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FINITE DIMENSIONAL GRADIENT LIE ALGEBRAS OF IMMERSIONS

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Dedicated to Franco Mercuri on occasion of his 60th birthday

Abstract

The subject of this paper is the group of diffeomorphisms generated by the flows of the gradient vector fields of the height functions on a manifold M immersed on an Euclidean space. A related object is the Lie algebra generated by those vector fields. The issue is to find those immersions such that the gradient Lie algebra is finite dimensional. It is proved that if M is a curve then it is analytic and the immersion must be on a plane. A classification of such curves is given. In higher dimension it is proved that the Lie algebra is semi-simple if M is compact and the holonomy group has no trivial subrepresentations. In this case the isotropy Lie algebra is parabolic, so that M is a covering of a flag manifold.

1 Introduction

Let $f : M^n \to \mathbb{R}^N$ be an immersion of the manifold M into the Euclidian space $\mathbb{R}^N$ with inner product $\langle \cdot, \cdot \rangle$. By the gradient group of $f : M \to \mathbb{R}^N$ we mean the group of diffeomorphisms generated by the flows of the gradients of the height functions. Precisely, for $v \in \mathbb{R}^N$ let $h_v(x) = \langle v, x \rangle$ be the corresponding height function and denote by $X^v = \mathrm{grad} h_v$ its gradient with respect to the induced Riemannian metric on M. It is well known, and easy to prove, that $X^v(x)$ is the orthogonal projection of v into T_xM (in fact, $dh_v(u) = \langle v, u \rangle = \langle \mathrm{proj}_{T_yM} v, u \rangle$ if u, v are tangent vectors).

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We assume throughout that M is *substantial*, which means that it is not contained in an affine subspace of $\mathbb{R}^n$. This implies that the map that associates $v \in \mathbb{R}^n$ to the gradient vector field X^v is injective.

Write X_t^v for the flow of X^v and perform all the possible compositions

$$\mathrm{gr}\,(f) = \{X_{t_1}^{v_1} \circ \cdots \circ X_{t_k}^{v_k} : v_i \in \mathbb{R}^N, t_i \in \mathbb{R}, k \geq 1\}.$$

The elements of $\mathrm{gr}\,(f)$ are local diffeomorphisms of M and we call $\mathrm{gr}\,(f)$ the *gradient group* of the immersion $f : M \to \mathbb{R}^N$. Of course the term "group" here is misleading since this set of local diffeomorphisms may not be a group due to the restrictions in the compositions. However if the vector fields X^v are complete (e.g. if M is compact) then $\mathrm{gr}\,(f)$ is a *bona fide* subgroup of the diffeomorphism group $\mathrm{Diff}\,(M)$ of M.

In this article we address the problem of finding those immersions $f : M \to \mathbb{R}^N$ such that $\mathrm{gr}\,(f)$ is finite dimensional.

Here by the dimension of $\mathrm{gr}\,(f)$ we understand the dimension of its Lie algebra in the following sense: Let us suppose once and for all that M and f are smooth ($\mathcal{C}^\infty$). Then the vector fields X^v are smooth as well. Hence it makes sense to take successive Lie brackets of these vector fields and generate a Lie subalgebra of the Lie algebra of vector fields of M. We denote this subalgebra by $\mathfrak{gr}\,(f)$, which is the smallest subalgebra containing X^v, $v \in \mathbb{R}^N$. We say that $\mathrm{gr}\,(f)$ is finite dimensional if $\dim \mathfrak{gr}\,(f) < \infty$, so that our problem is to determine the immersions $f : M \to \mathbb{R}^N$ such that $\mathfrak{gr}\,(f)$ is finite dimensional.

We note that by a classical theorem of Palais the flows of the vector fields $X \in \mathfrak{gr}\,(f)$ generate a Lie group G in case $\dim \mathfrak{gr}\,(f) < \infty$ and the vector fields $X \in \mathfrak{gr}\,(f)$ are complete. In this case the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{gr}\,(f)$ and the natural action of G on M turns M into a homogenous space such that $\mathfrak{gr}(f)$ is obtained by the infinitesimal action of $\mathfrak{g}$. It follows that $\mathrm{gr}(f) = G$ is a Lie group with Lie algebra $\mathfrak{gr}(f)$. Furthermore G is connected and its action on M is transitive and effective (by the very construction of G as a group of diffeomorphisms).

