichlet or Newmann boundary conditions have recently appeared. For homogeneous boundary conditions, see [7], [8] and references therein. For non–homogeneous boundary conditions, see [4], [10], [12] and the references therein. In [4], the same authors study a Dirichlet problem for a system of two ordinary differential equations which depends on two numerical parameters and with nonlinearity satisfying very general superlinear local growth conditions. Using the method of lower–upper solutions, a fixed point theorem of cone expansion/compression type and some degree–theoretic arguments, the authors prove the existence of one and then two positive solutions depending on the values of the two parameters. The nonexistence of solutions with respect to these parameters is also considered. These results are then applied to establish the existence and multiplicity of positive radial solutions for a certain class of semilinear elliptic systems in annular domains.

In [5], we study existence and multiplicity of positive solutions of the non–

homogeneous elliptic equation

$$\begin{cases} -\Delta u = q(|x|)f(u)\,, & \text{for} \quad |x| > 1 \ \text{ and } \ x \in \mathbb{R}^N, \\ u(x) = a\,, & \text{for} \quad |x| = 1\,, \\ u(x) \to b & \text{as} \quad |x| \to +\infty \end{cases}$$

where $N \geq 3$, the nonlinearity f is superlinear at zero and infinity, q is a non–trivial, non–negative function, and a and b are non–negative parameters. A typical model is given by $f(u) = u^p$, with $p > 1$. For related results about exterior domains, see also [14], [15] and [16] and references therein.

1.4 Organization of the Paper

This paper is organized as follows. In Section 2, we state three well known theorems which are crucial for proving our main result, Theorem 1.1. Section 3 is devoted to proving the existence of one positive solution when the parameters a and b are sufficiently small. In Section 4, we introduce a theorem of lower and upper solutions method for singular systems. Section 5 establishes a non–existence result, as well as an a priori estimate result used in Section 6 to prove the existence of two positive solutions.

2 Auxiliary Results

We next state the following three well known theorems (see e.g. [1], [2], [9], [13]).

Theorem A. Let X be a Banach space endowed with norm $\|\cdot\|$, and let $C \subset X$ be a cone in X. For $R > 0$, define $C_R = C \cap B[0,R]$, where $B[0,R]$ denotes the closed ball of radius R centered at the origin of X. Assume that $\mathcal{F} : C_R \to C$ is a completely continuous operator and that there exists $0 < r < R$ such that

(A1) $\|\mathcal{F}u\| < \|u\|$, for all $u \in \partial C_r$ and $\mathcal{F}u\| > \|u\|$, for all $u \in \partial C_R$ or

(A2) $\|\mathcal{F}u\| > \|u\|$, for all $u \in \partial C_r$ and $\|\mathcal{F}u\| < \|u\|$, for all $u \in \partial C_R$,

where $\partial C_R = \{u \in C : \|u\| = R\}$. Then $\mathcal{F}$ has a fixed point $u \in C$, with $r < \|u\| < R$.

Theorem B. Let X be a Banach space endowed with norm $\|\cdot\|$, and let C be a cone in X. For $r > 0$, define $C_r = \{u \in C : \|u\| < r\}$. Assume that $\mathcal{F} : \overline{C_r} \to C$ is a compact map such that $\mathcal{F}u \neq u$, for $u \in \partial C_r = \{u \in C : \|u\| = r\}$.

(B1) If $\|u\| \leq \|\mathcal{F}u\|$, for all $u \in \partial C_r$, then $i(\mathcal{F}, C_r, C) = 0$.

(B2) If $\|u\| \geq \|\mathcal{F}u\|$, for all $u \in \partial C_r$, then $i(\mathcal{F}, C_r, C) = 1$.

Theorem C. Let X be a Banach space; let C be a cone in X; and let Ω be a bounded open set in X. Let $0 \in \Omega$, and let $\mathcal{F} : C \cap \overline{\Omega} \to C$ be a compact operator. Suppose that $\mathcal{F}u \neq \lambda u$, for all $u \in C \cap \partial\Omega$ and all $\lambda \geq 1$. Then

$$i\,(\mathcal{F}, C \cap \Omega, C) = 1$$

where $i\,(\mathcal{F}, C \cap \Omega, C)$ denotes the *fixed point index* over Ω with respect to C for the compact operator $\mathcal{F}$.

