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A GEOMETRIC ITÔ FORMULA *

P. Catuogno

Abstract

Let M and N be manifolds equipped with connections Γ^M and Γ^N respectively and $F : M \to N$ be a smooth map. Let X be an M-valued semimartingale and Θ be an 1-form on N. We prove the following Itô formula in the context of Schwartz (second order) geometry,

$$\int \Theta \; d^{\Gamma^N} F(X) = \int F^* \Theta \; d^{\Gamma^M} X + \frac{1}{2} \int \beta_F^* \Theta(dX, dX)$$

where the integrals are in the Itô sense, and β_F is the fundamental form of F. Some applications are discussed.

1 Introduction

We recall the Itô formula for continuous semimartingales, which in the real valued case is

$$F(X_t) = F(X_0) + \int_0^t F'(X) dX + \frac{1}{2} \int_0^t F''(X) d[X, X] \qquad (1)$$

for X a continuous semimartingale and F a twice continuously differentiable function (see for instance Ph. Protter [11]). The equation (1) stands at the heart of stochastic calculus. It shows that $\mathcal{C}^2$ functions of continuous semimartingales are also semimartingales, and provides explicitly the Doob-Meyer decomposition of $f(X)$. Thus it allows calculations to be made, playing a role analogous to that of the fundamental theorem of ordinary calculus.

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The aim of this work is to extend the formula (1) to the stochastic differential geometry context. More precisely, we prove the following geometric Itô formula. Let M and N be manifolds endowed with connections Γ^M and Γ^N respectively and $F : M \to N$ be a smooth map. Let X be an M-valued semimartingale and Θ be an 1-form on N. Then

$$\int \Theta \; d^{\Gamma^N} F(X) = \int F^* \Theta \; d^{\Gamma^M} X + \frac{1}{2} \int \beta_F^* \Theta(dX, dX) \tag{2}$$

where the integrals are in the Itô sense, and β_F is the fundamental form of F. In the special case that $M = N = \mathbb{R}$ (equipped with the usual connection) and $\Theta = dx$, we recovery the classical Itô formula (2).

In the literature there are two versions of (2), one of J. M. Bismut for Itô processes (see [1] pp 407, Théorème 3.5) and another of J. R. Norris (see [10] pp 207). Both using covariant differentials and Stratonovich to Itô conversion rules. Our formulation and proof are different, we write the geometric Itô formula in terms of Itô integrals and give an intrinsic stochastic proof.

We apply the formula (2) to obtain the Bismut characterization of harmonic maps (see [8] pp 52), a stochastic characterization of the solutions of the heat equation, and we show that a smooth map is an harmonic Riemannian submersion if and only if sends Brownian motions into Brownian motions (this affirmation appear without proof in [10] pp 207).

The paper is organized as follows: In Section 2, we review some of the standard facts on Schwartz geometry and stochastic calculus on manifolds (see for instance M. Emery [4], [5], P. Meyer [7], [9] and L. Schwartz [13], [14]). In section 3 we prove our principal results.

2 Schwartz Geometry and Stochastic Calculus

Throughout this paper all the geometrical objects like manifolds, maps and functions will always be assumed to be smooth. As to manifolds and stochastic differential geometry, we shall use freely concepts and notations of Emery [4].

Let x be a point in a manifold M. The second order tangent space to M at x, $\tau_x M$ is the vector space of all differential operators on M at x of order at most two without a constant term. Let (U, x^i) be a local coordinate system around x. Every $L \in \tau_x M$ can be written in a unique way as

$$L = a^i D_i + a^{ij} D_{ij}$$

where $a^{ij} = a^{ji}$, $D_i = \dfrac{\partial}{\partial x^i}$ and $D_{ij} = \dfrac{\partial}{\partial x^i \partial x^j}$ are differential operators at x (we shall use the convention of summing over repeated indices). The elements of $\tau_x M$ are called second order tangent vectors at x, the elements of the dual vector space $\tau_x^* M$ are called second order forms at x. Every $\theta \in \tau_x^* M$ can be written in a unique way as

$$\theta = \theta_i d^2 xi + \theta_{ij} dx^i \cdot dx^j$$

where $\theta_{ij} = \theta_{ji}$ and $\{d^2x^i, 2dx^i \cdot dx^j : i \leq j\}$ is the dual basis of $\{D_i, D_{ij} : i \leq j\}$.

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ (respectively $\tau^* M = \bigcup_{x \in M} \tau_x^* M$) is canonically endowed with a vector bundle structure over M, it is called the second order tangent fiber bundle (respectively second order cotangent fiber bundle) of M.