This paper has two independent but complementary parts. In the first one we consider the case of curves ($\dim M = 1$) in $\mathbb{R}^N$ and determine all the immersed smooth curves such that the Lie algebra $\mathfrak{gr}(f)$ is finite dimensional. We prove in Section 2 that such a curve must be analytic, so that we can use Taylor expansions techniques. Now the basic result that makes a classification available is Corollary 3.3 which shows that $\dim \mathfrak{gr}(f) \leq 3$ if $\dim M = 1$ and $\mathfrak{gr}(f)$ is finite dimensional. This has as a consequence that the ambient Euclidean space $\mathbb{R}^N$ has dimension at most three. Afterwards (see Theorem 3.5) we show that dimension three is not allowed as well, so that a curve with finite dimensional gradient group is contained in a plane.

This way we proceed to classify the curves by the $\dim \mathfrak{gr}(f)$. The case is trivial if $\dim \mathfrak{gr}(f) = 1$ because M must be a piece of a straight line. If $\dim \mathfrak{gr}(f) = 2$ there exists essentially just one curve also, which is given explicitly in Theorem 4.1. On the other hand those curves having $\dim \mathfrak{gr}(f) = 3$ are given as the solutions of an ordinary differential equation (see the classification Theorem 6.2).

We start to look at higher dimensional manifolds at Section 7. We first prove a result relating $\mathrm{gr}(f)$ to the holonomy group of M, namely that the later is contained in the linear isotropy group of $\mathrm{gr}(f)$. This result opens the way to apply Lie group methods when the manifold M satisfies the following two conditions:

1. M is compact, and
2. the representation of the holonomy Lie algebra on the tangent space has no trivial subrepresentations.

The second of these conditons rules out the case where $\dim M = 1$, treated in the first part. This is because a 1-dimensional real representation of the holonomy algebra (a Lie subalgebra of $\mathfrak{so}(n)$) is trivial. Hence the first part of the paper complements the second one.

Now, the condition on the holonomy algebra implies that the isotropy subalgebra of the action of $\mathrm{gr}(f)$ on M coincides with its normalizer. This is used to prove that $\mathrm{gr}(f)$ is a semi-simple Lie group (by force noncompact) and has finite center. In our approach compactness of M is essential to ensure the existence of fixed points of some subgroups of $\mathrm{gr}(f)$. Afterwards we make an analysis of the gradient vector fields on homogeneous spaces to prove that the isotropy subalgebra of the action of $\mathrm{gr}(f)$ on M is parabolic, so that M is a covering of a flag manifold of $\mathrm{gr}(f)$.

We conclude this introduction by saying that one of the sources of the gradient group of an immersion goes back to the construction of the Brownian motion on M made by Itô [9] in the nineteen fourties. This is given as the solution of a stochastic differential equation whose coefficients are the gradient vector fields X^{e_i}, where e_i, $i = 1, \ldots, N$, is an orthonormal basis of $\mathbb{R}^N$. (The point here is that the second order operator $\sum_i (X^{e_i})^2$ is the Laplace-Beltrami operator of the metric on M.) Later developments showed that the flow of that stochastic differential equation evolves on the gradient group, so that the knowledge of $\mathrm{gr}(f)$ (or $\mathfrak{gr}(f)$) is relevant to understand the Brownian motion. To the best of our knowledge this group, and algebra, has not been studied extensively, in the finite or infinite dimensional case. They may prove to be interesting invariants of the immersions.

2 Real analytic curves

In this section we prove that every regular smooth curve such that $\dim \mathfrak{gr}(f) < \infty$ is real analytic.

Let $f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ be a regular smooth curve such that $\dim \mathfrak{gr}(f) < \infty$. Let $s : M \to I \subset \mathbb{R}$ be an arclength coordinate system. We say that f is real analytic if f_i, $i = 1, \ldots, n$, are real analytic functions with respect to s.

A vector field $X : M \to TM$ is written as $X(s) = \sigma(s) \frac{d}{ds}$ where $\sigma(s) = \langle X(s), f'(s) \rangle : M \to \mathbb{R}$ is a real function. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and put $X^i = X^{e_i} : M \to TM$, $i = 1, \ldots, n$, for the gradient of the height function with respect to e_i.

