Matemática Contemporânea, Vol 33, 01-22

ON THE LIFE AND WORK OF FRANCESCO MERCURI

M. H. Noronha **I. I. Onnis**

1 Introduction

Noronha, the first student supervised by Prof. Francesco Mercuri, and the last, Onnis, have written this article. Mercuri's other doctoral students include Yuriko Baldin, Fausto Souza, Christiam Figueroa, Helvecio de Castro, Jose Adonai Seixas, Guillermo Lobos, Ryuichi Fukuoka, Newton Santos, Martha Dussan, J. C. Almeida de Lima, and Leonardo Biliotti. Besides honoring our advisor by presenting this brief biography and some of his contributions to Mathematics in the area of Differential Geometry, we want to thank him - on behalf of all of his students - for his guidance, for everything he has taught us, and for being an example of a professor who cares deeply for his students.

The first section of this article follows Mercuri's graduate student days in Chicago and as a faculty member at UNICAMP, Brazil. The mathematical section describes some of the work he has done with collaborators and students. These are not presented chronologically and are not intended to cover all his work. We only want to highlight his creativity and talent for generating examples and finding connections among different aspects of a mathematical topic, as well as his remarkable generosity in sharing his ideas and the breadth of his mathematical knowledge.

2 On the life of Francesco Mercuri

2.1 From Rome to Brazil, via Chicago

Francesco Mercuri was born in Rome on July 7, 1946, the son of Marcella Grossetti, a druggist born in Rome, and Vincenzo Mercuri, an engineer born in the southern Italian region of Calabria (“almost Africa” as Francesco likes to joke). The eldest of four brothers, he spent his childhood in the San Giovanni district with his family, until he completed elementary school.

In 1956, Mercuri moved to his paternal grandmother’s house to begin secondary school, and lived there until completing his studies at Augusto Righi, the scientific high school. This school, located in a central district of Rome, was founded in 1946, just after the end of World War II, and is still considered one of the best schools in the city. During his high school years, a special person, Mrs. Diana Benincasa, his mathematics teacher, contributed decisively to his future choice for the course of studies in mathematics. As he says, “ I was deeply influenced by her enthusiasm and the pleasure she showed talking mathematics”. In 1964 Mercuri returned to his parents’ home and began his studies in mathematics at the University of Rome, today known as Università La Sapienza. In July 1969, he concluded his university studies (magna cum laude) defending a bachelor’s thesis titled “Teoria di Morse e applicazioni,” under the supervision of Claudio Procesi, one of the most influential Italian mathematicians. (Procesi is now a member of the Abel prize committee and was recently elected vice president of the International Mathematical Union for 2007-2010.) At that time, Procesi had just returned to the University of Rome from the University of Chicago where he completed his Ph.D. under the supervision of Israel Nathan Herstein. About Procesi, Mercuri says, “He is a brilliant mathematician, he knows about everything. He brought fresh air to the institution, new ideas and he attracted many students. He supervised dissertations in quite different areas such as Algebra, Algebraic Geometry, and Differential and Algebraic Topology. This period left an important mark on the mathematics at the University of Rome."

Influenced by Procesi's experience, many of his students from the University of Rome decided to go abroad for a Ph.D. program, which Italy lacked at that time. Sponsored by Procesi and Herstein and with a fellowship from the University of Chicago, Mercuri went to Chicago to study Differential Topology with Prof. Richard K. Lashof. "Besides being two top mathematicians, Procesi and Lashof share two other important features: They are enchanting persons, sometimes in different ways, and they have a global approach to Mathematics. Their attitudes surely had a great influence on my mathematical development," Mercuri says.

Mercuri remembers the Chicago days with much affection. He tells his friends many anecdotes about his life there, but few can be reported here. One was that he often practiced the most popular sport in Chicago at the time: Changing residences. "Every six months I moved, and almost always my roommates were non-mathematicians." One of his hobbies, known and appreciated by many, is cooking. However, during the Ph.D. period, he found practicing this art very difficult since, according to him, none of his friends excelled in domestic activities and never wanted to wash the dishes.

"The atmosphere at the University of Chicago was extremely exciting. Also, besides Mathematics, I had a quite active social life," he says. For instance, in the doctoral years, Mercuri founded a soccer team called "I Meio," which means "the best ones" in the Roman dialect. People of varied nationality composed the team: Italian, Brazilian, English and Spanish. "I Meio won the university championship easily", he remembers. Today Mercuri no longer plays any sport, but very much enjoys watching them, especially Soccer and Formula One.