The relation between second order geometry and stochastic calculus on manifolds is based in two fundamentals observations of L. Schwartz [13]: the Itô formula shows that M-valued semimartingales are well defined, and, secondly, means the Itô's differentials dX^i and $d[X^i, X^j]$ (where (x^i) is a local chart and X^i the i-th coordinate of the M-valued semimartingale X in this chart) behave under a change of coordinates as the coefficients of a second order tangent vector. This means that second order forms can be integrated along semimartingales. More formally,

Definition 1. A continuous random process X on a manifold M is a *semimartingale* if its composition $f(X)$ with any $f \in \mathcal{C}^\infty(M)$ is a real valued semimartingale.

Let $\Theta_{X_t} \in \tau^*_{X_t} M$ be an adapted stochastic second order form along X_t, an M-valued semimartingale. The integral of the form Θ along X was proposed by P. Meyer [7] (see also M. Emery [4] and [5]). Locally this integral can be describe as: let (U, x^i) be a local coordinate system in M. With respect to this chart the second order form Θ can be written as $\Theta_x = \theta_i(x) d^2 x^i + \theta_{ij}(x) dx^i \cdot dx^j$ where θ_i and $\theta_{ij} = \theta_{ji}$ are ($\mathcal{C}^\infty$ say) functions in M. Then the *integral* of Θ along X is defined by:

$$\int_0^t \Theta \; d_2 X = \int_0^t \theta_i(X_s) dX_s^i + \int_0^t \theta_{ij}(X_s) d[X^i, X^j]_s$$

We recall that a classical geometric *connection* Γ on M is equivalent to a section of the vector bundle $Hom(\tau M, TM)$ such that $\Gamma | TM = Id_{TM}$ (see for instance [7] pp 52).

Let M be a manifold endowed with a connection Γ. Let $\theta_{X_t} \in T^*_{X_t} M$ be an adapted stochastic 1-form along X_t, an M-valued semimartingale. The *Itô integral* of the form Θ along X is defined by:

$$\int_0^t \Theta \; d^\Gamma X = \int_0^t \Gamma^* \Theta \; d_2 X$$

where $\Gamma^*(x) : T^*_x M \to \tau^*_x M$ is the pull-back of $\Gamma(x)$ (see [7], [4] and [5]).

We say that a M-valued semimartingale X is a Γ-*martingale* if for any 1-form Θ the Itô integral $\int_0^t \Theta \, d^\Gamma X$ is a local martingale.

Let $F : M \to N$ be a smooth map, and $L \in \tau_x M$. We have that $F_*(x)L \in \tau_{F(x)}N$, the *differential* of F is given by

$$F_*(x)L(f) = L(f \circ F)$$

where $f \in \mathcal{C}^\infty(N)$. A covector $\theta \in \tau^*_{F(x)}N$ is pulled back into $F^*(x)\theta \in \tau^*_x M$ by

$$\langle F^*(x)\theta, L\rangle = \langle \theta, F_*(x)L\rangle$$

where $L \in \tau_x M$.

In terms of stochastic integrals the pull-back is well comported. In fact, let X be an M-valued semimartingale and $\Theta \in \tau^*_{F(X)}N$ be an adapted stochastic second order form along $F(X)$, we have that

$$\int F^*\Theta \; d_2 X = \int \Theta \; d_2 F(X). \tag{3}$$

Let L be a smooth section of τM. The *squared field operator* associated to L, denoted by QL, is the symmetric tensor given by

$$QL(f,g) = \frac{1}{2}(L(fg) - fL(g) - gL(f))$$

where $f, g \in C^\infty(M)$. We can consider $Q_x : \tau_x M \to T_x M \odot T_x M$ as the linear map defined by

$$Q_x(L = a^i D_i + a^{ij} D_{ij}) = a^{ij} D_i \odot D_j.$$

Pushing forward of second order vectors by smooth maps is related to the so called Schwartz morphisms between second order tangent vector bundles.

Definition 2. Let M and N be manifolds and take $x \in M$ and $y \in N$. A linear map $f : \tau_x M \to \tau_y N$ is called a *Schwartz morphism* if

i) $f(T_xM) \subset T_yN$ and

ii) for every $L \in \tau_x M$ we have that $Q(fL) = (f \otimes f)(QL)$.

We remark that if $F : M \to N$ is a smooth map between manifolds, then its differential is a Schwartz morphism.