Define the operator $\mathcal{F} : X \to X$ by

$$\mathcal{F}(\phi, \psi)(t) = (\mathcal{A}(\phi, \psi)(t), \mathcal{B}(\phi, \psi)(t)), \quad \text{for } t \in [0, 1]$$

where

$$\begin{aligned}
\mathcal{A}(\phi, \psi)(t) &= \int_0^1 G(t, \tau)\, g_1(\tau, \phi(\tau), \psi(\tau), a, b)\, d\tau \quad \text{and} \\
\mathcal{B}(\phi, \psi)(t) &= \int_0^1 G(t, \tau)\, g_2(\tau, \phi(\tau), \psi(\tau), a, b)\, d\tau.
\end{aligned}$$

Here $G(t,s)$ denotes the associated Green's function that is given by

$$G(t,s) = \begin{cases} s(1-t) & \text{for} \quad 0 \le s \le t \le 1\,, \\ t(1-s) & \text{for} \quad 0 \le t \le s \le 1\,. \end{cases} \tag{2.5}$$

Therefore, the System (S_{ab}) with boundary conditions (BC) is equivalent to the fixed point equation

$$\mathcal{F}(\phi,\psi) = (\phi,\psi)$$

in the usual Banach space $X = \mathcal{C}([0,1];\mathbb{R}) \times \mathcal{C}([0,1];\mathbb{R})$ endowed with the norm $||(u,v)|| = ||u||_\infty + ||v||_\infty$, where $||u||_\infty = \max_{t\in[0,1]} |u(t)|$. In this paper, we use topological methods in order to prove our main results. More precisely, we use a combination of fixed point techniques, degree theory, fixed point index theory, and the lower and upper solution method. For this, we consider the cone C in X defined by

$$C = \{(u,v) \in X \ : \ (u,v)(0) = (u,v)(1) = 0, \ \textit{and} \ \ u,v \ \textit{are} \ \textit{concave}\}.$$

Lemma 2.1 *Fixing $a, b \ge 0$, we have that $\mathcal{F} : X \to X$ is well defined, $\mathcal{F}(C) \subset C$, and $\mathcal{F}$ is completely continuous.*

We next state a technical result about some elementary properties of concave functions.

Lemma 2.2 *Suppose $u(t)$ is a non–negative, concave, continuous function defined on $[0,1]$. Then, for all $0 < t_0 < t_1 < 1$, we have*

$$\min_{t\in[t_0,t_1]} u(t) \ge t_0(1-t_1)||u||_\infty.$$

3 Existence of a Positive Solution for Small Parameters

In this section, we show the existence of a fixed point of $\mathcal{F}$ for a and b sufficiently small.

Lemma 3.1 *Suppose conditions* (H_1) *and* (H_2) *hold. Then there exist* $R_0 > 0$ *and* δ_0 *so that, for all* $(\phi, \psi) \in C_{R_0}$ *and all* (a, b) *with* $0 < a + b < \delta_0$ *, we have*

$$||\mathcal{F}(\phi, \psi)|| < ||(\phi, \psi)|| \,.$$

Proof. It follows from condition (H_2) that we may choose $\sigma \in (0, 1)$ so that

$$M^* \int_0^1 \tau(1-\tau)h(\tau)\, d\tau < 1 - \sigma \,.$$

Also, there exists $R > 0$ so that, for all $0 \le r + s + a + b \le R$ and $t \in (0, 1)$, we have

$$g_1(t, r, s, a, b) + g_2(t, r, s, a, b) \le M^* h(t)(r + s + a + b) \,.$$

Thus, for $(\phi, \psi) \in C_{(1-\sigma)R}$, $a + b \in (0, \sigma R)$, and $t, t' \in (0, 1)$, we have

$$\begin{aligned} \mathcal{A}(\phi, \psi)(t) + \mathcal{B}(\phi, \psi)(t') &\le M^* \ R \int_0^1 \tau(1-\tau)h(\tau)\, d\tau \\ &< (1 - \sigma)R \,. \end{aligned}$$

Taking $R_0 = (1 - \sigma)R$ and $\delta_0 = \sigma R$, for all $(\phi, \psi) \in C_{R_0}$ and for $0 < a + b < \delta_0$, we have

$$||\mathcal{F}(\phi, \psi)|| = ||\mathcal{A}(\phi, \psi)||_\infty + ||\mathcal{B}(\phi, \psi)||_\infty < R_0 = ||(\phi, \psi)|| \,.$$

