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SECOND VARIATION OF ENERGY FOR MINIMAL SURFACES IN RIEMANNIAN MANIFOLDS

J. D. Moore

Abstract

This article describes a formula for second variation of energy for two-dimensional parametrized minimal surfaces, in which the conformal structure on the two-dimensional domain is allowed to vary. Moreover, it shows that minimal surfaces with branch points have tangential Jacobi fields which are not visible via the standard formulae for second variation of area which is commonly used to study stability of minimal surfaces.

1 Introduction

The formula for second variation of area is one of the principal tools used to study stability of minimal submanifolds of Riemannian manifolds. However, the second variation formula which is in standard use (see Simons [11], Lawson [4], Micallef [6], Micallef and Wolfson [8], Fraser [3], as well as others) restricts attention to normal variation. This version of the second variation formula has the advantage that it can be extended to more general objects such as integral currents and varifolds. When the domain has two dimensions, however, stronger results are sometimes achievable using a parametrized theory in which closed minimal surfaces are regarded as maps from a compact surface Σ into a Riemannian manifold M, the maps being allowed to have branch points. In this context, one needs a formula for second variation for the two-variable energy function

$$E : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \longrightarrow \mathbb{R}, \tag{1}$$

where Σ is a closed oriented surface of genus g and $\mathcal{T}$ is the corresponding Teichmüller space.

In this article, we present such a second variation formula for the two-variable energy E and put it in a form suitable for application of complex analysis (see end of §4). When E is restricted to immersions and variations are normal, the second variation of E differs from the second variation of area only in "divergence terms," and we explain these divergence terms in §5. The second variation formula shows that if a parametrized minimal surface has branch points, these give rise to additional tangential Jacobi fields for E that are not visible when taking only normal variations. The number of such tangential Jacobi fields depend only on the total branching order, and can be calculated via the Riemann-Roch Theorem, as we describe in §7. This fact is key to the author's Bumpy Metric Theorem [9].

In §8, we present the key ideas behind the proof of this Bumpy Metric Theorem, in the hope that it will make this result more accessible.

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2 Preliminaries

Closely related to minimal surfaces are harmonic maps. If we fix a conformal structure ω on a closed oriented surface Σ, we can define the ω-energy

$$E_\omega : \mathrm{Map}(\Sigma, M) \to \mathbb{R} \quad \text{by} \quad E_\omega(f) = \frac{1}{2}\int_\Sigma |df|^2 dA. \tag{2}$$

Here the length of df and the area element dA are calculated with respect to the given Riemannian metric on M and any metric on Σ within the conformal equivalence class defined by ω. A critical point for E_ω is called an ω-harmonic map. In terms of a complex parameter z, an ω-harmonic map can be regarded as a solution to the equation,

$$\frac{D}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}\right) = 0; \quad \text{thus} \quad \frac{\partial f}{\partial z} \tag{3}$$

can be regarded as a holomorphic section of $\mathbf{E} = f^*TM \otimes \mathbb{C}$, when $\mathbf{E}$ is given the holomorphic structure defined by the Levi-Civita connection D; see Donaldson and Kronheimer [2], Proposition 2.1.56.

If (f, ω) is a crtical point for the two-variable energy (1), the map f is not only ω-harmonic, but also ω-conformal, that is

$$\left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \right\rangle = 0.$$

The only possible singularities of nonconstant conformal harmonic maps are branch points; each such branch point p lies in an open neighborhood $U \subset \Sigma$ such that

$$\frac{\partial f}{\partial z}\bigg| U = z^\nu g, \qquad g \in \Gamma(\mathbf{E}|U), \qquad g(p) \neq 0,$$

where z is a complex coordinate such that $z(p) = 0$ and ν is a positive integer called the branching order at p. The locally defined sections $\partial f/\partial z$ and g (near branch points) generate a line bundle $\mathbf{L} \subset \mathbf{E}$. The tangential component of $\mathbf{E}$ is

$$(f^*TM \otimes \mathbb{C})^\top = \mathbf{L} \oplus \bar{\mathbf{L}}.$$

It follows from (3) that $\mathbf{L}$ is a holomorphic subbundle of $\mathbf{E}$. When f has branch points, $\mathbf{L}$ is not the holomorphic tangent bundle to Σ and in fact $c_1(\mathbf{L})([\Sigma])$ is $2 - 2g$ plus the total branching order.

A standard calculation shows that the second variation for ω-energy is given by

$$d^2 E_\omega(f)(X, X) = \int_\Sigma \left[\|DX\|^2 - \langle \mathcal{K}(X), X\rangle\right] dA, \tag{4}$$

where in terms of the complex parameter $z = x_1 + ix_2$ on Σ,

$$\|DX\|^2 = \frac{1}{\lambda^2}\left[\left\|\frac{DX}{\partial x_1}\right\|^2 + \left\|\frac{DX}{\partial x_2}\right\|^2\right]$$

and

$$\langle \mathcal{K}(X), X\rangle = \frac{1}{\lambda^2}\left[\left\langle R\left(X, \frac{\partial f}{\partial x_1}\right)\frac{\partial f}{\partial x_1}, X\right\rangle + \left\langle R\left(X, \frac{\partial f}{\partial x_2}\right)\frac{\partial f}{\partial x_2}, X\right\rangle\right],$$

R being the Riemann-Christoffel curvature tensor of M. This second variation formula can be put in complex form as described in [7]:

$$d^2E_\omega(f) : \Gamma(\mathbf{E}) \times \Gamma(\mathbf{E}) \longrightarrow \mathbb{C}$$

is given by the formula

$$d^2E_\omega(f)(Z,\overline{Z}) = 4\int_\Sigma \left[\left|\frac{DZ}{\partial \bar{z}}\right|^2 - \left\langle R\left(Z, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \overline{Z}\right\rangle\right] dxdy, \tag{5}$$

when $z = x + iy$ is a complex parameter on Σ and R is the Riemann Christoffel curvature tensor of M. We can of course replace z by $\bar{z}$ in this formula and obtain

$$d^2E_\omega(f)(Z,\overline{Z}) = 4\int_\Sigma \left[\left|\frac{DZ}{\partial z}\right|^2 - \left\langle R\left(Z, \frac{\partial f}{\partial \bar{z}}\right)\frac{\partial f}{\partial z}, \overline{Z}\right\rangle\right] dxdy. \tag{6}$$

Note that in both cases the integrands are independent of the choice of complex parameter on Σ as well as the action of the diffeomorphism group on Σ.

3 Second variation of energy

In considering second variation of the two-variable energy (1), it is helpful to be explicit regarding dependence on the metric on Σ.

Let $\mathrm{Met}(\Sigma)$ denote the space of smooth Riemannian metrics on the compact oriented surface Σ, $\mathrm{Met}(M)$ the space of Riemannian metrics on an n-dimensional ambient manifold M. We can write a typical element $\eta \in \mathrm{Met}(\Sigma)$ in terms of fixed coordinates (x_1, x_2) on Σ as

$$\eta = \sum_{a,b=1}^{2} \eta_{ab}dx_a dx_b, \quad \text{and let} \quad (\eta^{ab}) = (\eta_{ab})^{-1}.$$

We can then define the energy function

$$E : \mathrm{Map}(\Sigma, M) \times \mathrm{Met}(\Sigma) \times \mathrm{Met}(M) \longrightarrow \mathbb{R}$$

by means of the coordinate formula

$$E(f, \eta, g) = \frac{1}{2} \int_{\Sigma} \sum_{a,b} \eta^{ab} \sqrt{\det(\eta_{ab})} \left\langle \frac{\partial f}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle dx_1 dx_2, \tag{7}$$

where

$$\left\langle \frac{\partial f}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle = g\left(\frac{\partial f}{\partial x_a}, \frac{\partial f}{\partial x_b} \right),$$

the integrand being independent of the choice of local coordinates.