It is easy to see that

$$X^i = f_i' \frac{d}{ds}. \tag{1}$$

The gradient vector fields X^i, $i = 1, \ldots, n$, form a linearly independent subset of $\mathfrak{gr}(f)$. We complement it to a basis $\{X^1, \ldots, X^d\}$ of $\mathfrak{gr}(f)$. The vector fields $X^{n+1}, \ldots, X^d$ are not necessarily gradient. We write these vector fields as $X^i = \sigma_i \frac{d}{ds}$. Their bracket is given by

$$[X^i, X^j] = (\sigma_i {\sigma_j}' - \sigma_j \sigma_i') \frac{d}{ds} \tag{2}$$

and these brackets are linear combinations with constant coefficients of $\{X^1, \ldots, X^d\}$. In the sequel we write $[\sigma_i, \sigma_j] = \sigma_i {\sigma_j}' - \sigma_j \sigma_i'$, that is, we identify X^i with σ_i.

Note that since f is given in arclength we have

$$\sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \left(f_i^{'}\right)^2 = 1. \tag{3}$$

We are ready to prove the following theorem.

Theorem 2.1 *Let $f : M \to \mathbb{R}^n$ be a regular smooth curve such that $\mathfrak{gr}(f)$ is finite dimensional. Then f is real analytic.*

Proof: We prove that the functions $\sigma_1, \ldots, \sigma_d$ are the solutions of an analytic system of ordinary differential equations. This implies that these functions are analytic. In particular this happens to $\sigma_1, \ldots, \sigma_n$, so that $f' = (\sigma_1, \ldots, \sigma_n)$ is analytic as well.

Fix $j = 1, \ldots, d$. Equation (2) implies that

$$\sum_{k=1}^{n} [\sigma_k, \sigma_j]\sigma_k = \left(\sum_{k=1}^{n} \sigma_k^2\right) \sigma_j^{'} - \sigma_j \left(\sum_{k=1}^{n} \sigma_k \sigma_k^{'}\right).$$

Equation (3) and its derivative implies that $\sum_{k=1}^{n} {\sigma_k}^2 = 1$ and $\sum_{k=1}^{n} \sigma_k \sigma_k^{'} = 0$, respectively. Therefore

$$\sum_{k=1}^{n} [\sigma_k, \sigma_j]\sigma_k = \sigma_j^{'}$$

for every $j = 1, \ldots, d$.

Now we write $\sum_{k=1}^{n} [\sigma_k, \sigma_j]\sigma_k$ in another fashion. The term $[\sigma_k, \sigma_j]$ is a linear combination with constant coefficients of $\sigma_1, \ldots, \sigma_d$ for every $j, k = 1, \ldots, d$. Then $\sum_{k=1}^{n} [\sigma_k, \sigma_j]\sigma_k$ is a sum of products of $\sigma_1, \ldots, \sigma_d$ for every $j = 1, \ldots, d$.

Therefore $\sigma_1, \ldots, \sigma_d$ is the solution of the system of ordinary differential equations

$$\begin{cases} \sigma_1'(s) &= \sum_{l,m=1}^{d} a_{1lm}\sigma_l(s)\sigma_m(s) \\ \sigma_2'(s) &= \sum_{l,m=1}^{d} a_{2lm}\sigma_l(s)\sigma_m(s) \\ \vdots & \vdots \qquad \vdots \\ \sigma_d'(s) &= \sum_{l,m=1}^{d} a_{dlm}\sigma_l(s)\sigma_m(s) \end{cases} \tag{4}$$

with their respective initial conditions, where a_{jlm} are suitable real numbers. It is well known that (4) has a unique real analytic solution (see, for instance, [1]), which settles the theorem.

$\square$

3 Restrictions on the dimensions

Let $f : M^1 \to \mathbb{R}^n$ be a smooth regular curve. Let $s : M \to I \subset \mathbb{R}$ be an arclength coordinate system of M. Theorem 2.1 states that f is real analytic. In particular, Equations (1) and (2) imply that the vector fields in $\mathfrak{gr}(f)$ are real analytic. Let $X \in \mathfrak{gr}(f)$ and fix $x_0 \in M$. We say that $X = \sigma\frac{d}{ds}$ has a singularity of order m at x_0 if the Taylor series of σ centered at $s(x_0)$ is written as

$$\sigma(s) = \sum_{i=m}^{\infty} a_i(s - s(x_0))^i$$

with $a_m \neq 0$. Set

$$\mathcal{O}_m(x_0) = \{X \in \mathfrak{gr}(f) : X \text{ has a singularity of order } \geq m \text{ at } x_0\}.$$

Note that $\mathcal{O}_m(x_0)$ is a subalgebra of $\mathfrak{gr}(f)$.