□

Lemma 3.2 *Assume that conditions* (H_1) *and* (H_3) *hold. Then there exists* $R_1 > R_0$ *so that, for all* $(\phi, \psi) \in C_{R_1}$ *, we have*

$$||\mathcal{F}(\phi, \psi)|| > ||(\phi, \psi)|| \,.$$

Proof. For otherwise, there would exist an increasing sequence $r_n \to +\infty$ and a sequence $\{(\phi_n, \psi_n)\}$ in C so that the real sequence $\{r_n\}$ defined by $\|(\phi_n, \psi_n)\| = r_n$ would satisfy

$$\|\mathcal{F}(\phi_n, \psi_n)\| \leq \|(\phi_n, \psi_n)\| \, .$$

We consider two cases:

Case 1: $\|\phi_n\|_\infty / r_n \to 0$ as $n \to +\infty$. Consequently, $\|\psi_n\|_\infty / r_n \to 1$ as $n \to +\infty$. Combining the monotonicity of the nonlinearities, the concavity of ϕ_n and ψ_n, and Lemma 2.2 we would have

$$\begin{aligned}
\|\mathcal{F}(\phi_n, \psi_n)\| &\geq \int_\gamma^\delta G(1/2, \tau)[\, g_1(\tau, \gamma(1-\delta)\|\phi_n\|_\infty \, , \gamma(1-\delta)\|\psi_n\|_\infty, 0, 0) \\
&\quad + g_2(\tau, \gamma(1-\delta)\|\phi_n\|_\infty \, , \gamma(1-\delta)\|\psi_n\|_\infty, 0, 0)] \, d\tau \\
&= \int_\gamma^\delta G(1/2, \tau) \frac{J_n(\tau)}{\gamma(1-\delta)\|\psi_n\|_\infty} \cdot \gamma(1-\delta)\|\psi_n\|_\infty \; d\tau \\
&= \gamma(1-\delta) \, M_n \, \frac{\|\psi_n\|_\infty}{r_n} \; r_n
\end{aligned}$$

where

$$\begin{aligned}
J_n(\tau) &= g_1(\tau, \gamma(1-\delta)\|\phi_n\|_\infty \, , \gamma(1-\delta)\|\psi_n\|_\infty \, , 0 \, , 0) \\
&\quad + g_2(\tau, \gamma(1-\delta)\|\phi_n\|_\infty \, , \gamma(1-\delta)\|\psi_n\|_\infty, 0, 0)
\end{aligned}$$

and

$$M_n = \int_\gamma^\delta G(1/2 \, , \tau) \frac{J_n(\tau)}{\gamma(1-\delta)\|\psi_n\|_\infty} \; d\tau \longrightarrow +\infty$$

by assumption (H_3). Therefore, we would have

$$1 \geq \gamma(1-\delta) M_n \frac{\|\psi_n\|_\infty}{r_n} \; ,$$

which is impossible.

Case 2: $\|\phi_n\|_\infty / r_n \to a > 0$ **as** $n \to +\infty$. Similarly, in this case we would have

$$\begin{aligned} \|\mathcal{F}(\phi_n, \psi_n)\| &\geq \int_\alpha^\beta G(1/2,\tau) \frac{S_n(\tau)}{\alpha(1-\beta)\|\phi_n\|_\infty} \cdot \alpha(1-\beta)\|\phi_n\|_\infty \, d\tau \\ &= \alpha(1-\beta)\, N_n \, \frac{\|\phi_n\|_\infty}{r_n} \, r_n \end{aligned}$$

where

$$\begin{aligned} S_n(\tau) &= g_1(\tau, \alpha(1-\beta)\|\phi_n\|_\infty\,, \alpha(1-\beta)\|\psi_n\|_\infty\,, 0\,, 0) \\ &\quad + g_2(\tau, \alpha(1-\beta)\|\phi_n\|_\infty\,, \alpha(1-\beta)\|\psi_n\|_\infty, 0, 0) \end{aligned}$$

and

$$N_n = \int_\alpha^\beta G(1/2,\tau)\, \frac{S_n(\tau)}{\alpha(1-\beta)\|\phi_n\|_\infty}\, d\tau \longrightarrow +\infty$$

by assumption (H_3). Therefore, we would have

$$1 \geq \alpha(1-\beta)\, N_n \, \frac{\|\phi_n\|_\infty}{r_n},$$

which is impossible.

$\square$

Taking into account Lemmas 3.1 and 3.2, the following is a direct consequence of Theorem A.