We can fix the metric $g \in \mathrm{Met}(M)$, and consider the function

$$E_g : \mathrm{Map}(\Sigma, M) \times \mathrm{Met}(\Sigma) \longrightarrow \mathbb{R}, \qquad E_g(f, \eta) = E(f, \eta, g). \tag{8}$$

One can show that a critical point for the energy (8) is a pair (f, η) where f is harmonic, as well as conformal with respect to the metric η on Σ and the fixed metric $\langle \cdot, \cdot \rangle$ on M. Moreover, one verifies directly that the energy E does not change when the metric η is multiplied by a conformal factor, that is E is invariant under Weyl rescalings.

A direct calculation shows that the second derivative of energy at a critical point for the energy is a sum of three terms. If X is a section of f^*TM and $\dot{\eta}$ is a trace-free variation of the metric on Σ,

$$\begin{aligned} d^2 E_g(f, \eta)((X, \dot{\eta}), (X, \dot{\eta})) &= d^2 E_\omega(f)(X, X) \\ &+ 2 \int_{\Sigma} \sum_{a,b} \frac{d}{dt} \left(\eta^{ab} \sqrt{\det(\eta_{ab})} \right) \Big|_{t=0} \left\langle \frac{DX}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle dx_1 dx_2 \\ &+ \frac{1}{2} \int_{\Sigma} \sum_{a,b} \frac{d^2}{dt^2} \left(\eta^{ab} \sqrt{\det(\eta_{ab})} \right) \Big|_{t=0} \left\langle \frac{\partial f}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle dx_1 dx_2. \end{aligned} \tag{9}$$

We can think of these three terms as second-order partial derivatives in the three directions:

$$\mathrm{Map}(\Sigma, M) \times \mathrm{Map}(\Sigma, M), \quad \mathrm{Map}(\Sigma, M) \times \mathrm{Met}(\Sigma), \quad \mathrm{Met}(\Sigma) \times \mathrm{Met}(\Sigma).$$

As mentioned in the previous section, the first term in (9) is

$$\int_\Sigma \left[\left\| \frac{DX}{\partial x_1} \right\|^2 + \left\| \frac{DX}{\partial x_2} \right\|^2 \right] dx_1 dx_2 - \int_\Sigma \langle \mathcal{K}(X), X \rangle dA.$$

To evaluate the other two terms, we take a metric variation of the form

$$\eta_{ab}(t) = \lambda^2 \delta_{ab} + t\dot{\eta}_{ab}, \tag{10}$$

where we can assume that $(\dot{\eta}_{ab})$ satisfies the trace-free condition $\dot{\eta}_{11} + \dot{\eta}_{22} = 0$; the trace term $\dot{\eta}_{11} + \dot{\eta}_{22}$ does not affect the energy, and if we were to include it in the calculation its contribution would simply cancel out of the formula in the end. Differentiation then yields

$$\frac{d}{dt}\eta(t)\bigg|_{t=0} = 0, \qquad \frac{d^2}{dt^2}\eta(t)\bigg|_{t=0} = 2\dot{\eta}_{11}\dot{\eta}_{22} - 2\dot{\eta}_{12}\dot{\eta}_{12}, \tag{11}$$

where we use the symbol η to denote $\det(\eta_{ab})$. Using the formulae

$$\frac{d}{dt}(\eta^{-1/2}) = -\frac{1}{2}\eta^{-3/2}\frac{d\eta}{dt}, \qquad \frac{d^2}{dt^2}(\eta^{-1/2}) = -\frac{1}{2}\eta^{-3/2}\frac{d^2\eta}{dt^2} + \frac{3}{4}\eta^{-5/2}\left(\frac{d\eta}{dt}\right)^2,$$

we can differentiate the components appearing in (7),

$$\begin{pmatrix} \sqrt{\eta}\eta^{11} & \sqrt{\eta}\eta^{12} \\ \sqrt{\eta}\eta^{21} & \sqrt{\eta}\eta^{22} \end{pmatrix} = \eta^{-1/2} \begin{pmatrix} \eta_{22} & -\eta_{12} \\ -\eta_{21} & \eta_{11} \end{pmatrix},$$

obtaining

$$\frac{d}{dt}\begin{pmatrix} \sqrt{\eta}\eta^{11} & \sqrt{\eta}\eta^{12} \\ \sqrt{\eta}\eta^{21} & \sqrt{\eta}\eta^{22} \end{pmatrix} = \frac{d}{dt}(\eta^{-1/2}) \begin{pmatrix} \eta_{22} & -\eta_{12} \\ -\eta_{21} & \eta_{11} \end{pmatrix} - \eta^{-1/2} \begin{pmatrix} \dot{\eta}_{11} & \dot{\eta}_{12} \\ \dot{\eta}_{21} & \dot{\eta}_{22} \end{pmatrix},$$

with a similar result for the second derivative. Evaluation at $t = 0$ and use of (11) yields the results

$$\frac{d}{dt}\left(\sqrt{\eta}\eta^{ab}\right)\bigg|_{t=0} = -\lambda^{-2}\dot{\eta}_{ab}, \tag{12}$$

$$\frac{d^2}{dt^2}\left(\sqrt{\eta}\eta^{ab}\right)\Big|_{t=0} = \lambda^{-4}(\dot{\eta}_{11}^2 + \dot{\eta}_{12}^2)\delta_{ab}. \tag{13}$$

It now follows from (12) that the second term in (9) is

$$-2\int_\Sigma \sum_{a,b} \frac{\dot{\eta}_{ab}}{\lambda^2}\left\langle \frac{DX}{\partial x_a}, \frac{\partial f}{\partial x_b}\right\rangle dx_1 dx_2, \tag{14}$$

while from (13) and the trace-free condition we conclude that the third term is

$$\int_\Sigma \frac{1}{\lambda^4}(\dot{\eta}_{11}^2 + \dot{\eta}_{12}^2)\sigma^2 dx_1 dx_2 = \frac{1}{2}\int_\Sigma \sum_{a,b}\left|\frac{\dot{\eta}_{ab}}{\lambda^2}\right|^2 \left\|\frac{\partial f}{\partial x_b}\right\|^2 dx_1 dx_2. \tag{15}$$

Adding the three terms together yields our second variation formula

$$\begin{aligned} d^2 E_g(f,\eta)((X,\dot{\eta}),(X,\dot{\eta})) &= d^2 E_{\omega,g}(f)(X,X) \\ &\quad - 2\int_\Sigma \sum_{a,b} \frac{\dot{\eta}_{ab}}{\lambda^2}\left\langle \frac{DX}{\partial x_a}, \frac{\partial f}{\partial x_b}\right\rangle dx_1 dx_2 \\ &\quad + \int_\Sigma \frac{1}{\lambda^4}(\dot{\eta}_{11}^2 + \dot{\eta}_{12}^2)\sigma^2 dx_1 dx_2, \end{aligned} \tag{16}$$

where

$$\sigma^2 = \left|\frac{\partial f}{\partial x_1}\right|^2 = \left|\frac{\partial f}{\partial x_2}\right|^2 = 2\left|\frac{\partial f}{\partial z}\right|^2.$$

The group $\mathrm{Diff}^+(\Sigma)$ of orientation-preserving diffeomorphisms of Σ acts on $\mathrm{Met}(\Sigma)$ by

$$(\eta,\phi) \in \mathrm{Met}(\Sigma) \times \mathrm{Diff}^+(\Sigma) \quad \mapsto \quad \phi^*\eta \in \mathrm{Met}(\Sigma).$$