During his mathematical studies at the University of Chicago, Mercuri started to work on some problems in Riemannian Geometry, but in 1972 he had to interrupt his doctoral studies to return to Italy to fulfill military service in Orvieto, a city in the Umbria region. He likes to say that the Italian army could not do without him. When he finished 15 months in the army, in November 1973, the academic year at the University of Chicago had already started, so the fellowship was unavailable for that year. From November 1973 until October 1974, he developed research activities and taught Differential Geometry as a "Professore Incaricato" at the Lecce University.

In September 1974 he decided to give up his position in Italy and return to Chicago to finish his Ph.D. This decision was considered totally crazy by some friends, since in a short time he would have become an associate professor, a stable position, through a test where only his curriculum would have been evaluated and a few publications would have been enough to pass.

Before going back to Chicago, Mercuri attended the mathematical summer school called "Corso C.I.M.E. (Centro Internazionale Matematico Estivo)" held in June 1974 in Varenna, Italy. He met W. Klingenberg, a professor at the University of Bonn, who was a lecturer at that meeting. After some fruitful conversations, Klingenberg suggested that he should work on a problem about closed geodesics. In this connection, Mercuri explains that although Klingenberg was not his official advisor, he considered Mercuri (as well as Manfredo P. do Carmo) as his own student and his suggestions were extremely important in Mercuri's thesis.

The problem that interested Klingenberg and many other mathematicians concerned the existence of infinitely many closed geodesics on a compact Riemannian manifold (the problem is still open in its full generality). In 1972, Katok had produced an example of a Finsler metric on the sphere S^2, arbitrarily closed to the standard Riemannian metric, with only two closed geodesics. So it would be interesting to extend the results known for the Riemannian case, in particular the critical point theory, to the Finsler case trying to understand the differences. When Mercuri went back to the University of Chicago in September 1974, he talked to Lashof about the problem suggested by Klingenberg, commenting that it looked too easy. Lashof briefly answered, "Klingenberg is an excellent and experienced mathematician!" A year later, in 1975, Mercuri defended his Ph.D. thesis, titled *Closed geodesics on Finsler manifolds,* but officially the Ph.D. degree was not awarded until 1976.

2.2 The Brazilian period

Some Brazilian mathematicians whom he met in Chicago invited Mercuri to visit the State University of Campinas (UNICAMP) in Brazil. He decided to accept this invitation, because it would have been very difficult at that time to find a good position in Italy. He arrived in Campinas in January 1976. The director of IMECC, the Institute of Mathematics, Statistics and Computer Science, was then Professor Ubiratan D'Ambrosio, who had significant international experience and was trying to start a center of excellence. Through his international contacts he was able to attract many young Ph.D.s from various countries. Mercuri's initial plan was to stay in Brazil just two or three years, publish a few papers, and return to Italy when the job situation improved. But he had two important experiences: He saw the possibility to collaborate with the creation of a mathematics department with high quality research and he met Elizabeth ("Betty"), whom he wed in July 1977. "She is a great woman and gave me a lot of support, motivation, and two wonderful children, Elena and Vincenzo Leonardo," he says. So he decided to stay in Brazil.

Since he didn't yet speak Portuguese, Mercuri (affectionately called "Franco") initially taught some graduate classes in English, but soon he was also teaching basic undergraduate courses. Everybody remembers his lectures: a neat blackboard, clear speech with numerous examples and counterexamples, mention of recent results and open problems, but also a language which he calls "portuliano"-Portuguese with many Italian words. He says that once a dear friend of his, Richard Pfister, asked his wife Betty which language Francesco used to speak to his children. Betty answered, "Neither Italian nor Portuguese, but the language you are listening to!"

In the early 1980's Mercuri was the graduate program coordinator of the Mathematics Department. These were very exciting years, with a large group of relatively young professors trying to organize a high-quality program. There were many different ideas of directions to take, many discussions, and finally the program was structured. Nowadays this program is, together with that of IMPA, the best-ranked program in Brazil. Mercuri credits the arrival of Professor Djairo Guedes de Figueiredo, a world leading mathematician, as fundamental. The program attracts students from various countries. As a result, Mercuri has supervised students from Chile, Peru, Colombia and Italy, besides Brazilian students.