Theorem 3.3 *There exists* $\delta_0 > 0$ *so that, for all* a *and* b *satisfying* $0 < a + b < \delta_0$, *the operator* $\mathcal{F}$ *has a fixed point* $(\phi, \psi) \in C$ *verifying* $R_0 < ||(\phi, \psi)|| < R_1$.

Combining the maximum principle with hypothesis (H_1) we obtain that ϕ and ψ are positive functions.

4 Upper and Lower Solutions

In this section, we establish the classical upper–lower solutions method for obtaining non–negative solutions of our class of singular systems. For this, consider the system

$$\begin{array}{rcl} u''(t) + f(t,u(t),v(t)) & = & 0, \\ v''(t) + g(t,u(t),v(t)) & = & 0\,, \quad \text{for } \; t \in (0,1)\,, \\ (u(0),v(0)) = (0,0), \;\; (u(1),v(1)) & = & (0,0) \end{array} \tag{P}$$

where both of the functions f and g satisfy assumption (H_1).

As usual, we will say that (u,v) is an *upper solution of* System (P) if (u,v) verifies the following inequalities:

$$\begin{array}{rcl} u''(t) + f(t,u(t),v(t)) & \leq & 0, \\ v''(t) + g(t,u(t),v(t)) & \leq & 0, \quad \text{for } \; t \in (0,1)\,, \\ (u(0),v(0)) & \geq & (0,0), \;\; (u(1),v(1)) \;\; \geq \;\; (0,0)\,. \end{array} \tag{4.6}$$

Similarly, we define a *lower solution* of System (P) replacing "greater than or equal to" with "less than or equal to".

The upper–lower solutions method is established in the next lemma. (See also [11].)

Lemma 4.1 *Let* $(\underline{u},\underline{v})$ *(resp.* $(\bar{u},\bar{v})$ *) be a lower (resp. an upper) solution of System* (P)*. Moreover, we suppose*

$$(0,0) \leq (\underline{u},\underline{v}) \leq (\bar{u},\bar{v})\,.$$

Then System (P) *has a non–negative solution* (u,v) *verifying*

$$(\underline{u},\underline{v}) \leq (u,v) \leq (\bar{u},\bar{v})\,.$$

Proof. Let

$$\begin{aligned} M(u,v)(t) &= \int_0^1 G(t,\tau) f(\tau, u(\tau), v(\tau))\, d\tau, \\ N(u,v)(t) &= \int_0^1 G(t,\tau) g(\tau, u(\tau), v(\tau))\, d\tau \quad \text{and} \\ T(u,v)(t) &= (M(u,v)(t), N(u,v)(t)) \,. \end{aligned}$$

Thus System (P) is equivalent to the fixed point equation

$$T(u,v) = (u,v)$$

in the Banach space $X = \mathcal{C}([0,1];\mathbb{R}) \times \mathcal{C}([0,1];\mathbb{R})$ endowed with the norm $||(u,v)|| = ||u||_\infty + ||v||_\infty$.

We need to introduce the auxiliary operator $\hat{T}$ defined by

$$\hat{T}(u,v)(t) = (\hat{M}(u,v)(t)\,, \hat{N}(u,v)(t))$$

where

$$\begin{aligned} \hat{M}(u,v)(t) &= \int_0^1 G(t,\tau) f(\tau, \xi(t,u)\,,\, \zeta(\tau,v))\, d\tau \qquad \text{and} \\ \hat{N}(u,v)(t) &= \int_0^1 G(t,\tau)\, g(\tau, \xi(t,u)\,,\, \zeta(\tau,v))\, d\tau, \end{aligned}$$

and

$$\begin{aligned} \xi(t,u) &= \max\{\underline{u}(t), \min\{u, \overline{u}(t)\}\} \qquad \text{and} \\ \zeta(t,v) &= \max\{\underline{v}(t), \min\{v, \overline{v}(t)\}\} \,. \end{aligned}$$

It is not difficult to see that the operator $\hat{T}$ has the following three properties:

(a) The operator $\hat{T}$ is bounded and completely continuous.

(b) If the pair $(u,v) \in X$ is a fixed point of $\hat{T}$, then (u,v) is a fixed point of T, with $(\underline{u}, \underline{v}) \le (u,v) \le (\overline{u}, \overline{v})$.