This group also acts on $\mathrm{Map}(\Sigma, M)$ by $(f,\phi) \mapsto f\circ\phi$, and the energy E is invariant under the product action

$$((f,\eta),\phi) \in \mathrm{Map}(\Sigma,M)\times\mathrm{Met}(\Sigma)\times\mathrm{Diff}^+(\Sigma) \quad \mapsto \quad (f\circ\phi, \phi^*\eta).$$

As mentioned before, the energy is also invariant under Weyl rescalings of the metric, $(\eta_{ab}) \to (\lambda^2\eta_{ab})$, so the energy function (8) has an infinite-dimensional symmetry group and a corresponding infinite-dimensional family of Jacobi fields. To simplify the analysis, we let $\mathrm{Conf}(\Sigma)$ denote the space of equivalence classes of metrics under Weyl rescalings, and note that there is an isomorphism between $\mathrm{Conf}(\Sigma)$ and the space $\mathrm{Met}_0(\Sigma)$, the subspace of metrics on Σ which have constant curvature and total area one. We can thus eliminate the Weyl rescalings by restricting E to $\mathrm{Met}_0(\Sigma)$. To eliminate the symmetries coming from the diffeomorphism group, we can divide out by the normal subgroup $\mathrm{Diff}_0(\Sigma)$ of $\mathrm{Diff}^+(\Sigma)$ consisting of diffeomorphisms isotopic to the identity. The energy descends to a function defined on the quotient

$$\frac{\mathrm{Map}(\Sigma, M) \times \mathrm{Conf}(\Sigma)}{\mathrm{Diff}_0(\Sigma)} \cong \mathrm{Map}(\Sigma, M) \times \mathcal{T},$$

an isomorphism between the two sides being established by choice of a section of the action of $\mathrm{Conf}(\Sigma) \cong \mathrm{Diff}_0(\Sigma)$ on $\mathrm{Met}_0(\Sigma)$. (See Chapter 2 of [10] for construction of such a section.) The resulting map

$$E_g : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \longrightarrow \mathbb{R}$$

has a finite-dimensional group of symmetries and is well-adapted to study by techniques from global analysis.

4 Complexification

To best understand the complex form of the second variation formula (16), it is convenient to describe the complexification of the tangent space to the space of metrics on Σ in terms of canonical bundles. If $\mathrm{Met}(\Sigma)$ denotes the space of Riemannian metrics on a closed oriented surface Σ, and η is an element of $\mathrm{Met}(\Sigma)$,

$$T_\eta(\mathrm{Met}(\Sigma)) \otimes \mathbb{C} = \mathbf{K}^{(2,0)} \oplus \mathbf{K}^{(1,1)} \oplus \mathbf{K}^{(0,2)}, \tag{17}$$

where $\mathbf{K}$ is the canonical bundle over Σ, $\bar{\mathbf{K}}$ is its conjugate and

$$\mathbf{K}^{(p,q)} = \mathbf{K}^p \otimes \bar{\mathbf{K}}^q.$$

Of course, the holomorphic tangent bundle of Σ is isomorphic to $\bar{\mathbf{K}}$. Covariant differentiations yield operators

$$D' : \Gamma(\mathbf{K}^{(p,q)}) \to \Gamma(\mathbf{K}^{(p+1,q)}), \quad D'' : \Gamma(\mathbf{K}^{(p,q)}) \to \Gamma(\mathbf{K}^{(p,q+1)}).$$

Note that sections of the summand $\mathbf{K}^{(2,0)}$ are quadratic differentials on Σ.

The real inner product on $T\Sigma$ extends to a complex bilinear inner product $\langle \cdot, \cdot \rangle$ on the sum of all $\mathbf{K}^{(p,q)}$'s, which pairs $\mathbf{K}^{(p,q)}$ with $\mathbf{K}^{(q,p)}$, or to a Hermitian inner product with respect to which all the $\mathbf{K}^{(p,q)}$'s are orthogonal. The real structure determines a conjugation $C : \mathbf{K}^{(p,q)} \to \mathbf{K}^{(q,p)}$.

In the context of Fréchet manifolds, the Lie algebra of the group $\mathrm{Diff}(\Sigma)$ of diffeomorphisms of Σ is just the space of vector fields on Σ. Complex-valued vector fields are sections of $\mathbf{K} \oplus \bar{\mathbf{K}}$, and as described in [1], the action of $\Gamma(\mathbf{K} \oplus \bar{\mathbf{K}})$ on $T_\eta(\mathrm{Met}(\Sigma))$ is given by

$$D' + D'' : \Gamma(\mathbf{K}^{(1,0)}) \to \Gamma(\mathbf{K}^{(2,0)}) \oplus \Gamma(\mathbf{K}^{(1,1)}),$$
$$D' + D'' : \Gamma(\mathbf{K}^{(0,1)}) \to \Gamma(\mathbf{K}^{(1,1)}) \oplus \Gamma(\mathbf{K}^{(0,2)}).$$

Thus for example, if one restricts attention to the holomorphic tangent bundle $\bar{\mathbf{K}}$, the key piece of this action is

$$D'' : Z \mapsto \frac{DZ}{\partial \bar{z}} d\bar{z},$$

The other component $D'(Z)$ simply affects the scale factor in the metric represented by $\mathbf{K}^{(1,1)}$. The cokernel of $D' : \Gamma(\mathbf{K}^{(1,0)}) \to \Gamma(\mathbf{K}^{(2,0)})$ can be thought of as the cotangent space to Teichmüller space consisting of holomorphic quadratic differentials, which is isomorphic to the tangent space consisting of extremal Beltrami differentials.

To analyze the second term in the variation formula (16), we set $\phi = (1/2)(\dot\eta_{11} - \dot\eta_{22}) - i\dot\eta_{12}$, and take the complex bilinear inner product between the two expressions

$$\sum \frac{DX}{\partial x_a} dx_a = \frac{DX}{\partial z} dz + \frac{DX}{\partial \bar z} d\bar z \tag{18}$$

and

$$\sum \dot\eta_{ab} \frac{\partial f}{\partial x_b} dx_a = \phi \frac{\partial f}{\partial \bar z} dz + \bar\phi \frac{\partial f}{\partial z} d\bar z. \tag{19}$$

Using the fact that $\langle dx_a, dx_b\rangle = (1/\lambda^2)\delta_{ab}$, we see that the second term of (16) is -2 times the inner product of the left hand sides of (18) and (19). Thus, taking the inner product of the right-hand sides, we see that the second term is

$$-4\int_\Sigma \left\langle \frac{DX}{\partial z}, \frac{\bar\phi}{\lambda^2}\frac{\partial f}{\partial z}\right\rangle dx_1dx_2 - 4\int_\Sigma \left\langle \frac{DX}{\partial \bar z}, \frac{\phi}{\lambda^2}\frac{\partial f}{\partial \bar z}\right\rangle dx_1dx_2. \tag{20}$$

Here ϕ transforms under change of complex parameter in such a way that

$$\phi dz^2 \quad \leftrightarrow \quad \frac{2\bar\phi}{\lambda^2}\frac{d\bar z}{dz} \tag{21}$$

is a quadratic or Beltrami differential, a section of $\mathbf{K}^{(2,0)}$ or $\mathbf{K}^{(-1,1)}$ respectively. The third term in (16) is

$$\int_\Sigma \frac{1}{\lambda^4}(|\dot\eta_{11}|^2 + |\dot\eta_{12}|^2)\sigma^2 dx_1dx_2 = \int_\Sigma \left(\left|\frac{\bar\phi}{\lambda^2}\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\phi}{\lambda^2}\frac{\partial f}{\partial \bar z}\right|^2 \right) dx_1dx_2.$$