Definition 3. Let M and N be manifolds equipped with connections Γ^M and Γ^N respectively and $F : M \to N$ be a smooth map. The section α_F of $\tau^* M \otimes F^* TN$ is defined by

$$\alpha_F = \Gamma^N \circ F_* - F_* \circ \Gamma^M.$$

The *fundamental form* of F, β_F is the unique section of $(TM \odot TM)^* \otimes F^* TN$ such that $\alpha_F = \beta_F \circ Q$. The map F is said to be *affine* if its fundamental form vanishes i.e. $\beta_F = 0$. In the case that M is a Riemannian manifold and Γ^M is the Levi-Civita connection, the *tension field* of F, $\tau_F : M \to TN$ is given by

$$\tau_F = tr \beta_F.$$

The map F is said to be *harmonic* if its tension field vanish i.e. $\tau_F = 0$.

The following linear algebra lemma shows that β_F is well defined.

Lemma 1. *Let E be a vector bundle and α be a section of $\tau^* M \otimes E$ such that $\alpha|TM = 0$. Then there exists an unique section β of $(TM \odot TM)^* \otimes E$ such that $\alpha = \beta \circ Q$.*

Proof: Since $Ker\ Q = TM \subset Ker\ \alpha$, the lemma follows from the first isomorphism theorem (see [12] pp 67).

□

Proposition 1. *Let M, N and L be manifolds equipped with connections Γ^M, Γ^N and Γ^L respectively. Let $F : M \to N$ and $G : N \to L$ be smooth maps. Then*

$$\beta_{G\circ F} = G_* \circ \beta_F + \beta_G \circ (F_* \otimes F_*)$$

Proof: We first compute $\alpha_{G\circ F}$.

$$\begin{aligned}
\alpha_{G\circ F} &= \Gamma^L \circ (G \circ F)_* - (G \circ F)_* \circ \Gamma^M \\
&= \Gamma^L \circ G_* \circ F_* - G_* \circ \Gamma^N \circ F_* + G_* \circ \Gamma^N \circ F_* - G_* \circ F_* \circ \Gamma^M \\
&= G_* \circ (\Gamma^N \circ F_* - F_* \circ \Gamma^M) + (\Gamma^L \circ G_* - G_* \circ \Gamma^N) \circ F_* \\
&= G_* \circ \alpha_F + \alpha_G \circ F_*
\end{aligned}$$

From the definition of β, using the expression for $\alpha_{G\circ F}$ and the fact that F_* is a Schwartz morphism, we have that

$$\begin{aligned}
\beta_{G\circ F} \circ Q &= \alpha_{G\circ F} \\
&= G_* \circ \alpha_F + \alpha_G \circ F_* \\
&= G_* \circ \beta_F \circ Q + \beta_G \circ Q \circ F_* \\
&= (G_* \circ \beta_F + \beta_G \circ F_* \otimes F_*) \circ Q.
\end{aligned}$$

This establish the formula.

□

3 The geometric Itô formula

We can now formulate our main result.

Theorem 1. *Let M and N be manifolds equipped with connections Γ^M and Γ^N respectively and $F : M \to N$ be a smooth map. Let X be an M-valued semimartingale and Θ be an 1-form on N. Then*

$$\int \Theta \ d^{\Gamma^N} F(X) = \int F^*\Theta \ d^{\Gamma^M} X + \frac{1}{2} \int \beta_F^* \Theta(dX, dX)$$

Proof: We calculate

$$\begin{array}{rcl}
\int \Theta\ d^{\Gamma^N} F(X) & = & \int (\Gamma^N)^* \Theta\ d_2 F(X) \\
& = & \int F^* (\Gamma^N)^* \Theta\ d_2 X \\
& = & \int F^* (\Gamma^N)^* \Theta\ d_2 X + \int (\Gamma^M)^* F^* \Theta\ d_2 X - \int (\Gamma^M)^* F^* \Theta\ d_2 X \\
& = & \int (\Gamma^M)^* F^* \Theta\ d_2 X + \int \Big(F^* (\Gamma^N)^* \Theta - (\Gamma^M)^* F^* \Theta \Big)\ d_2 X \\
& = & \int F^* \Theta\ d_2^{\Gamma^M} X + \int \alpha_F^* \Theta\ d_2 X \\
& = & \int F^* \Theta\ d_2^{\Gamma^M} X + \frac{1}{2} \int \beta_F^* \Theta (dX, dX),
\end{array}$$

where we use the formula (3) in the second line. The last equality is a consequence of the lemma below.

□

Lemma 2. $\int \alpha_F^* \Theta\ d_2 X = \frac{1}{2} \int \beta_F^* \Theta(dX, dX)$

Proof: By the definition of β_F, we have that

$$\frac{1}{2} \int \beta_F^* \Theta(dX, dX) = \int Q^* \beta_F^* \Theta\ d_2 X = \int (\beta_F \circ Q)^* \Theta\ d_2 X = \int \alpha_F^* \Theta\ d_2 X.$$

For the first equality see [4] Proposition 6.31.