(c) If $(u,v) = \lambda \hat{T}(u,v)$, with $0 \le \lambda \le 1$, then $||(u,v)|| \le K_3$, where K_3 does not depend on either λ or $(u,v) \in X$.

The proof now follows from the topological degree of Leray–Schauder. (See [1, Corollary 8.1, p. 61].)

□

The following is an application of the preceding result.

Proposition 4.2 *Assume that System* $(S_{a_2 b_2})$ *has a non–negative solution and that*

$$(0,0) \leq (a_1, b_1) \leq (a_2, b_2)\,.$$

Then System $(S_{a_1 b_1})$ *has a non–negative solution.*

Proof. Let the pair (u_2, v_2) be a non–negative solution of System $(S_{a_2 b_2})$. Since g_1 and g_2 are non–decreasing functions in the last two variables, we have that (u_2, v_2) is an upper solution and that $(0,0)$ is a lower solution of System $(S_{a_1 b_1})$. The conclusion results from Lemma 4.1.

□

5 A priori Bounds and Non–existence

This section is devoted to establishing a priori estimates for the positive solutions of System (S_{ab}).

Lemma 5.1 *Suppose that conditions* (H_0), (H_1) *and* (H_3) *hold. Then there exists a positive constant* $K > 0$ *such that, for every positive solution* (u, v) *of System* (S_{ab})*, we have*

$$||(u,v)|| \leq K$$

where K *may be chosen independent of* a *and* b*.*

The proof is analogous to that of Lemma 3.2.

Remark 5.2 *Assume that (ϕ,ψ) is a positive solution of System (S_{ab}), using the same argument as in the proof of Lemma 3.2 we find*

$$\begin{aligned}\|\mathcal{F}(\phi,\psi)\| &= \|\mathcal{A}(\phi,\psi)\|_\infty + \|\mathcal{B}(\phi,\psi)\|_\infty \\ &\geq \int_\eta^\xi G(1/2,\tau)[\,g_1(\tau,0,0,a,b)+g_2(\tau,0,0,a,b)]\,d\tau.\end{aligned}$$

From Lemma 5.1 and hypotheses (H_3) we conclude that there exists a $\rho>0$ such that, for all $(a,b)\in(0,+\infty)\times(0,+\infty)$ with $|(a,b)|>\rho$, the System (S_{ab}) has no positive solutions.

We next define the set

$$A=\{a>0:\ \text{System } (S_{ab}) \text{ has a positive solution for some } b>0\}\,.$$

From Theorem 3.3 and Remark 5.2 we conclude that A is non–empty and bounded. Thus

$$0<\bar{a}=\sup A<+\infty\,.$$

Using the upper–lower solutions method, we see that for all $a\in(0,\bar{a})$, there exists $b>0$ such that System (S_{ab}) has a positive solution. We now define the function $\Gamma:[0,\bar{a}]\to[0,+\infty)$ by

$$\Gamma(a)=\sup\{b>0: \text{System } (S_{ab}) \text{ has a positive solution}\}\,.$$

By Proposition 4.2 we know that the function Γ is non–increasing. Moreover, $\Gamma\,(0)>0$ as is easily verified. We claim that $\Gamma(a)$ is attained. In fact, it suffices to use Lemma 5.1 and the compactness of the operator $\mathcal{F}$. Finally, it follows from the definition of the function Γ that the System (S_{ab}) has at least one positive solution for $0\le b\le\Gamma(a)$, and furthermore that it has no positive solutions for $b>\Gamma(a)$, which proves parts (i) and (ii) of Theorem 1.1, respectively.

6 Existence of Two Positive Solutions

In this section, we establish existence of two positive solutions of System (S_{ab}), which corresponds to proving part *(iii)* of Theorem 1.1. For this, we will assume that the nonlinearities g_1 and g_2 are increasing.

Fix $a \in [0, \bar{a}]$, and let $(\overline{\phi}, \overline{\psi})$ be the solution of Problem $(S_{a\Gamma(a)})$ which is obtained using Proposition 4.2. Our next result allows us to establish another solution of System (S_{ab}) for $0 < b < \Gamma(a)$.