Thus if Z is a section of $f^*TM \otimes \mathbb{C}$,

$$\begin{aligned} d^2E(f,\omega)((Z,\dot\eta),(\bar Z,\dot\eta)) &= d^2E_\omega(f)(Z,\bar Z) \\ &- 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial \bar z}, \frac{\phi}{\lambda^2}\frac{\partial f}{\partial \bar z}\right\rangle dx_1dx_2 - 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar\phi}{\lambda^2}\frac{\partial f}{\partial z}\right\rangle dx_1dx_2 \\ &+ \int_\Sigma \left(\left|\frac{\bar\phi}{\lambda^2}\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\phi}{\lambda^2}\frac{\partial f}{\partial \bar z}\right|^2 \right) dx_1dx_2. \end{aligned} \tag{22}$$

To allow $\dot{\eta}_{ab}$ to be complex, we replace ϕ by ψ in the first term on the right in (19), which corresponds to specifying two quadratic differentials ϕdz^2 and $\bar{\psi} d\bar{z}^2$, sections of $\mathbf{K}^{(2,0)}$ and $\mathbf{K}^{(0,2)}$ respectively. Note that

$$\left(Z, \phi dz^2, \bar{\psi} d\bar{z}^2\right) \in (T_f \mathrm{Map}(\Sigma, M) \otimes \mathbb{C}) \oplus \Gamma(\mathbf{K}^{(2,0)}) \oplus \Gamma(\mathbf{K}^{(0,2)}),$$

has conjugate $\left(\bar{Z}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)$, since conjugation interchanges $\mathbf{K}^{(2,0)}$ and $\mathbf{K}^{(0,2)}$. The second variation formula now becomes

$$\begin{aligned} d^2 E(f,\omega)\left(\left(Z, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) &= d^2 E_\omega(f)(Z, \bar{Z}) \\ - 4\mathrm{Re} \int_\Sigma \left\langle \frac{DZ}{\partial \bar{z}}, \frac{\phi}{\lambda^2} \frac{\partial f}{\partial \bar{z}} \right\rangle dx_1 dx_2 &- 4\mathrm{Re} \int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{\psi}}{\lambda^2} \frac{\partial f}{\partial z} \right\rangle dx_1 dx_2 \\ &+ \int_\Sigma \left(\left| \frac{\bar{\phi}}{\lambda^2} \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\psi}{\lambda^2} \frac{\partial f}{\partial \bar{z}} \right|^2 \right) dx_1 dx_2. \quad (23) \end{aligned}$$

Substitution of (5) into this formula yields:

Second Variation Theorem. *The second variation of the two-variable energy E is given by*

$$\begin{aligned} d^2 E(f,\omega)&\left(\left(Z, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) \\ &= \int_\Sigma \left[\left| 2\frac{DZ}{\partial \bar{z}} - \frac{\bar{\phi}}{\lambda^2} \frac{\partial f}{\partial z} \right|^2 - 4 \left\langle R\left(Z, \frac{\partial f}{\partial z}\right) \frac{\partial f}{\partial \bar{z}}, \bar{Z} \right\rangle \right] dx_1 dx_2 \\ &\quad - 4Re \int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{\psi}}{\lambda^2} \frac{\partial f}{\partial z} \right\rangle dx_1 dx_2 + \int_\Sigma \left| \frac{\psi}{\lambda^2} \frac{\partial f}{\partial \bar{z}} \right|^2 dx_1 dx_2. \quad (24) \end{aligned}$$

Here Z is a section of $f^* TM \otimes \mathbb{C}$ and $(\phi dz^2, \bar{\psi} d\bar{z}^2)$ is an element of $T_\omega \mathcal{T} \otimes \mathbb{C}$.

Had we used (6) instead, we would have obtained

$$\begin{aligned} d^2 E(f,\omega)&\left(\left(Z, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) \\ &= \int_\Sigma \left[\left| 2\frac{DZ}{\partial z} - \frac{\psi}{\lambda^2} \frac{\partial f}{\partial \bar{z}} \right|^2 - 4 \left\langle R\left(Z, \frac{\partial f}{\partial \bar{z}}\right) \frac{\partial f}{\partial z}, \bar{Z} \right\rangle \right] dx_1 dx_2 \\ &\quad - 4\mathrm{Re} \int_\Sigma \left\langle \frac{DZ}{\partial \bar{z}}, \frac{\phi}{\lambda^2} \frac{\partial f}{\partial \bar{z}} \right\rangle dx_1 dx_2 + \int_\Sigma \left| \frac{\bar{\phi}}{\lambda^2} \frac{\partial f}{\partial z} \right|^2 dx_1 dx_2. \quad (25) \end{aligned}$$

From these formulae, we see quite clearly how the action of the diffeomorphism group enters into the second variation formula, complexifications of infinitesimal diffeomorphisms being sections of $\mathbf{L} \oplus \bar{\mathbf{L}}$ or of $\mathbf{K}^{(0,1)} \oplus \mathbf{K}^{(1,0)}$.

5 Relationship to second variation of area

We now describe the relationship between (16) and (24) and the classical formulae presented in Simons [11] and Micallef and Wolfson [8] for second variation of area under normal variations. Of course, if $f : \Sigma \to M$ is an immersion, second derivative of area along normal deformations must agree with the second derivative of energy along deformations through an appropriate family of conformal harmonic maps, in which the conformal structure changes in accordance with the immersion.

To see how this works out, we suppose that $X^\perp$ is a section of the normal bundle $(f^*TM)^\perp$ to f, and let

$$\sum_{a,b=1}^{2} \eta_{ab} dx_a dx_b = \lambda^2 (dx_1^2 + dx_2^2)$$

denote the metric on Σ induced by f, described in terms of isothermal coordinates (x_1, x_2) on Σ. A variation in the direction of $X^\perp$ will produce a corresponding variation $\sigma(X^\perp) = (\dot{\eta}_{ab})$ in the induced metric on Σ, which is given by the formula

$$\dot{\eta}_{ab} = \left[\left\langle \frac{DX^\perp}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle + \left\langle \frac{\partial f}{\partial x_a}, \frac{DX^\perp}{\partial x_b} \right\rangle \right] = -2 \left\langle X^\perp, \frac{D}{\partial x_a} \left(\frac{\partial f}{\partial x_b} \right) \right\rangle, \quad (26)$$

the last equality using the fact that $X^\perp$ is perpendicular to $\partial f/\partial x_1$ and $\partial f/\partial x_2$. Note that since f is harmonic, it follows from the right-hand expression that $\dot{\eta}_{11} + \dot{\eta}_{22} = 0$, so the variation is infinitesimally area preserving. Since the deformation $(X^{\perp}, \sigma(X^{\perp}))$ is through conformal maps, it is to be expected that this will result in the most negative possible second variation.