At the beginning of the 1990's there were many students in Differential Geometry at UNICAMP and at the University of São Paulo. Mercuri, together with some colleagues from USP, started a joint seminar in Differential Geometry, which is still active and is considered a basic meeting point for graduate students and researchers from throughout the state. The seminar frequently hosts lecturers from other institutions in Brazil and abroad.

For his academic activities Mercuri was awarded the *Zeferino Vaz* prize in 1997, the highest prize at UNICAMP, and later, in March 2005, the *Ordem Nacional do Merito Cientifico* prize, one of the highest at the national level.

Mercuri officially retired in 2003, but he still teaches, supervises students, and is active in research. The only difference is that he is no longer involved in administration and it is easier for him to travel. Besides his participation in international congresses, his friends and collaborators in Brazil and abroad often invite him for seminars, joint research work, and to lecture mini-courses. Everybody enjoys his lectures. They are clear, well organized, and the results he presents are always interesting, current, and relevant to Differential Geometry. He goes directly to the heart of the problem, in the simplest way, making people believe that all is easy!

He never returned to Chicago but goes to Italy every year. There, besides doing Mathematics, he visits his family and friends and enjoys beautiful places, good company, food and....a good glass of wine!

3 An Overview of Mercuri's Mathematical work

3.1 The closed geodesics problem

It is a classical question in Riemannian Geometry if there are always infinitely many closed geodesics (geometrically distinct and non trivial) on any compact Riemannian manifold. A positive answer was given, under additional condition on the topology of the manifold, by Gromoll and Mayer in [14]. They proved:

Theorem 3.1 *Let M be a compact simply connected Riemannian manifold. If the rational cohomology of M is not generated, as a ring, by only one element, then M admits infinitely many non constant geometrically distinct closed geodesics.*

Remark *Also, a positive answer, under a "generic" condition on the metric, was given in [17].*

In [18] Katok produced an example of a Finsler metric on S^2, arbitrarily close to the standard Riemannian metric, with only two closed geodesics. In order to understand the difference between the Riemannian and the Finsler case, Mercuri studied in his doctoral thesis at the University of Chicago the critical point theory for the closed geodesics problem for Finsler metrics. His thesis was partially published in [19]. Unfortunately, as he says, he was able to prove the Palais-Smale condition for the Finsler energy, one of the main points in the theory. The "unfortunately" is referred to the fact that the result does not show any difference between the two cases. When he arrived in Brasil he stopped working on the problem, for reasons that will be clear later on, but the problem remained in his mind. After some time he proposed to a student of his, F. Souza, to prove the Gromoll-Mayer Theorem for the Finsler case. Souza did it in his Doctoral thesis with the help of a Morse Theory with low differentiability (a peculiarity of the Finsler case) that Mercuri established in [25]. It is worth mentioning that the difference between the Riemannian and Finsler case is still not completely clear, despite the fact that quite a few papers have been published on the subject recently.

In his last paper (see [6]), in collaboration with his former student L. Biliotti and his colleague P. Piccione, Mercuri went back to the problem of existence of closed geodesics, this time for semi-Riemannian metrics. Here the situation is quite different because the energy functional is not bounded below, does not satisfy the Palais-Smale condition, and the Morse index is infinite. However, with the help of previous work on the Morse Index Theorem and via the Maslow index (see [26]) and under additional hypotheses on the metric, they were able to prove a Gromoll-Mayer type theorem.

3.2 Positively curved submanifolds of Euclidean spaces

When Mercuri arrived in Brazil, one of the most popular topics of research was the geometry of submanifolds of space forms. He started soon to be interested in the subject, combining his knowledge of Morse Theory, his favorite subject in mathematics, with a long standing interest: The topology of positively curved manifolds.