Lemma 6.1 *For each* $0 < b < \Gamma(a)$, *there exists* $\varepsilon_0 > 0$ *so that, for all* $0 < \varepsilon \leq \varepsilon_0$ *and all* $t \in [0,1]$, *we have*

$$\overline{\phi}_\varepsilon(t) > \int_0^1 G(t,s) g_1(s, \overline{\phi}_\varepsilon(s), \overline{\psi}_\varepsilon(s), a, b)\, ds$$

and

$$\overline{\psi}_\varepsilon(t) > \int_0^1 G(t,s) g_2(s, \overline{\phi}_\varepsilon(s), \overline{\psi}_\varepsilon(s), a, b)\, ds$$

where $\overline{\phi}_\varepsilon(t) = \overline{\phi}(t) + \varepsilon$ *and* $\overline{\psi}_\varepsilon(t) = \overline{\psi}(t) + \varepsilon$.

Proof. Fix $\delta \in (0, 1/2)$. Since g_1 is increasing, we have that, for each $0 < b < \Gamma(a)$, we may find a positive constant $I = I(b)$ such that, for all $s \in [\delta, 1-\delta]$, we have

$$g_1(s, \overline{\phi}(s), \overline{\psi}(s), a, \Gamma(a)) - g_1(s, \overline{\phi}(s), \overline{\psi}(s), a, b) \geq I > 0\,.$$

By the uniform continuity of g_1, there exists $\varepsilon_0 > 0$ so that, for all $s \in [\delta, 1-\delta]$ and all $0 < \varepsilon \leq \varepsilon_0$, we have

$$\left| g_1(s, \overline{\phi}(s) + \varepsilon, \overline{\psi}(s) + \varepsilon, a, b) - g_1(s, \overline{\phi}(s), \overline{\psi}(s), a, b) \right| < \frac{I}{2}\,.$$

Next we define

$$\zeta_\varepsilon(t,s) = G(t,s)[g_1(s,\overline{\phi}_\varepsilon(s),\overline{\psi}_\varepsilon(s),a,b) - g_1(s,\overline{\phi}(s),\overline{\psi}(s),a,b)]$$

and

$$\eta(t,s) = G(t,s)[g_1(s,\overline{\phi}(s),\overline{\psi}(s),a,\Gamma(a)) - g_1(s,\overline{\phi}(s),\overline{\psi}(s),a,b)]\,.$$

Assume $0 < \varepsilon \leq \varepsilon_0$. Then

$$\begin{aligned}\overline{\phi}_\varepsilon\,(t) &> \int_0^1 G(t,s)\,g_1(s,\overline{\phi}(s),\overline{\psi}(s),a,\Gamma(a))\,ds\\ &= \int_0^1 G(t,s)\,g_1(s,\overline{\phi}_\varepsilon(s),\overline{\psi}_\varepsilon(s),a,b)\,ds\\ &- \int_0^1 \zeta_\varepsilon(t,s)\,ds + \int_0^1 \eta_(t,s)\,ds\,.\end{aligned}$$

Since $\eta(t,s)$ is positive and $\eta(t,s) - \zeta_\varepsilon(t,s) > \frac{I}{2}\,G(t,s)$, for $s \in [\,\delta, 1-\delta]$, we have

$$\begin{aligned}\overline{\phi}_\varepsilon\,(t) &> \int_0^1 G(t,s)\,g_1(s,\overline{\phi}_\varepsilon(s),\overline{\psi}_\varepsilon(s),a,b)\,ds - \int_0^\delta \zeta_\varepsilon(t,s)\,ds\\ &- \int_{1-\delta}^1 \zeta_\varepsilon(t,s)\,ds + \frac{I}{2}\int_\delta^{1-\delta} G(t,s)\,ds.\end{aligned}$$

It is not difficult to show that Lebesgue's Dominated Convergence Theorem implies that $\int_0^\delta \zeta_\varepsilon(t,s)\,ds + \int_{1-\delta}^1 \zeta_\varepsilon(t,s)\,ds$ converges to zero, uniformly in t, as ε tends to zero. Thus, for ε sufficiently small, we have

$$\overline{\phi}_\varepsilon\,(t) > \int_0^1 G(t,s)\,g_1(s,\overline{\phi}_\varepsilon(s),\overline{\psi}_\varepsilon(s),a,b)\,ds$$

uniformly in $t \in [0,1]$.
A similar computation holds for $\overline{\psi}_\varepsilon$.

□

We are now in a position to prove part (iii) of Theorem 1.1, or in other words show the existence of two positive solutions of System (S_{ab}) for $0 < b < \Gamma(a)$, where $a \in [0, \bar{a}]$ is fixed.

Proof of part (iii) of Theorem 1.1.