Before proceeding we note the following fact which is an easy consequence of the bracket formula (2).

Proposition 3.1 *If X and Y are vector fields of a smooth curve M then $[X,Y]=0$ if and only if they are linearly dependent (over $\mathbb{R}$).*

Theorem 3.2 *Let $f: M^1 \to \mathbb{R}^n$ be a real analytic regular curve. If there exist a point $x_0 \in M$ such that $\dim(\mathcal{O}_2(x_0)) \geq 2$, then $\mathfrak{gr}(f)$ is infinite dimensional.*

Proof: Let $X = \sigma \frac{d}{ds}$ and $Y = \tau \frac{d}{ds}$ be two independent vector fields in $\mathcal{O}_2(x_0)$. These vector fields are written respectively by

$$\sigma(s) = \sum_{i=m_X}^{\infty} a_i (s - s(x_0))^i$$

and

$$\tau(s) = \sum_{i=m_Y}^{\infty} b_i (s - s(x_0))^i$$

where the orders m_X and m_Y of the singularities of X and Y, respectively are ≥ 2. An easy computation with brackets shows that $[X,Y]$ has singularity at x_0 of order $\geq m_X + m_Y - 1 > \max\{m_X, m_Y\}$. Also $[X,Y] \neq 0$ by Proposition 3.1. Because of the orders at x_0 the vector fields X, Y and $[X,Y]$ are linearly independent.

Repeating the argument with X and $[X,Y]$ we produce another non trivial vector field $[X,[X,Y]]$, which singularity has order greater than the order of the singularity of $[X,Y]$ at x_0. Proceeding successively we get a sequence X, $[X,Y]$, $[X,[X,Y]]$, $[X,[X,[X,Y]]]$, $[X,[X,[X,[X,Y]]]]$ and so on, of non trivial vector fields with strictly increasing orders of singularities at x_0. Therefore $\mathcal{O}_2(x_0)$ is an infinite dimensional subspace of $\mathfrak{gr}(f)$, proving the theorem.

□

Corollary 3.3 *Let $f: M^1 \to \mathbb{R}^n$ be a real analytic regular curve. Then $\mathfrak{gr}(f)$ is infinite dimensional if $\dim \mathfrak{gr}(f) > 3$.*

Proof: Fix $x_0 \in M$ and suppose that we have in $\mathfrak{gr}(f)$ four linearly independent vector fields $X_j = \sigma_j \frac{d}{ds}$. Write the Taylor series about $s(x_0)$ as $\sigma_j\,(s) = \sum_{i=0}^{\infty} a_{ji}.(s - s(x_0))^i$. Making suitable linear combinations of X_j (similar to the Gauss elimination of a system of linear equations) we get new vector fields $\widetilde{X}_j = \widetilde{\sigma}_j \frac{d}{ds}$ with

$$\begin{cases} \widetilde{\sigma}_1(s) &= \sum_{i=m_1}^{\infty} b_{1i}.(s - s(x_0))^i \\ \widetilde{\sigma}_2(s) &= \sum_{i=m_2}^{\infty} b_{2i}.(s - s(x_0))^i \\ \widetilde{\sigma}_3(s) &= \sum_{i=m_3}^{\infty} b_{3i}.(s - s(x_0))^i \\ \widetilde{\sigma}_4(s) &= \sum_{i=m_4}^{\infty} b_{4i}.(s - s(x_0))^i \end{cases}$$

with $m_1 < m_2 < m_3 < m_4$ and $b_{jm_j} \neq 0$ for $j = 1, \ldots, 4$. Therefore $\dim(\mathcal{O}_2(x_0)) \geq 2$ and the result follows.

□

As a consequence we get the following restriction on the dimension of the ambient Euclidean space.

Corollary 3.4 *Let $f : M \to \mathbb{R}^n$ be a regular smooth curve such that $\mathfrak{gr}(f)$ is finite dimensional. Then $f(M)$ is contained in an affine tridimensional subspace of $\mathbb{R}^n$.*

Proof: In fact let $A \subset \mathbb{R}^n$ be the affine subspace spanned by M and denote by V the vector subspace parallel to A. Then the map $v \in V \mapsto X^v \in \mathfrak{gr}(f)$ is injective, so that $\dim V \leq 3$.