It follows from (26) that

$$\left\langle \frac{DX^{\perp}}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle = \frac{1}{2}\dot{\eta}_{ab}, \tag{27}$$

and hence the second and third terms (14) and (15) in the second variation formula are

$$-\int_{\Sigma} \sum_{a,b} \left| \frac{\dot{\eta}_{ab}}{\lambda} \right|^2 dx_1 dx_2 \quad \text{and} \quad \frac{1}{2}\int_{\Sigma} \sum_{a,b} \left| \frac{\dot{\eta}_{ab}}{\lambda} \right|^2 dx_1 dx_2$$

respectively. It follows that the second variation is given by

$$\begin{aligned} d^2E(f,\eta)((X^{\perp},\dot{\eta}),(X^{\perp},\dot{\eta})) = &\int_{\Sigma} \sum_a \left\| DX^{\perp} \right\|^2 dA \\ &- \frac{1}{2}\int_{\Sigma} \sum_{a,b} \left| \frac{\dot{\eta}_{ab}}{\lambda^2} \right|^2 dA - \int_{\Sigma} \left[\langle \mathcal{K}(X), X \rangle\right] dA. \end{aligned}$$

A direct calculation shows that

$$\sum_{a,b} \left| \frac{\dot{\eta}_{ab}}{\lambda^2} \right|^2 = 4 \sum_{a,b} \left| \frac{1}{\lambda^2} \left\langle \frac{DX^{\perp}}{\partial x_a}, \frac{\partial f}{\partial x_b} \right\rangle \right|^2 = 4 \left\| (DX^{\perp})^{\top} \right\|^2,$$

from which we obtain the real version of the classical formula for second variation of area ([11], Theorem 3.2.2 or [4], Theorem 32):

$$\begin{aligned} &d^2E(f,\omega)\big((X^{\perp},\sigma(X^{\perp})),(X^{\perp},\sigma(X^{\perp}))\big) \\ &\quad = \int_{\Sigma} \left[\|DX^{\perp}\|^2 - 2\|(DX^{\perp})^{\top}\|^2 - \langle \mathcal{K}(X^{\perp}), X^{\perp}\rangle\right] dA \\ &\qquad = \int_{\Sigma} \left[\|(DX^{\perp})^{\perp}\|^2 - \langle \mathcal{B}(X^{\perp}) + \mathcal{K}(X^{\perp}), X^{\perp}\rangle\right] dA, \end{aligned} \tag{28}$$

where

$$\langle \mathcal{B}(X^{\perp}), X^{\perp}\rangle = \|(DX^{\perp})^{\top}\|^2,$$

a term which, as one easily verifies, depends only on the second fundamental form of f. From this second variation formula, we obtain a formally self-adjoint Jacobi operator

$$L^{\perp} : \Gamma((f^*TM)^{\perp}) \to \Gamma((f^*TM)^{\perp}),$$

such that

$$d^2E(f,\omega)\left((X_1^{\perp}, \sigma(X_1^{\perp})), (X_2^{\perp}, \sigma(X_2^{\perp}))\right) = \int_{\Sigma} \langle L^{\perp}(X_1^{\perp}), X_2^{\perp}\rangle dA,$$

which we call the *normal Jacobi operator*. We call a solution to $L^{\perp}(X^{\perp}) = 0$ a *normal Jacobi field*.

How can we describe normal variations in the context of the complex vector bundle $\mathbf{E}$? The first step is to introduce a holomorphic bundle of codimension one in $\mathbf{E}$,

$$\mathbf{L}^{\perp} = \{v \in \mathbf{E} : \langle v, w\rangle = 0 \text{ for all } w \in \mathbf{L}\ \}.$$

We can then define the holomorphic normal bundle $\mathbf{N}(f)$ to f by

$$\mathbf{N}(f) = \mathbf{L}^{\perp}/\mathbf{L}.$$

We can choose the first quadratic differential ϕdz^2 to cancel the tangential projection of $DZ^{\perp}/\partial\bar{z}$ and (24) becomes

$$\begin{aligned} d^2E(f,\omega)&\left(\left(Z^{\perp}, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}^{\perp}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) \\ &= 4\int_{\Sigma} \left[\left|\left(\frac{DZ^{\perp}}{\partial\bar{z}}\right)^{\perp}\right|^2 - \left\langle R\left(Z^{\perp}, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \bar{Z}^{\perp}\right\rangle\right] dx_1 dx_2 \\ &\quad - 4\mathrm{Re}\int_{\Sigma}\left\langle \frac{DZ^{\perp}}{\partial z}, \frac{\bar{\psi}}{\lambda^2}\frac{\partial f}{\partial z}\right\rangle dx_1dx_2 + \int_{\Sigma}\left|\frac{\psi}{\lambda^2}\frac{\partial f}{\partial\bar{z}}\right|^2 dx_1dx_2. \end{aligned} \tag{29}$$

Here $Z^{\perp}$ denotes a section of $\mathbf{N}(f)$ and diffeomorphism invariance ensures that we can assume that ψdz^2 is a holomorphic differential.

The penultimate term in (29) can be interpreted in terms of the quadratic differential

$$\zeta dz^2 = -4\left\langle Z^{\perp}, \frac{D}{\partial z}\left(\frac{\partial f}{\partial z}\right)\right\rangle dz^2 = 4\left\langle \frac{DZ^{\perp}}{\partial z}, \frac{\partial f}{\partial z}\right\rangle dz^2.$$

Then (29) becomes

$$\begin{aligned} &d^2E(f,\omega)\left(\left(Z^{\perp}, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}^{\perp}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) \\ &\quad = 4\int_{\Sigma}\left[\left|\left(\frac{DZ^{\perp}}{\partial \bar{z}}\right)^{\perp}\right|^2 - \left\langle R\left(Z^{\perp}, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \bar{Z}^{\perp}\right\rangle\right] dx_1 dx_2 \\ &\qquad\qquad - \mathrm{Re}\frac{\zeta\bar{\psi}}{\lambda^4} dA + \int_{\Sigma}\frac{|\psi|^2}{2\lambda^4} dA, \qquad (30) \end{aligned}$$

where we have assumed that the metric $\lambda^2(dx^2 + dy^2)$ on Σ is the metric $\sigma^2(dx^2 + dy^2)$ induced by f. The sum of the last two terms is minimized when $\zeta = \psi$, and then (30) becomes

$$\begin{aligned} &d^2E(f,\omega)\left(\left(Z^{\perp}, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}^{\perp}, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) \\ &= 4\int_{\Sigma}\left[\left|\left(\frac{DZ^{\perp}}{\partial \bar{z}}\right)^{\perp}\right|^2 - \left\langle R\left(Z^{\perp}, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \bar{Z}^{\perp}\right\rangle\right] dx_1 dx_2 - \frac{1}{2}\int_{\Sigma}\frac{|\zeta|^2}{\lambda^4} dA. \end{aligned} \qquad (31)$$

Note that

$$\left\langle \frac{\partial Z^{\perp}}{\partial z}, \frac{\partial f}{\partial \bar{z}}\right\rangle = -\left\langle Z^{\perp}, \frac{\partial^2 f}{\partial z \partial \bar{z}}\right\rangle = 0,$$

so

$$\left(\frac{\partial Z^{\perp}}{\partial z}\right)^{\top} = \frac{2}{\lambda^2}\left\langle\left(\frac{\partial Z^{\perp}}{\partial z}\right)^{\top}, \frac{\partial f}{\partial z}\right\rangle \frac{\partial f}{\partial \bar{z}} = -\frac{2}{\lambda^2}\left\langle Z^{\perp}, \frac{D}{\partial z}\left(\frac{\partial f}{\partial z}\right)\right\rangle \frac{\partial f}{\partial \bar{z}}$$

and

$$4\left|\left(\frac{\partial Z^{\perp}}{\partial z}\right)^{\top}\right|^2 dx_1 dx_2 = 4\left|-\frac{\zeta}{2\lambda^2}\frac{\partial f}{\partial z}\right|^2 dx_1 dx_2 = \frac{|\zeta|^2}{2\lambda^4} dA,$$

so (30) can be rewritten as

$$\begin{aligned} & d^2E(f,\omega)\left(\left(Z^\perp, \phi dz^2, \bar{\psi} d\bar{z}^2\right), \left(\bar{Z}^\perp, \psi dz^2, \bar{\phi} d\bar{z}^2\right)\right) \\ & = 4\int_\Sigma \left[\left| \left(\frac{DZ^\perp}{\partial \bar{z}} \right)^\perp \right|^2 - \left\langle R\left(Z^\perp, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \bar{Z}^\perp \right\rangle - \left| \left(\frac{\partial Z^\perp}{\partial z} \right)^\top \right|^2 \right] dx_1 dx_2, \end{aligned} \tag{32}$$

which is just the formula for second variation of area presented by Micallef and Wolfson [8].