He was strongly influenced by a beautiful paper by J. D. Moore, (see [19]). Moore proved that an orientable compact submanifold of an Euclidean space with positive sectional curvature and codimention two is an homotopy sphere. This theorem generalizes a previous result of A. Weinstein, [34]), and is based on the seminal work of Chern and Lashof on minimal absolute total curvature of submanifolds of Euclidean spaces, ([9]), and on a Theorem of Gallot and Meyer on compact and simply connected manifolds of nonnegative curvature operator ([13]). Mercuri, in collaboration with his student Y. Baldin, generalized Moore's result for the case of nonnegative sectional curvature, proving in [3] the following Theorem:

Theorem 3.2 *Let M be a compact connected Riemannian manifold of dimen-*

sion $n \geq 3$, *with nonnegative sectional curvature and* $f : M \to \mathbf{R}^{n+2}$ *an isometric immersion. Then:*

1. *If the sectional curvature at some point is positive,* M *is an homotopy sphere.*
2. *If* M *is simply connected then either* M *is an homotopy sphere or* M *is the product of two spheres and* f *is the product of two convex embeddings.*
3. *If* M *is not simply connected, then* M *is diffeomorphic to the product of a compact manifold with a circle or* $n = 3$ *and* M *is a finite quotient of* S^3.

One of the main points was the equivalence between positive sectional curvature and positive curvature operator. Recall that the curvature operator of a Riemannian manifold with curvature tensor R is the operator:

$$\mathcal{R} : \Lambda^2(T_pM) \to \Lambda^2(T_pM), \qquad \langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)W, Z \rangle,$$

where $\Lambda^2(T_pM)$ is the space of exterior 2-forms. It is clear that positive curvature operator implies positive sectional curvature, but the converse is not always true. But for codimension two submanifolds of Euclidean spaces, the converse holds (see [34]).

Mercuri then started looking for conditions under which the positivity of the curvature operator is equivalent to the positivity of the sectional curvature. The simplest case is the following:

Definition 3.3 *A Riemannian manifold is said to have* pure *curvature operator if for every* $p \in M$ *there is an orthonormal basis* $\{X_1, ..., X_n\}$ *of the tangent space such that the 2-forms* $X_i \wedge X_j$ *are eigenvectors of the curvature operator* $\mathcal{R}$.

Although very restrictive, the condition of pure curvature operator holds for various classes of Riemannian manifolds such as all 3-manifolds, locally conformally flat manifolds, and for submanifolds of space forms with flat normal connection. For the case of manifolds with pure nonnegative curvature operator, Mercuri, Derdzinski, and a former student Noronha proved that if M is compact and simply connected, then it is the Riemannian product of Real Cohomology Spheres (see [10]). It is worth pointing out here, that this simple result is a consequence of major theorems proved in the 1980's such as the Micallef-Moore Theorem for manifolds of positive isotropic curvature ([29]), Hamilton's Ricci flow for nonnegative operators ([15]), and the solution of the Frankel conjecture by Siu-Yau ([33]).

Mercuri wrote an excellent survey paper (see [21]) where he beautifully describes how the results in [29], [15], [33] can be put together to give a complete classification of manifolds of nonnegative curvature operator.

Also in the 1980's a new and important notion of curvature was introduced. In [29], Micallef and Moore introduced the concept of *curvature on totally isotropic two-planes for manifolds of dimension* ≥ 4. We will call it, for brevity, *isotropic curvature*. This curvature plays a role in the study of stability of harmonic 2-spheres similar to the one that the sectional curvature does in the study of stability of geodesics. In fact, studying the index of harmonic 2-spheres in a compact Riemannian manifold M, Micallef and Moore proved that if M has positive isotropic curvature then the homotopy groups $\pi_i(M)$ vanish for $2 \leq i \leq [n/2]$. Therefore, if $\pi_1(M)$ is finite then the Betti numbers $\beta_i(M)$ are zero for $1 \leq i \leq n-1$.

For nonnegative isotropic curvature, the situation is more complex. A topological classification for compact and simply connected manifolds with nonnegative isotropic curvature exits only in dimension 4: It is either homeomorphic to a sphere, or biholomorphic to the complex projective space $\mathbf{CP}^2$ or is a product of two surfaces where one of them is homeomorphic to a sphere. For higher dimensions there are only partial results. Mercuri and Noronha worked in the case of hypersurfaces, proving the following result (see [22]):

Theorem 3.4 *Let $f : M^n \to \mathbf{R}^{n+1}$, $n \geq 4$, be an isometric immersion of a compact manifold M with nonnegative isotropic curvature. Then the homology groups $H_i(M, \mathbf{Z}) = 0$ for $2 \leq i \leq n-2$ and the fundamental group $\pi_1(M)$ is a free group on β_1 elements. Moreover, for for any natural number β there exist a compact hypersurface with nonnegative isotropic curvature, with $\beta_1 = \beta$.*