□

Therefore a substantial curve with finite dimensional gradient group is immersed in $\mathbb{R}^3$. Our next result improve this by lowering the dimension to two.

Theorem 3.5 *Let $f : M \to \mathbb{R}^3$ be a substantial regular smooth curve. Then $\mathfrak{gr}(f)$ is infinite dimensional.*

Proof: Assume by contradiction that $\dim \mathfrak{gr}(f) < \infty$. There exist a non-flat point $p \in M$. Without loss of generality we can consider that $f(p)$ is the origin of $\mathbb{R}^3$, that f is tangent to the x-axis at p and that the principal normal has the same direction as the y-axis at p. In addition we can make a dilation in such a way that the curvature of the curve at p is equal to one. Let $s : M \to I \subset \mathbb{R}$ be the arclength coordinate system such that $f(0) = (0, 0, 0)$ and $f'(0) = e_1$.

Denote the curvature function of f by κ. Note that $\dim \mathfrak{gr}(f) = 3$ because the basic gradient vector fields $X^i = \sigma_i \frac{d}{ds}$, $i = 1, 2, 3$, form a linearly independent subset of $\mathfrak{gr}(f)$.

We have also that $\sigma_1(0) = 1$, $\sigma_2(0) = \sigma_3(0) = 0$, $\sigma_1'(0) = \sigma_3'(0) = 0$ and $\sigma_2'(0) = \kappa(p) = 1$ due to the arrangements made to the curve and the coordinate system. The Taylor series of $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1$ implies that $\sigma_1''(0) = -1$. Then the Taylor series of σ_1, σ_2 and σ_3 are given by

$$\begin{aligned}
\sigma_1(s) &= 1 - \tfrac{1}{2}s^2 + \sum_{i=3}^{\infty} \frac{\sigma_1^{(i)}(0)}{i!} s^i \\
\sigma_2(s) &= s + \sum_{i=2}^{\infty} \frac{\sigma_2^{(i)}(0)}{i!} s^i \\
\sigma_3(s) &= \sum_{i=2}^{\infty} \frac{\sigma_3^{(i)}(0)}{i!} s^i .
\end{aligned}$$

The Lie brackets are given by

$$\begin{array}{rcl}[\sigma_1,\sigma_2] &=& \sigma_1+\sigma_2''(0)\sigma_2+A\sigma_3\\ [\sigma_1,\sigma_3] &=& \sigma_3''(0)\sigma_2+B\sigma_3\\ [\sigma_2,\sigma_3] &=& \frac{1}{2}\sigma_3''(0)\sigma_3.\end{array}$$

So that the Jacobi identity gives

$$\left(\sigma_3''(0)-\frac{1}{2}(\sigma_3''(0))^2\right)\sigma_2+\left(B+\frac{\sigma_2''(0)\sigma_3''(0)}{2}\right)\sigma_3=0, \tag{5}$$

which implies that both coefficients are equal to zero.

The coefficient of σ_2 in (5) gives that either $\sigma_3''(0)=0$ or $\sigma_3''(0)=2$. But the first case is impossible, because this would imply that σ_2 is proportional to σ_3. Hence $\sigma_3''(0)=2$. The other coefficient in (5) gives $B=-\sigma_2''(0)$. Thus the brackets become

$$\begin{array}{rcl}[\sigma_1,\sigma_2] &=& \sigma_1+\sigma_2''(0)\sigma_2+A\sigma_3\\ [\sigma_1,\sigma_3] &=& 2\sigma_2-\sigma_2''(0)\sigma_3\\ [\sigma_2,\sigma_3] &=& \sigma_3.\end{array}$$

We can use the same procedure used in Theorem 2.1 (see (4)) in order to isolate σ_1', σ_2' and σ_3'. We get

$$\begin{array}{rcl}\sigma_1' &=& -\sigma_1\sigma_2-\sigma_2''(0)\sigma_2^2-(A+2)\sigma_2\sigma_3+\sigma_2''(0)\sigma_3^2\\ \sigma_2' &=& \sigma_1^2+\sigma_2''(0)\sigma_2\sigma_3+A\sigma_1\sigma_3-\sigma_3^2\\ \sigma_3' &=& 2\sigma_1\sigma_2-\sigma_2''(0)\sigma_1\sigma_3+\sigma_2\sigma_3.\end{array} \tag{6}$$