Invariance under diffeomorphisms of Σ shows that all that really matters in (32) is the projection of ζ into the space of holomorphic quadratic differentials.

6 Kähler metrics

Variations with nontrivial tangential components make an important appearance in the theory of holomorphic curves in a Kähler manifold, or more generally, in an almost Kähler manifold (as studied in [5]). Recall that a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is called *almost Hermitian* if it is endowed with an almost complex structure J such that

$$\langle Jv, Jw\rangle = \langle v, w\rangle, \qquad \text{for all } v, w \in T_pM.$$

Given any almost Hermitian manifold $(M, \langle\cdot,\cdot\rangle, J)$ we define its *Kähler form* to be the two-form ω such that

$$\omega(v, w) = \langle Jv, w\rangle, \qquad \text{for all } v, w \in T_pM.$$

The almost Hermitian manifold $(M, \langle\cdot,\cdot\rangle, J)$ is said to be *almost Kähler* if this form is closed. It is said to be *Kähler* if the Kähler form is closed and J is integrable, or equivalently, if J is parallel with respect to the Levi-Civita connection ∇. If $(M, \langle\cdot,\cdot\rangle, J)$ is any Hermitian manifold, we can extend $\langle\cdot,\cdot\rangle$ and J complex linearly to $TM \otimes \mathbb{C}$ and let

$$T^{1,0}M = \{v \in TM \otimes \mathbb{C} : Jv = \sqrt{-1}v\},$$
$$T^{0,1}M = \{v \in TM \otimes \mathbb{C} : Jv = -\sqrt{-1}v\},$$

called the *holomorphic* and *antiholomorphic tangent bundles* respectively. These bundles are interchanged by the conjugation operation $z \mapsto \bar{z}$ and are *isotropic*—that is

$$v, w \in T^{1,0}_p M \quad \Rightarrow \quad \langle v, w\rangle = 0.$$

Note that $(1/2)(\mathrm{id} - \sqrt{-1}J)$ and $(1/2)(\mathrm{id} + \sqrt{-1}J)$ are orthogonal projections taking sections of $TM \otimes \mathbb{C}$ into sections of $T^{1,0}M$ and $T^{0,1}M$, and hence if $f : \Sigma \to M$ is a smooth map,

$$\left|\frac{\partial f}{\partial z}\right|^2 = \left|\left(\frac{\partial f}{\partial z}\right)^{(1,0)}\right|^2 + \left|\left(\frac{\partial f}{\partial z}\right)^{(0,1)}\right|^2,$$

where

$$\left(\frac{\partial f}{\partial z}\right)^{(1,0)} = \frac{1}{2}\left(\frac{\partial f}{\partial z} - \sqrt{-1}J\frac{\partial f}{\partial z}\right), \quad \left(\frac{\partial f}{\partial z}\right)^{(0,1)} = \frac{1}{2}\left(\frac{\partial f}{\partial z} + \sqrt{-1}J\frac{\partial f}{\partial z}\right).$$

We say that $f : \Sigma \to M$ is *J-holomorphic* if

$$J\left(\frac{\partial f}{\partial z}\right) = \sqrt{-1}\frac{\partial f}{\partial z} \quad \text{or equivalently,} \quad \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial z}\right)^{(1,0)}.$$

Such a map is holomorphic in the usual sense if M is a Kähler manifold, and agrees in with the notion of J-holomorphic curves studied in symplectic geometry. Similarly, f is *anti-J-holomorphic* if

$$J\left(\frac{\partial f}{\partial z}\right) = -\sqrt{-1}\frac{\partial f}{\partial z}.$$

In an almost Hermitian manifold, we can divide the energy,

$$E(f,\omega) = 2\int_\Sigma \left|\frac{\partial f}{\partial z}\right|^2 dxdy$$

into two components, $E(f,\omega) = E'(f,\omega) + E''(f,\omega)$, both nonnegative, where

$$E'(f,\omega) = 2\int_\Sigma \left|\left(\frac{\partial f}{\partial z}\right)^{(1,0)}\right|^2 dxdy, \quad E''(f,\omega) = 2\int_\Sigma \left|\left(\frac{\partial f}{\partial z}\right)^{(0,1)}\right|^2 dxdy.$$

Thus (f,ω) is J-holomorphic if and only if $E''(f,\omega) = 0$.If $f : \Sigma \to M$ is a smooth map into an almost Hermitian manifold $(M, \langle\cdot,\cdot\rangle, J)$, then

$$E'(f,\omega) = \frac{1}{2}\left(E(f,\omega) + \int_\Sigma f^*\omega\right), \qquad E''(f,\omega) = \frac{1}{2}\left(E(f,\omega) - \int_\Sigma f^*\omega\right),$$

where ω is the Kähler form. This fact is a direct consequence of the calculation:

$$\begin{aligned} 2\left|\left(\frac{\partial f}{\partial z}\right)^{(1,0)}\right|^2 &= \left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar z}\right\rangle - \frac{1}{2}\left\langle \sqrt{-1}J\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar z}\right\rangle \\ &= \frac{1}{4}\left[\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2\right] + \frac{1}{2}\omega\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right). \end{aligned}$$

If $(M, \langle\cdot,\cdot\rangle, J)$ is *almost Kähler*, the integral of ω is constant, and E' and E'' have exactly the same critical points as E; that is, their critical points are conformal harmonic maps. In this case, a J-holomorphic curve (f,ω) is both harmonic and conformal, the latter because $T^{1,0}M$ is isotropic. Moreover, a J-holomorphic curve is automatically stable and minimizes energy within its homotopy class.

If M is Kähler, the holomorphic tangent and cotangent bundles are preserved by parallel translation, and hence if Z and W are sections of $T^{1,0}M$, then

$$\langle \nabla_X\nabla_Y W, Z\rangle = 0 \quad \text{for all } X \text{ and } Y \qquad \Rightarrow \qquad \langle R(X,Y)W, Z\rangle = 0.$$

Thus if we define the curvature operator

$$\mathcal{R} : \Lambda^2 TM \otimes \mathbb{C} \to \Lambda^2 TM \otimes \mathbb{C} \quad \text{by} \quad \langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)W, Z \rangle,$$

we see that $\mathcal{R}$ vanishes on $\Lambda^2(T^{1,0}M)$ and $\Lambda^2(T^{0,1}M)$.

Kähler metrics are far from being generic, and one expects nontrivial Jacobi fields along algebraic curves which admit nontrivial deformations; these are common in smooth algebraic varieties. Suppose that $f : \Sigma \to M$ is a holomorphic curve and $(Z, \phi dz^2)$ is a pair, where Z is a section of the holomorphic tangent bundle $f^* T^{1,0}M$ and ϕdz^2 is a holomorphic quadratic differential which satisfy the equation

$$\frac{DZ}{\partial \bar{z}} = \frac{\bar{\phi}}{2\lambda^2} \frac{\partial f}{\partial z}. \tag{33}$$

We can then set $\psi = 0$ and obtain a Jacobi field $(Z, \phi dz^2, 0)$ for (24). To see this, we need to check that

$$d^2 E(f, \omega) \left(\left(Z, \phi dz^2, 0 \right), \left(\bar{Z}_1, \psi_1 dz^2, \bar{\phi}_1 d\bar{z}^2 \right) \right) = 0$$

for all $(\bar{Z}_1, \psi_1 dz^2, \bar{\phi}_1 d\bar{z}^2)$. But the curvature term vanishes by the argument in the preceding paragraph. Thus the equation (24) implies that the only terms that could be nonzero are those which appear in

$$d^2 E(f, \omega) \left(\left(Z, \phi dz^2, 0 \right), \left(0, \psi_1 dz^2, 0 \right) \right),$$

and it is apparent from (33) that the only term in this expression that does not immediately vanish is

$$-4 \mathrm{Re} \int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{\psi}_1}{\lambda^2} \frac{\partial f}{\partial z} \right\rangle dx_1 dx_2.$$

This term vanishes because $f^* T^{1,0}M \oplus f^* T^{0,1}M$ is a parallel unitary direct sum decomposition of $(f^* TM) \otimes \mathbb{C}$, and $f^* T^{1,0}M$ is an isotropic subbundle of $\mathbf{E}$ which contains $\partial f / \partial z$.