This result shows that compact hypersurfaces with nonnegative isotropic curvature look very much like conformally flat hypersurfaces (see next subsection). In fact, the examples constructed to show that any β appears as the first Betti number of a compact hypersurface with nonnegative isotropic curvature are conformally flat. Actually, Mercuri and Noronha proved in the same paper the following result for compact conformally flat manifolds:

Theorem 3.5 *Let M^n, $n \geq 4$, be a compact conformally flat manifold with nonnegative isotropic curvature. Then either M is flat or $\beta_i(M) = 0$ for $3 \leq i \leq n-3$. Moreover if $\beta_2(M) \neq 0$ then either M is flat or is isometrically covered by the product of a hyperbolic plane with an $(n-2)$-sphere with its standard metric.*

The result above combined with Moore's result on conformally flat submanifolds of Euclidean spaces (see [30]) gives us the following:

Corollary 3.6 *Let $f : M^n \to \mathbf{R}^{n+p}$, $2 \leq p \leq n/2-1$, be an isometric immersion of a compact, orientable conformally flat manifold M with nonnegative isotropic curvature. Then $H_i(M;\mathbf{Z}) = 0$ for $p \leq i \leq n-p$.*

3.3 Conformally flat and cohomogeneity one hypersurfaces of the Euclidean space

In 1983, Mercuri spent the (Brazilian) summer at IMPA, Rio de Janeiro. At that time, do Carmo and Dajczer where working on conformally flat hypersurfaces of Euclidean spaces. There was an incomplete classification of those hypersurfaces, essentially due to Kulkarni, and, studying the local problem, do Carmo and Dajczer found new examples. Mercuri then decided to apply Morse Theory to those results in order to obtain global descriptions of those hypersurfaces. In collaboration with do Carmo and Dacjzer, the following result was obtained:

Theorem 3.7 *Let M^n, $n \geq 4$ be a compact, conformally flat manifold and $f : M^n \to \mathbf{R}^{n+1}$ an isometric immersion. Then M is the union of umbilical pieces, connected by tubes, diffeomorphic to $[0,1] \times S^{n-1}$, foliated by $(n-1)$-dimensional round spheres.*

Standard examples of conformally flat hypersurfaces are the hypersurfaces of revolution. Do Carmo and Dajczer caracterized those hypersurfaces in terms of the structure of the second fundamental form (see [7]). Later on, Podestà and Spiro characterized them in terms of the intrinsic geometry (see [32]). Recall that a Riemannian manifold is said to be of cohomogeneity k with respect to a closed subgroup G of its isometry group, if G acts effectively and isometrically with principal orbits of codimension k. The result of Podestà and Spiro states that a compact cohomogeneity one hypersurface M of dimension $n \geq 4$ with the property that the principal orbits are umbilical in M is a hypersurface of revolution.

The crux of their proof is that the umbilicity of the orbits implies the constancy of their sectional curvature. In a joint work with Asperti and Noronha, the converse was proved, that is, if the principal orbits have constant sectional curvature then they are umbilical submanifolds of M and therefore, if M is compact and of dimension at least 4, M is a hypersurface of revolution.

With another former student, J. A. Seixas, Mercuri studied in [28], the case of complete hypersurfaces. It turns out that Podesta-Spiro's theorem does not hold in this case. This can be easily seen by considering suitable non totally geodesic isometric immersions of $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$. However, with the assumptions that the group G is compact and the connected components of the flat part are bounded, Mercuri and Seixas were able to get the same conclusion, namely, that if the principal orbits are umbilical submanifolds then M is a hypersurface of revolution.

As remarked above, hypersurfaces of revolution are conformally flat and of cohomogeneity one. It was then natural to ask if those two conditions imply that the hypersurface is of revolution. The answer is negative as shown by the following example: Consider the group $G = SO(2) \times SO(n-1)$ acting in the standard way on $\mathbf{R}^{n+1} = \mathbf{R}^2 \times \mathbf{R}^{n-1}$. In the $\{e_1, e_{n+1}\}$ plane take a circle, centered in $(\lambda, 0), \lambda > 0$, and of radius $r < \lambda$. Letting G act on this circle we obtain a compact conformally flat cohomogeneity one hypersurface, which is not of revolution. The question is completely answered in the result below proved by Mercuri and Noronha in [23].