□

The following result is well known in the literature (see [4] pp 40 proposition 4.32, [2] pp 234 among others).

Proposition 2. *Let M and N be manifolds equipped with connections Γ^M and Γ^N respectively and $F : M \to N$ be a smooth map. Then F is affine if and only if, for every Γ^M-martingale X, $F(X)$ is a Γ^N-martingale.*

Proof: It is clear from the theorem and the definitions.

□

Corollary 1. *Let M be a Riemannian manifold, N be a manifold endowed with a connection Γ^N and $F : M \to N$ be a smooth map. Let B be an M-valued Brownian motion and Θ be an 1-form on N. Then*

$$\int \Theta \ d^{\Gamma^N} F(B) = \int F^*\Theta \ d^{\Gamma^M} B + \frac{1}{2}\int \tau_F^*\Theta(B) \ dt$$

where Γ^M is the Levi-Civita connection associated to the Riemannian metric.

Proof: By the definition of τ_F, we have that

$$\int \beta_F^*\Theta(dB, dB) = \int tr\beta_F^*\Theta(B) \ dt = \int \tau_F^*\Theta(B) \ dt.$$

For the first equality see [4] Proposition 5.18. The result follows from the geometric Itô formula.

□

Corollary 2. *Let M be a Riemannian manifold, N be a manifold endowed with a connection Γ^N and $F : M \times [0,T] \to N$ be a smooth map. Let B be an M-valued Brownian motion and Θ be an 1-form on N. Then*

$$\int \Theta \ d^{\Gamma^N} F(B,S) = \int F_S^*\Theta \ d^{\Gamma^M} B + \int \Big(- \frac{dF_S}{dt} + \frac{1}{2}\tau_{F_S}\Big)^*\Theta(B) \ dt$$

where S_t is the finite variation processes $T - t$ and Γ^M is the Levi-Civita connection associated to the Riemannian metric.

Proof: We consider in $M \times [0,T]$ the product connection $\Gamma^N \times \Gamma$ where Γ is the standard connection of the interval $[0,T]$, the geometric Itô formula gives

$$\int \Theta \ d^{\Gamma^N} F(B,S) = \int F^*\Theta \ d^{\Gamma^M\times\Gamma}(B,S) + \frac{1}{2}\int \beta_F^*\Theta(d(B,S)d(B,S)). \quad (4)$$

By the good properties of the Itô integral respect to the product connection (see [5] Proposition 3.15 pp 50),

$$\begin{aligned}\int F^*\Theta\ d^{\Gamma^M\times\Gamma}(B,S) &= \int F(\cdot,S)^*\Theta\ d^{\Gamma^M}B + \int F(B,\cdot)^*\Theta\ d^{\Gamma}S \qquad (5)\\ &= \int F_S^*\Theta\ d^{\Gamma^M}B - \int \Theta(\frac{dF}{dt}(B,S))\ dt.\end{aligned}$$

As S is a finite variation processes an easy calculation shows that

$$\begin{aligned}\int \beta_F^*\Theta(d(B,S),d(B,S)) &= \int \beta_{F_S}^*\Theta(dB,dB) \qquad (6)\\ &= \int \tau_{F_S}^*\Theta(B)\ dt\end{aligned}$$

Substituting (5) and (6) in (4) we conclude that

$$\int \Theta\ d^{\Gamma^N}F(B,S) = \int F_S^*\Theta\ d^{\Gamma^M}B + \int \Big(-\frac{dF_S}{dt} + \frac{1}{2}\tau_{F_S}\Big)^*\Theta(B)\ dt.$$

□

As a direct consequence of Corollary 1, we have the following result of Bismut (see e.g. [8] pp 54, [2] pp 224 and [4] Proposition 5.28).

Proposition 3. *Let M be a Riemannian manifold, N be a manifold equipped with a connection Γ^N and $F: M \to N$ be a smooth map. Then F is harmonic if and only if, for every M-valued Brownian motion B, $F(B)$ is a Γ^N-martingale.*

Proof: Let F be harmonic and B be an M-valued Brownian motion. Let Θ be an 1-form on N. From $\tau_F = 0$ and the Corollary 1, it follows that

$$\int \Theta\ d^{\Gamma^N}F(B) = \int F^*\Theta\ d^{\Gamma^M}B.$$

Hence $\int \Theta\ d^{\Gamma^N}F(B)$ is a local martingale.