In a deformation through algebraic curves, the quadratic differential in the Jacobi equation allows for a change in conformal structure on Σ.

7 Tangential Jacobi fields

The second variation formula for the two-variable energy gives rise to the possibility of tangential Jacobi fields, Jacobi fields which are tangential to the surface Σ. Suppose that $(Z, \phi dz^2)$ is a pair, where Z is a section of the line bundle $\mathbf{L}$ described in §2 and ϕdz^2 is a holomorphic quadratic differential which satisfy the equation

$$\frac{DZ}{\partial \bar{z}} = \frac{\bar{\phi}}{2\lambda^2} \frac{\partial f}{\partial z}. \tag{34}$$

We can then set $\psi = 0$ and obtain a Jacobi field $(Z, \phi dz^2, 0)$ for (24), by an argument almost identical to that presented in the previous section. Indeed, we need to check that

$$d^2 E(f, \omega) \left(\left(Z, \phi dz^2, 0 \right), \left(\bar{Z}_1, \psi_1 dz^2, \bar{\phi}_1 d\bar{z}^2 \right) \right) = 0$$

for all $(\bar{Z}_1, \psi_1 dz^2, \bar{\phi}_1 d\bar{z}^2)$. But the curvature term vanishes when Z is a section of $\mathbf{L}$, no matter what Z_1 is, and (34) implies that the only terms that could be nonzero are those which appear in

$$d^2 E(f, \omega) \left(\left(Z, \phi dz^2, 0 \right), \left(0, \psi_1 dz^2, 0 \right) \right).$$

Indeed, we see that the only term in this expression that does not immediately vanish is

$$-4 \mathrm{Re} \int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{\psi}_1}{\lambda^2} \frac{\partial f}{\partial z} \right\rangle dx_1 dx_2,$$

and this term vanishes because $\mathbf{L} \oplus \bar{\mathbf{L}}$ is a parallel decomposition of $(f^*TM)^\top \otimes \mathbb{C}$, the tangential part of $(f^*TM) \otimes \mathbb{C}$, and

$$\left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \right\rangle = 0 \quad \text{or equivalently,} \quad \langle \mathbf{L}, \mathbf{L} \rangle = 0.$$

Conversely, it is not difficult to show that all Jacobi fields $(Z, \phi dz^2, \bar{\psi} d\bar{z}^2)$ in which Z is a section of $\mathbf{L}$ are of the form $(Z, \phi dz^2, 0)$ where ϕdz^2 is a holomorphic quadratic differential which satisfies (34), or equivalently,

$$D''(Z) = \iota\left(\bar{\phi} d\bar{z}^2\right), \quad \text{where} \quad \iota : \Gamma(\bar{\mathbf{K}} \otimes \bar{\mathbf{K}}) \subset \Gamma(\mathbf{L} \otimes \bar{\mathbf{K}}) \tag{35}$$

is the inclusion. Let $\mathcal{J}(\mathbf{L})$ denote the complex linear space of pairs $(Z, \phi dz^2)$ satisfying this equation. By a slight abuse of terminology, we say that elements of $\mathcal{J}(\mathbf{L})$ are *tangential Jacobi fields*.

Whenever a parametrized minimal surface (f, ω) has a nontrivial branch locus, the branch points give rise to nontrivial elements of $\mathcal{J}(\mathbf{L})$. Recall that the first Chern class of the line bundle $\mathbf{L}$ depends on the total branching order ν; if g is the genus of Σ and ν is the total branching order,

$$c_1(\mathbf{L})[\Sigma] = 2 - 2g + \nu.$$

Lemma. *The complex dimension of the space of tangential Jacobi fields is given by the formula:*

$$dim_{\mathbb{C}} \mathcal{J}(\mathbf{L}) = \begin{cases} 3 + \nu & \text{if } g = 0, \\ 1 + \nu & \text{if } g = 1, \\ \nu & \text{if } g \geq 2. \end{cases}$$

The proof (following [9]) is an application of the Riemann-Roch theorem to the operator

$$D'' : \Gamma(\mathbf{L}) \to \Gamma(\mathbf{L} \otimes \mathbf{K}^{0,1}).$$

When the total branching order ν is zero, the cokernel of D'' is just the space of holomorphic quadratic differentials, and the only solutions to (34) are the holomorphic sections $\mathcal{O}(\mathbf{K^{-1}})$ of the holomorphic tangent bundle $\mathbf{K^{-1}}$. This space of holomorphic sections is just the tangent space to the identity component G of the group of conformal automorphisms, which is $PSL(2,\mathbb{C})$ when $g=0$, $S^1 \times S^1$ when $g=1$ and trivial when $g \geq 2$. Thus when $\nu = 0$, $\dim_{\mathbb{C}} \mathcal{J}(\mathbf{L})$ is three, one or zero when the genus g is zero, one or ≥ 2 respectively.

Suppose next that (f,ω) is a conformal harmonic map with a single branch point p of multiplicity one. Then $\mathbf{L} = \bar{\mathbf{K}} \otimes \zeta_p$, where ζ_p is the point bundle at p, and according to the Riemann-Roch theorem, the analytic index of D'' is given by the formula,

$$\dim \mathcal{O}(\mathbf{L}) - \dim \mathcal{O}(\mathbf{L}^{-1} \otimes \mathbf{K}) = 1 - g + c_1(\mathbf{L})[\Sigma].$$

Either the dimension of the kernel of D'' increases by one which yields a new element of $\mathcal{J}(\mathbf{L})$, or the dimension of the image of

$$D'' : \Gamma(\bar{\mathbf{K}} \otimes \zeta_p) \to \Gamma(\bar{\mathbf{K}} \otimes \bar{\mathbf{K}} \otimes \zeta_p)$$

decreases by one. In the latter case, at least one nonzero antiholomorphic quadratic differential is covered by D'', which also gives a new element of $\mathcal{J}(\mathbf{L})$. (The space of new elements of $\mathcal{J}(\mathbf{L})$ obtained increasing ν by one is at most one-dimensional, because given two such solutions, a nontrivial linear combination would vanish at p and therefore lie within $\mathcal{J}(\bar{\mathbf{K}})$.)

The general case of the Lemma is proven by induction on ν. Each time ν is increased by one, either a new holomorphic section of $\mathbf{L}$ is produced or a new antiholomorphic quadratic differential is covered by D'', in either case leading to an increase of one in the dimension of $\mathcal{J}(\mathbf{L})$.

Thus a minimal surface (f,ω) with total branching order ν always has a ν-dimensional space of tangential Jacobi fields in addition to the space of Jacobi fields generated by the conformal automorphisms of Σ. If (f,ω) is a branched cover of a minimal surface without branch points, these Jacobi fields have a very natural interpretation; they form the tangent space to a critical submanifold of branched covers. It is clear that tangential Jacobi fields must play a key role in elucidating the geometry of prime minimal surfaces with nontrivial branch locus.

8 Generic metrics on M

The goal of this section is to present the geometric ideas underlying a key application of the second variation formula, the Bumpy Metric Theorem of [9], without giving the complete details of the proof.