Theorem 3.8 *Let $f : M^n \to \mathbf{R}^{n+1}$ be an isometric immersion of a compact,* *cohomogeneity one conformally flat Riemannian manifold, $n \geq 4$. Then M^n is of revolution or, up to normalizations, is the example discussed above.*

Besides hypersurfaces of revolution, there are other hypersurfaces of cohomogeneity one. For example we can consider a group $G \subseteq SO(n+1)$ acting on $\mathbf{R}^{n+1}$ with cohomogeneity two. Those groups are classified and the quotient space is an angular sector in a 2-plane in $\mathbf{R}^{n+1}$. Taking a curve in this sector, with suitable conditions on the boundary, and letting G act on this curve, we obtain an hypersurface of cohomogeneity one. Those examples were called the *standard examples* (in particular hypersurfaces of revolution are standard examples for $G = SO(n)$). In a recent joint paper with Podestá, Seixas, and Tojeiro, (see [27]) cohomogeneity one hypersurfaces of the Euclidean space were completely classified. They proved:

Theorem 3.9 *Let $f : M^n \to \mathbf{R}^{n+1}$ be an isometric immersion of a complete cohomogeneity one Riemannian G-manifold, G compact. If $n \geq 5$ and the connected components of the flat part are bounded or, if $n \geq 3$ and M is compact, then either f is rigid, or $f(M)$ is a hypersurface of revolution. In either case, $f(M)$ is one of the standard examples described above.*

3.4 Minimal surfaces

The study of minimal surfaces is a very popular theme of research in Brazil. A few years ago, in one of his visit to Fortaleza, Mercuri started working with J. L. M. Barbosa in a classical problem of minimal surfaces. Ossermann had shown in the 1960's that the Gauss map of a complete non flat minimal surface of finite total curvature in $\mathbf{R}^3$ omits at most 3 points on the unit sphere and it was soon conjectured that it could not omit more than 2 points. Ossermann's proof relies on the Weierstrass representation formula for minimal surfaces in $\mathbf{R}^3$ (see below) and on the calculations of the Euler characteristic of a Riemann surface in terms of zeros and poles of holomorphic functions and their differentials. Mercuri found a new proof of Ossermann's Theorem which uses only topological properties of complete minimal surfaces of finite total curvature. This proof was not considered sufficiently interesting, at that time, and was left on the side. A few years later Mercuri suggested to one of his student, R. Fukuoca, to look again at this problem and at his proof. Mercuri and Fukuoca soon realized that the ideas behind that proof could be extended to the case of higher dimensional minimal hypersurfaces. Immediately after, in a joint work with Barbosa and Fukuoca, the so called *hypersurfaces of finite geometric type* were introduced. They constitute a class of hypersurfaces of Euclidean spaces that, from the Differential Topology viewpoint, have the same properties of complete minimal surfaces of finite total curvature. In [5], the authors proved the analogue of Ossermann's theorem for surfaces of finite geometric type as well as other results for higher dimensional minimal hypersurfaces of finite geometric type, which gave a topological characterization of even dimensional catenoids.

At the same time there was a growing interest in the study of surfaces of 3-dimensional homogeneous manifolds. Mercuri, in a joint work with his student C. Figueroa and his colleague R. Pedrosa, classified the surfaces of constant mean curvature in the Heisenberg group that are invariant for 1-parameter subgroups (see [12]). This, together with the work of U. Abresch and H. Rosenberg on the generalized Hopf differentials (see [1]), led to the solution of the isoperimetric problem for the Heisenberg group. They also found families of complete minimal surfaces that are graphs on the orthogonal complement of the center. This naturally led to the question of classifying complete minimal graphs in the style of the Bernstein Theorem [1]. Such a classification has recently been obtained by I. Fernandez and P. Miura in [11].

Mercuri continued his work on minimal surfaces in 3-dimensional homogeneous manifolds. In collaboration with S. Montaldo and M. P. Piu, he obtained a Weierstrass type formula for minimal surfaces in Riemannian manifolds (see [20]).