Consider the set

$$\Omega = \left\{(\phi,\psi) \in X : \ -\varepsilon < \phi\,(t) < \overline{\phi}_\varepsilon\,(t)\,, \ \ -\varepsilon < \psi\,(t) < \overline{\psi}_\varepsilon\,(t)\ ,\ \text{for} \ \ t \in [0,1]\right\}$$

where $\overline{\phi}_\varepsilon$ and $\overline{\psi}_\varepsilon$ are the functions of Lemma 6.1. It is not hard to see that Ω is bounded and open in X, and that $0 \in \Omega$. Note that one solution of System (S_{ab}) belongs to $\overline{C \cap \Omega}$. Also, we known that $\mathcal{F} : C \cap \overline{\Omega} \to C$ is a compact operator.

Let $(\phi,\psi) \in C \cap \partial\Omega$. It follows that there exists a $t_0 \in (0,1)$ such that one of the following two cases hold: $\phi\,(t_0) = \overline{\phi}_\varepsilon\,(t_0)$ or $\psi\,(t_0) = \overline{\psi}_\varepsilon\,(t_0)$. In the case $\phi\,(t_0) = \overline{\phi}_\varepsilon\,(t_0)$, it follows from Lemma 6.1 that, for all $\lambda \geq 1$, we have

$$\begin{aligned}
\mathcal{A}(\phi,\psi)(t_0) &= \int_0^1 G(t_0,s)\,g_1(s,\phi(s),\psi(s),a,b)\,ds \\
&\leq \int_0^1 G(t_0,s)\,g_1(s,\overline{\phi}_\varepsilon(s),\overline{\psi}_\varepsilon(s),a,b)\,ds \\
&< \overline{\phi}_\varepsilon\,(t_0) = \phi\,(t_0) \leq \lambda\,\phi\,(t_0)\ .
\end{aligned}$$

Similarly, $\mathcal{B}(\phi,\psi)(t_0) < \lambda\psi(t_0)$ in the case $\psi\,(t_0) = \overline{\psi}_\varepsilon\,(t_0)$. Hence $\mathcal{F}(\phi,\psi) \neq \lambda(\phi,\psi)$, for all $(\phi,\psi) \in C \cap \partial\Omega$ and all $\lambda \geq 1$. Now according to Theorem C, we have

$$i\,(\mathcal{F}, C \cap \Omega, C) = 1.$$

On the other hand, a slight change in the proof of Lemma 5.1 shows the existence of an $r > 0$ sufficiently large, say $r > R_1$, where R_1 is as in Theorem 3.3, so that

$$\|\mathcal{F}(\phi,\psi)\| > \|(\phi,\psi)\|$$

for every $\|(\phi,\psi)\| = r$ and every $(\phi,\psi) \in C$.

Let $R = \max\{K+1, r, \|(\overline{\phi}_\varepsilon, \overline{\psi}_\varepsilon)\|\}$, where K is as in Lemma 5.1. Set

$$C_R = \{(\phi,\psi) \in C : \|(\phi,\psi)\| < R\}.$$

Then Lemma 5.1 implies that $\mathcal{F}(\phi,\psi) \neq (\phi,\psi)$, for $(\phi,\psi) \in \partial C_R$. Consequently, part *(B1)* of Theorem B implies $i\,(\mathcal{F}, C_R, C) = 0$.

Now by the additivity property of the fixed point index we obtain

$$i\,(\mathcal{F}, C\cap\Omega, C) + i\left(\mathcal{F}, C_R\backslash\overline{C\cap\Omega}, C\right) = i\,(\mathcal{F}, C_R, C) = 0.$$

Since $i\,(\mathcal{F}, C\cap\Omega, C) = 1$, we conclude $i\left(\mathcal{F}, C_R\backslash\overline{C\cap\Omega}, C\right) = -1$. Therefore, $\mathcal{F}$ has another fixed point in $C_R\backslash\overline{C\cap\Omega}$, which was to be shown.

Remark 6.2 *Detailed proofs, more examples and comments can be found in [6].*

Open Problem 1 *Does the conclusion of Theorem 1.1 hold for problems involving $p-$Laplacian operator and without the non-decreasing assumption on the nonlinear terms?.*

Open Problem 2 *Do (1.2), (1.3) and (1.4) have infinitely many solutions for $a,\ b > 0$ small enough?*

Open Problem 3 *It would be interesting to study multiplicity of positive solutions of elliptic systems in more general domains or more general boundary conditions.*

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