If we calculate $[\sigma_1,\sigma_2]$ using (6), we get

$$[\sigma_1,\sigma_2]=\sigma_1+\sigma_2''(0)\sigma_2+A\sigma_3-2\sigma_1\sigma_3^2-2d\sigma_2\sigma_3^2+2\sigma_2^2\sigma_3-A\sigma_3^3$$

what implies that $-2\sigma_1\sigma_3^2-2d\sigma_2\sigma_3^2+2\sigma_2^2\sigma_3-A\sigma_3^3=O(s^4)$ is a non-trivial element of $\mathfrak{gr}(f)$. But Theorem 3.2 states that if $\dim\mathcal{O}_2(x))\geq 2$ for some $x\in M$, then $\mathfrak{gr}(f)$ is infinite dimensional, arriving at a contradiction.

□

4 Two dimensional groups

In view of Theorem 3.5 we consider from now on curves in $\mathbb{R}^2$. When $\dim \mathfrak{gr}(f) = 1$ it is easy to see that the curve must be a piece of straight line in Euclidean space. The next theorem shows that in case $\dim \mathfrak{gr}(f) = 2$ there is also just one possible curve.

Theorem 4.1 *Let $f : M \to \mathbb{R}^2$ be a regular smooth curve such that $\dim \mathfrak{gr}(f) = 2$. Then there exist an arclength parameter s such that f is a piece of the curve*

$$s \longmapsto (2\arctan(e^s) - \pi/4, \log(\cosh(s)))$$

composed with isometries and dilations of $\mathbb{R}^2$.

Proof: Take a non-flat point $p \in M$ which is assumed to be the origin of the plane and such that the curve is tangent to the x-axis at p. In addition we can make a dilation so that the curvature at p is equal to one. Let $s : M \to I \subset \mathbb{R}$ be a arclength coordinate system such that $f(0) = (0,0)$. The basic gradient vector fields are $X^1 = \sigma_1 \frac{d}{ds} = f_1' \frac{d}{ds}$ and $X^2 = \sigma_2 \frac{d}{ds} = f_2' \frac{d}{ds}$. We fix the orientation on M given by the normal vector field $(-\sigma_2, \sigma_1)$. Using elementary theory of curves in Euclidean spaces, we have that the Taylor series of X^1 and X^2 about 0 are

$$\sigma_1(s) = 1 - \frac{1}{2}s^2 + O(s^3)$$

and

$$\sigma_2(s) = s + \frac{\sigma_2''(0)}{2}s^2 + O(s^3).$$

Therefore the curvature is given by

$$\kappa = [\sigma_1, \sigma_2] = \sigma_1 + \sigma_2''(0)\sigma_2$$

due to de dimension of $\mathfrak{gr}(f)$.

Now we are going to get a more convenient basis for $\mathfrak{gr}(f)$. Consider the gradient of the height function with respect to the vectors $(\cos\theta, -\sin\theta)$ and $(\sin\theta, \cos\theta)$, which are given respectively by $X_1^\theta = \cos\theta X^1 - \sin\theta X^2 = \sigma_1^\theta \frac{d}{ds}$ and $X_2^\theta = \sin\theta X^1 + \cos\theta X^2 = \sigma_2^\theta \frac{d}{ds}$. It is easy to see that $\kappa^\theta := [X_1^\theta, X_2^\theta] = [X^1, X^2] = \kappa$.

We can choose θ such that $(1, \sigma_2''(0))$ (the first two terms of the Taylor series of κ) is linearly dependent with $(\cos\theta, -\sin\theta)$ (the first two terms of the Taylor series of σ_1^θ) and such that $\cos\theta > 0$. Then σ_1^θ and κ are linearly dependent and

$$[\sigma_1^\theta, \sigma_2^\theta] = \sigma_1^\theta(\sigma_2^\theta)' - (\sigma_1^\theta)'\sigma_2^\theta = \sec\theta\sigma_1^\theta. \tag{7}$$