A parametrized minimal surface (f, ω) is said to be *prime* if it is not a branched cover of a minimal surface (which may be nonorientable) of strictly lower energy. Algebraic curves in Kähler manifolds provide many examples of prime minimal surfaces that do contain branch points. However, when a Riemannian manifold is given a generic Riemannian metric all of its prime parametrized minimal surfaces are free of branch points. That follows from the Main Theorem of [9]:

Bumpy Metric Theorem. *Suppose that M is a compact connected smooth manifold of dimension at least four. Then for a generic choice of Riemannian metric on M, all prime parametrized minimal surfaces are free of branch points and lie on nondegenerate critical submanifolds of $Map(\Sigma, M) \times \mathcal{T}$ which are orbits for the G-action on $Map(\Sigma, M) \times \mathcal{T}$, where G is the identity component of the group of conformal automorphisms of Σ.*

In the statement of the theorem, generic choice of metric means a countable intersection of open dense subsets of $\mathrm{Met}_k(\Sigma)$, where $\mathrm{Met}_k(\Sigma)$ is the space of

L^2_k-metrics on M and k ranges throughout the positive integers. A critical submanifold is said to be nondegenerate if the Hessian is nondegenerate on the normal space to the submanifold.

The proof is an application of Smale's infinite-dimensional generalization of Sard's theorem [12]. We briefly sketch the key ideas of the proof under the simplifying assumption that all branch points are simple; that is all have branching order one. We consider

$$E : \mathcal{M} \times \mathrm{Met}(M) \to \mathbb{R}, \quad \text{where} \quad \mathcal{M} = \mathrm{Map}(\Sigma, M) \times \mathcal{T}.$$

The Euler-Lagrange equation for E defines a map

$$F : \mathcal{M} \times \mathrm{Met}(M) \to T\mathcal{M}$$

and we let

$$\begin{aligned}\mathcal{P}_s = \{((f,\omega),g) \in \mathcal{M} \times \mathrm{Met}(M) : F((f,\omega),g) = 0 \text{ and} \\ (f,\omega) \text{ is prime with at worst simple branch points}\}.\end{aligned}$$

The strategy is to show that

1. $\mathcal{P}_s$ is a submanifold, and
2. the projection $\mathcal{P}_s \to \mathrm{Met}(\Sigma)$ is a Fredholm map, the Fredholm index being the real dimension of G.

To carry this program out, we need to establish a series of lemmas:

Lemma 1. *A prime parametrized minimal surface (f,ω) is somewhere injective; in other words, there is a point $p \in \Sigma$ such that $f^{-1}(f(p)) = \{p\}$.*

It then follows from the Aronszajn unique continuation theorem that $f^{-1}(f(p)) = \{p\}$ holds for p belonging to an open dense set of points.

Lemma 2. *When the dimension of M is at least four, the only singularities of a prime minimal surface (f, ω) aside from branch points are transverse self-intersections, when the metric on M is generic.*

To show that $\mathcal{P}$ is a submanifold we must construct deformations $\dot{g}$ in the metric g such that the vertical projections of $D_2((f,\omega), g)(\dot{g})$ cover a complement to the direct sum of the tangent space to the orbit of the G-action and the range of the Jacobi operator

$$L : T_{(f,\omega)}\mathcal{M} \to T_{(f,\omega)}\mathcal{M}.$$

The cokernel of this operator is the space of Jacobi fields, so we must essentially cover all Jacobi fields orthogonal to G-orbits with metric variations. Normal Jacobi fields could be covered by variations in the metric with techniques similar to those used to prove a bumpy metric theorem for minimal imbeddings [13]. Thus the key issue is how to cover tangential Jacobi fields with metric variations.

Note that the energy (2) is invariant under actions of diffeomorphisms on both domain and range: if $\{\phi_t : t \in \mathbb{R}\}$ is a one-parameter group of diffeomorphisms of Σ and $\{\psi_t : t \in \mathbb{R}\}$ is a one-parameter group of diffeomorphisms of M, then

$$E(\psi_t \circ f, \eta, \psi_{-t}^* g) = E(f, \eta, g) = E(f \circ \phi_t, \phi_t^* \eta, g).$$

We are interested in the case where the vector field X on M generated by $\{\psi_t : t \in \mathbb{R}\}$ restricts to a section of $\mathbf{L}$.

If $f : \Sigma \to M$ is an imbedding, any diffeomorphism ψ of M which takes $f(\Sigma)$ to itself restricts to a diffeomorphism ϕ of Σ such that $\psi \circ f = f \circ \phi$. Conversely, for any diffeomorphism ϕ on Σ, there corresponds a family of diffeomorphisms ψ on M such that $\psi \circ f = f \circ \phi$ Hence a one-parameter group $\{\psi_t : t \in \mathbb{R}\}$

of diffeomorphisms of M which is tangent to $f(\Sigma)$ can be "cancelled" by a one-parameter group of diffeomorphisms of Σ itself:

$$E(f, \eta, \psi_t^* g) = E(\psi_t \circ f, \eta, g) = E(f, \phi_{-t}^* \eta, g).$$

In constrast, if f is a prime minimal surface with nontrivial branch locus, any diffeomorphism ψ of M which takes $f(\Sigma)$ to itself must fix the image of the branch points. Hence the only diffeomorphisms ϕ of Σ which can satisfy relations of the form $\psi \circ f = f \circ \phi$ are those which lie within

$$\mathrm{Diff}_{0,D}(\Sigma) = \{\phi \in \mathrm{Diff}_0(\Sigma) : \phi(p_k) = p_k, 1 \leq k \leq m\},$$

where

$$D = \nu_1 p_1 + \cdots \nu_m p_m$$

is the divisor of branch points of f, ν_k being the order of the branch point p_k. Indeed, a one-parameter family $\{\psi_t : t \in \mathbb{R}\}$ of diffeomorphismss of M tangent to $f(\Sigma)$ can be cancelled by a corresponding one-parameter family of diffeomorphisms of Σ itself precisely when it preserves the conformal structure on the punctured Riemann surface

$$\Sigma - \{p_1, \ldots, p_m\}.$$

The tangential deformations in the metric on M which do not preserve the conformal structure on the punctured Riemann surface correspond to meromorphic quadratic differentials on Σ with simple poles at some of the branch points. Lemmas 1 and 2 allow us to realize these deformations in the metric on Σ, after multiplying by suitable cutoff functions. Using such deformations, it is then possible to perturb away extra Jacobi fields that arise because of simple branch points:

Lemma 3. *Suppose that the compact Riemannian manifold M has dimension at least four, and that (f, ω) is a prime minimal surface with only simple branch points. If X is any Jacobi field in $T_{(f,\omega)}\mathcal{M}$ which is perpendicular to the orbit of the G-action, then there is a deformation $\dot{g}$ in the metric on M such that,*

$$\langle \pi_V \circ D_2((f, \omega), g)(\dot{g}), X \rangle \neq 0.$$

It is this Lemma that allows one to show that $\mathcal{P}_s$ is a submanifold. It is then relatively straightforward to check that the projection $\pi : \mathcal{P}_s \to \mathrm{Met}(\Sigma)$ is Fredholm and that the Fredholm index is the dimension of the space of conformal automorphisms of Σ. According to the Sard-Smale theorem, it then follows that for generic choice of $g_0 \in \mathrm{Met}(M)$, $\pi^{-1}(g_0)$ is a disjoint union of connected nondegenerate submanifolds of $\mathcal{M} \times \mathrm{Met}(M)$, each being an orbit for the action of the group G of automorphisms. The parametrized minimal surfaces lying in these submanifolds cannot have branch points because their only Jacobi fields are those generated by the G-action.

Of course, there are several additional details that are needed to flesh out this sketch of the proof, and we must also treat the case in which branch points are not simple.

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Department of Mathematics
University of California
Santa Barbara, CA, USA 93106
E-mail: moore@math.ucsb.edu