The (local version of) the Weierstrass representation formula for minimal surfaces in $\mathbf{R}^3$ states that, if $\Omega \subset C$ is a simply connected domain and $f : \Omega \to \mathbf{R}^n$ is a conformal minimal immersion, then the complex tangent vector $\partial f/\partial z := \sum \phi_i \partial/\partial x_i$ has the following properties:

$$\sum |\phi_i| > 0, \text{(i.e. } f \text{ is an immersion)}, \ \sum \phi_i^2 = 0, \text{(i.e. } f \text{ is conformal)},$$

$$\partial\phi_i/\partial\overline{z} = 0 \ \text{ (i.e. } f \text{ is minimal)}.$$

Conversely, given functions $\phi_i : \Omega \to \mathbf{C}$ with the above properties, then the map:

$$f : \Omega \to \mathbf{R}^3, \qquad f_i(z) = 2Real \int_{z_0}^{z} \phi_i dz,$$

is a conformal minimal immersion.

In the case that the ambient space is a general Riemannian manifold with metric g_{ij}, the same structure works, if the above conditions on the ϕ_i's are replaced by the following:

$$\sum g_{ij}\phi_i\overline{\phi_j} > 0, \qquad \sum g_{ij}\phi_i\phi_j = 0, \qquad \partial\phi_i/\partial\overline{z} + \Gamma^i_{kl}\phi_k\overline{\phi}_l = 0,$$

were the Γ^i_{kl}'s are the Christoffel symbols of the Riemannian connection of the manifold.

[1]The classical Bernstein Theorem states that, in $\mathbf{R}^3$, a complete minimal graph is linear.

In addition to bringing the theory of holomorphic functions to the subject, the Weierstrass representation is a powerful machine to produce examples of minimal surfaces, provided that one finds solutions for the PDE involved. Such solutions are not easy to find if the ambient space is a general Riemannian manifold. However, in the case of special ambient spaces such as the Heisenberg group $\mathbf{H}_3$ and $\mathbf{H}^2 \times \mathbf{R}$, where $\mathbf{H}^2$ is the hyperbolic plane, Mercuri, Montaldo, and Piu (see [20] were able to find explicit solutions and hence, by integrating these solutions, they found examples of minimal surfaces in those spaces.

Also, in [24], Mercuri and Onnis used the Weierstrass representation described above to give a positive answer to the Björling problem in 3-dimensional Lie groups:

Theorem 3.10 *Let M be a 3-dimensional Lie group with a left invariant metric, $\gamma : [0,1] \to M$ a real analytic curve and $N : [0,1] \to TM$, an analytic vector field along γ and normal to it. Then there exist $\epsilon > 0$ and a conformal minimal immersion $\Gamma : [0,1] \times (-\epsilon, \epsilon) \to M$ such that $\Gamma(t,0) = \gamma(t)$ and N is the normal to Γ along γ.*

References

[1] Abresch, U.; Rosenberg, H., *Generalized Hops differentials*, Mat. Contemp. 28 (2005), 1-28.

[2] Asperti, A. C.; Mercuri, F.; Noronha, M. H., *Cohomogeneity one manifolds and hypersurfaces of revolution*, Bolletino dellaUnione Matematica Italiana (7) 11-B (1997), 199 -215.

[3] Baldin, Y.; Mercuri, F., *Isometric immersions in codimension two with nonnegative curvature*, Math. Z. 173, (1980), 111-117.

[4] Baldin, Y.; Mercuri, F., *Codimension two nonorientable submanifolds with nonnegative curvature*, Proc. AMS 103, (1988), 918-920.

[5] Barbosa, J. L.; Fukuoka, R.; Mercuri, F., *Immersions of finite geometric type in Euclidean spaces*, Ann. Global Anal. Geom., 22 (2002), 301-315.

[6] Biliotti, L.; Mercuri, F.; Piccione, P., *On a Gromoll-Meyer type theorem in globally hyperbolic stationary spacetimes*, to appear in Comm. in Anal. Geom.

[7] do Carmo, M.; Dajczer, M., *Rotation hypersurfaces in spaces of constant curvature,* Trans. A.M.S in Mathematics, 107, 1985.

[8] do Carmo, M.; Dajczer, M.; Mercuri, F., *Compact conformally flat hypersurfaces,* Trans.A.M.S. 288, (1985), 189-203.

[9] Chern, S. S.; Lashof, R. K., *On the total curvature of immersed manifolds*, Amer. J. Math., 79, (1957), 306-318.

[10] Derdzinski, A.; Mercuri, F.; Noronha, M. H., *Manifolds with pure nonnegative curvature operator*, Bol. Soc. Bras. Matem. 18, (1987), 13-22.

[11] Fernandez, I.; Mira, P., *Holomorphic quadratic differentials and the Bernstein problem in the Heisenberg group.* preprint (2007).