This equation, together with

$$(\sigma_1^\theta)^2 + (\sigma_2^\theta)^2 = 1 \tag{8}$$

and its derivative

$$\sigma_1^\theta(\sigma_1^\theta)' + \sigma_2^\theta(\sigma_2^\theta)' = 0 \tag{9}$$

are enough to find σ_1^θ and σ_2^θ explicitly. In fact, first of all we multiply Equation (7) by σ_1^θ, Equation (9) by σ_2^θ, sum the resulting equations (considering Equation (8)) in order to get

$$(\sigma_2^\theta)' = \sec\theta(1 - (\sigma_2^\theta)^2) \tag{10}$$

which must satisfy the initial condition $\sigma_2^\theta(0) = \sin\theta$. Proceeding in a similar way, we can see that

$$(\sigma_1^\theta)' = -\sec\theta\sigma_1^\theta\sigma_2^\theta \tag{11}$$

with the initial condition $\sigma_1^\theta(0) = \cos\theta$. Equations (10) and (11) are explicitly solvable and f is a piece of the curve

$$s \mapsto (2\arctan(e^s) - \pi/4, \log(\cosh(s))), \tag{12}$$

composed with isometries and dilations of $\mathbb{R}^2$, as we wanted to prove.

□

5 A one-parameter family of plane curves

In this section we study some properties of a one-parameter family of curves. These curves are essentially all the regular smooth curves such that $\dim \mathfrak{gr}(f)$ $= 2$ or 3.

Let $f = (f_1, f_2) : M \to \mathbb{R}^2$ be a smooth regular curve. The orientation of f will be given by the normal vector field $(-\sigma_2, \sigma_1)$ where, as before $X^1 = \sigma_1 \frac{d}{ds}$ and $X^2 = \sigma_2 \frac{d}{ds}$ are the basic gradient vector fields. Then the curvature function of f is given by $\kappa = [\sigma_1, \sigma_2]$. We are interested to study curves that satisfies the following system of ordinary differential equations

$$\left\{ \begin{array}{rcl} f_1'(s) & = & \sigma_1(s) \\ f_2'(s) & = & \sigma_2(s) \\ \sigma_1'(s) & = & -\sigma_2(s)\kappa(s) \\ \sigma_2'(s) & = & \sigma_1(s).\kappa(s) \\ \kappa'(s) & = & C\sigma_1(s)\sigma_2(s) \\ f_1(0) & = & 0 \\ f_2(0) & = & 0 \\ \sigma_1(0) & = & 1 \\ \sigma_2(0) & = & 0 \\ \kappa(0) & = & 1 \end{array} \right. \tag{13}$$

where $C \in \mathbb{R}$ is a fixed parameter.

For every $C \in \mathbb{R}$, the solution of (13) is a real analytic curve. It is a curve such that in a neighborhood of $s = 0$ it is placed above the x-axis and it has its concavity pointed upwards. Therefore there exist $\varepsilon > 0$ such that $\sigma_1(s)$ and $\sigma_2(s)$ are strictly positive for every $s \in (0, \varepsilon)$. If we multiply the third and the fifth equations of (13) we get

$$C\sigma_1\sigma_1'\sigma_2 = -\sigma_2\kappa\kappa'$$

which in the interval $(0, \varepsilon)$ can be simplified to

$$C\sigma_1\sigma_1' = -\kappa\kappa'.$$

We can integrate both sides in order to get

$$C\sigma_1^2 = -\kappa^2 + D \tag{14}$$

for some $D \in \mathbb{R}$. But σ_1 and κ are real analytic. Therefore (14) holds everywhere with $D = C + 1$ due to the initial conditions. Then

$$C\sigma_1^2 = -\kappa^2 + C + 1. \tag{15}$$

Analogously we can multiply the fourth and the fifth equations of (13) in order to get

$$C\sigma_2^2 = \kappa^2 - 1. \tag{16}$$

Note that the curve is a circle if $C = 0$. Also, if $C = -1$, then the solution of (13) is given by $s \mapsto (2\arctan(e^s) - \pi/4, \ln(\cosh(s)))$, which is the curve studied in Section 4.

Therefore we suppose that $C \neq -1$. In these cases it is possible to prove that the solution is locally the inverse of an elliptic function, although this information is not of much help for our purposes.

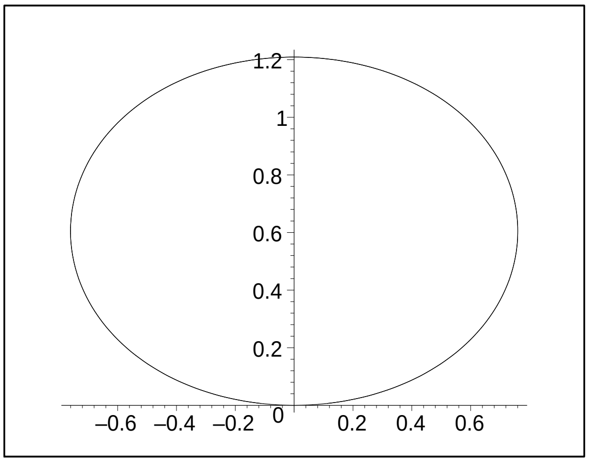

Figure 1: $C = 3$

Let us make a qualitative analysis of these curves.