Conversely, if F transforms Brownian motions into Γ^N-martingales, from the Corollary 1 and the Doob-Meyer decomposition we have that $\tau_F^*\Theta = 0$ for every 1-form Θ on N. We conclude that $\tau_F = 0$.

□

From the Corollary 2 and the Doob-Meyer decomposition we obtain the following stochastic characterization of the solutions of the heat equation,

Proposition 4. *Let M be a Riemannian manifold, N be a manifold endowed with a connection Γ^N and $F : M \times [0,T] \to N$ be a smooth map. Then F is solution of the heat equation $\frac{dF_t}{dt} = \frac{1}{2}\tau_{F_t}$ if and only if, for every M-valued Brownian motion B, $F(B_t, T-t)$ is a Γ^N-martingale.*

We recall that a smooth map $F : M \to N$ between Riemannian manifolds is a *Riemannian submersion* if $F_*(x)|(Ker\ F_*)^\perp$ is an isometry for every $x \in M$ (see [3]).

The following proposition states that a smooth map is a harmonic Riemannian submersion if and only if it sends Brownian motions into Brownian motions (see [10] pp 207). Let us mention a simple consequence of this, a smooth map $F : M \to M$ is an isometry if and only if sends Brownian motions into Brownian motions.

Proposition 5. *Let M and N be Riemannian manifolds and $F : M \to N$ be a smooth map. Then F is a harmonic Riemannian submersion if and only if, for every M-valued Brownian motion B, $F(B)$ is a N-valued Brownian motion.*

Proof: Let F be a harmonic Riemannian submersion and B be an M-valued Brownian motion, from the above proposition, we have that $F(B)$ is a Γ^N-martingale. Let $f \in \mathcal{C}^\infty(N)$, we first observe that $\nabla(f \circ F) = F^\dagger(\nabla f)$ where $F^\dagger$

is the adjoint of F_*. Applying the Levy characterization of Brownian motions (see e.g. [4] Proposition 5.18) we obtain that

$$[f \circ F(B), f \circ F(B)] = [(f \circ F)(B), (f \circ F)(B)] = \int \|\nabla(f \circ F)\|^2(B)dt.$$

On the other hand,

$$\|\nabla(f \circ F)\|^2 = \|F^\dagger(\nabla f)\|^2 \circ F = \|(\nabla f)\|^2 \circ F$$

since $F^\dagger$ is an isometry. We conclude that

$$[f \circ F(B), f \circ F(B)] = \int \|(\nabla f)\|^2(F(B))dt,$$

and a new application of the Levy characterization gives the result.

Conversely, if F transforms M-valued Brownian motions into N-valued Brownian motions, from the above proposition we have that $\tau_F = 0$. Let $\Theta = df$, then

$$\int df\ d^{\Gamma^N} F(B) = \int d(f \circ F)\ d^{\Gamma^M} B.$$

From the properties of the Itô integral (see [4] Proposition 7.34) and the fact that B and $F(B)$ are Brownian motions,

$$f \circ F(B) - f \circ F(B_0) - \frac{1}{2}\int \Delta_N f(F(B))dt = f \circ F(B) - f \circ F(B_0) - \frac{1}{2}\int \Delta_M (f \circ F)(B)dt.$$

It follows that $(\Delta_N f) \circ F = \Delta_M(f \circ F)$ and taking the product $f = g \cdot h$, we obtain that

$$< \nabla(g \circ F), \nabla(h \circ F) >=< \nabla g, \nabla h > \circ F.$$

We conclude that $F_*|(KerF_*)^\perp$ is an isometry, i.e. F is a Riemannian submersion.

$\square$

Example 1. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a smooth map such that it transforms Brownian motions into Brownian motions, we have that F is a harmonic Riemannian submersion. Applying Theorem 2.1 of [15] we have that $F^{-1}(x)$, for $x \in \mathbb{R}^2$ are 1-dimensional minimal submanifolds of $\mathbb{R}^3$, hence $F^{-1}(x)$is a line. Let H be a plane orthogonal to $F^{-1}(x)$, and P_H be the orthogonal projector onto H. Obviously, P_H is an harmonic Riemannian submersion and $F = (F|H) \circ P_H$. Identifying H with $\mathbb{R}^2$ we have that $F|H$ send $\mathbb{R}^2$-Brownian motions into $\mathbb{R}^2$-Brownian motions, else is an isometry. We obtain that

$$F(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1, a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2)$$

where $\{(a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23})\}$ is an orthonormal set.

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Departamento de Matemática
Universidade Estadual de Campinas
13.081-970 - Campinas - SP, Brazil
E-mail: pedrojc@ime.unicamp.br