[12] Figueroa, C.; Mercuri, F.; Pedrosa, R., *Invariant surfaces in the Heisenberg group.* Ann. Mat. Pura appl. 177,(1999), 173-194.

[13] Gallot, S.; Meyer, D., *Opérateur de courbure et Laplacien des formes différentelles d'une variété Riemanniene*, J. Math. Pures et Appl. 54, (1975), 285-304.

[14] Gromoll, D.; Mayer, W., *Periodic geodesics on compact Riemannian manifolds*, J. Diff. Geo. 3 (1969), 493-510.

[15] Hamilton, R., *Four-manifolds with positive curvature operator*, Proc. AMS 70 (1978), 72-76.

[16] Katok, A. B., *Ergotic perturbations of degenerate integrable Hamiltonian systems* Math. USSR, Izvestija 7, (1973), 535-571.

[17] Klingenberg, W.; Takens, F., *Generic properties of geodesic flows*, Math. Ann. 197 (1972), 323–334.

[18] Kobayashi, S., *Compact homogeneous hypersurfaces,* Trans.A.M.S. 88, (1958), 137-143.

[19] Mercuri, F., *The critical points theory for the closed geodesics problem*, Math. Z., 156, (1977), 231-245.

[20] Mercuri, F.; Montaldo, S.; Piu, P., *A Weierstass representation formula for minimal surfaces in $\boldsymbol{H}_3$ and $\boldsymbol{H}^2 \times \boldsymbol{R}$*, Acta Math. Sin. 22 (2006), 1603-1612.

[21] Mercuri, F.; Noronha, M. H., *On the topology of complete Riemannian manifolds with nonnegative curvature operator*, Rendiconti Seminario Facoltà di Scienze, Università di Cagliari (Italy), Vol. 63, 2 (1993), 1- 23.

[22] Mercuri, F.; Noronha, M. H., *Low codimensional submanifolds of Euclidean spaces with non negative isotropic curvature*, Trans. Amer. Math. Soc. 348 (19996), 2711-2714.

[23] Mercuri, F.; Noronha, M. H., *Conformall flatness, cohomogenety one and hypersurfaces of revolution*, Differential Geom. Appl. 9 (1998), 143-149.

[24] Mercuri, F.; Onnis, I., *On the Björling problem in a tridimensional Liegroup*, preprint.

[25] Mercuri, F.; Palmieri, G., *Morse theory with low differentiability* Boll. U.M.I., 7 1-B, (1987), 621-631.

[26] Mercuri, F.; Piccione, P.; Tausk, D., *Stability ofthe conjugate index, degenerate conjugate points and the Maslow index in semi Riemannian geometry*, Pacific J. Math. 26 (2002), 375-400.

[27] Mercuri, F.; Podestá, F.; Seixas, J. A. P.; Tojeiro, R., *Cohomogeneity one hypersurfaces of Euclidean spaces*, Comment. Math. Helv. 81(2006), 471-479.

[28] Mercuri, F.; Seixas, J. A. P., *Hypersurfaces of cohomogeneity one and hypersurfaces of revolution*, Differential Geom. Appl. 20 (2004), 255-239.

[29] Micallef, M.; Moore, J. D., *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. of Math 127, (1988), 199-227.

[30] Moore, J. D., *Conformally flat submanifolds of Euclidean spaces*, Math. Ann. 225 (1977), 89-97.

[31] Moore, J. D., *Codimension two submanifolds of positive curvature*, Proceedings of the AMS, 70 (1978), 72-74.

[32] Podestá, F.; Spiro, A., *Cohomogeneity one hypersurfaces of the Euclidean space,* Ann. Global Anal. and Geom. 13, (1995), 169-184.

[33] Siu, Y. T.; Yau, S. T., *Compact Kahler manifolds with positive bisectional curvature*, Invent. Math 59,(1980), 189-204.

[34] Weinstein, A., *Positively curved n-manifolds in* $\boldsymbol{R}^{n+2}$, J. Diff. Geom. 4 (1970), 1-4.

Department of Mathematics
California State University Northridge
Northridge, CA, 91330-8313, USA
E-mail: maria.noronha@csun.edu

Departamento de Matemática
ICMC - USP
Caixa Postal 668
13560-970, São Carlos, SP, Brasil,
E-mail: onnis@icmc.usp.br