If $C > -1$, then $\kappa \in [1, 1 + C]$ (or $\kappa \in [1 + C, 1]$) due to (16) and $|\sigma_2| \leq 1$. The positive lower bound of κ and the geometry of f near $s = 0$ implies that there exist a first point $s = T > 0$ such $\sigma_1(T) = 0$ and $\sigma_2(T) = 1$. Write $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) = f|_{[0,T]}$. We claim that the image of f is diffeomorphic to a circle, that it has horizontal and vertical symmetry (see Figure 1) and that the image of $\widetilde{f}$ is exactly one fourth of the image of f. In fact, define $\widehat{f} : [0, 4T] \to \mathbb{R}^2$ by

$$\widehat{f}(s) = \begin{cases} \widetilde{f}(s) & \text{if} \quad s \in [0, T] \\ (\widetilde{f}_1(2T - s), 2\widetilde{f}_2(T) - \widetilde{f}_2(2T - s)) & \text{if} \quad s \in [T, 2T] \\ (-\widetilde{f}_1(s - 2T), \widetilde{f}_2(2T) - \widetilde{f}_2(s - 2T)) & \text{if} \quad s \in [2T, 3T] \\ (-\widetilde{f}_1(4T - s), \widetilde{f}_2(4T - s)) & \text{if} \quad s \in [3T, 4T]. \end{cases}$$

Direct calculations show that $\widehat{f}$ satisfies the system of equations (13) and it image is diffeomorphic to a circle. Moreover, the analytic extension of $\widehat{f}$ to $\mathbb{R}$ is f, what proves that the image of f is diffeomorphic to a circle.

If $C < -1$, then $|\sigma_2| \leq \sqrt{-C^{-1}}$ due to Equation (16). This implies that

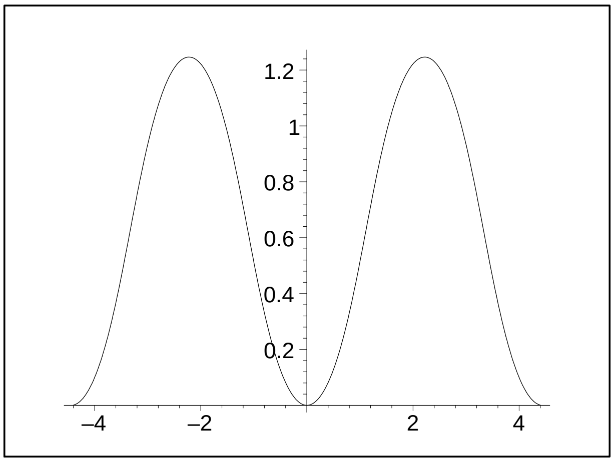

Figure 2: $C = -2$

σ_1 never vanishes. More precisely, $\sigma_1(M) \in [\sqrt{1 + C^{-1}}, 1]$. We claim that κ vanishes at some point. In fact, if this is not the case, then the fourth equation of (13) would imply that σ_2 is always increasing. Fifth equation of (13) would imply that $\kappa'(s) \leq A < 0$ for $s \geq \varepsilon' > 0$ and κ would vanish at some point, what gives the contradiction.

Let $s = T > 0$ be the first positive point such that $\kappa(T) = 0$. As in the case $C > -1$, we will rebuilt f from $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) = f|_{[0,T]}$ using some symmetries. In this case, f is similar to the sine function (see Figure 2 (It is vertically distorted)) and $\widetilde{f}$ is exactly one fourth of a cycle of the periodic function. Define $\widehat{f} : [0, 4T] \to \mathbb{R}^2$ by

$$\widehat{f}(s) \quad = \quad \begin{cases} \widetilde{f}(s) & \text{if} \quad s \in [0,T] \\ (2\widetilde{f}_1(T) - \widetilde{f}_1(2T - s), 2\widetilde{f}_2(T) - \widetilde{f}_2(2T - s)) & \text{if} \quad s \in [T, 2T] \\ (2\widetilde{f}_1(T) + \widetilde{f}_1(s - 2T), 2\widetilde{f}_2(2T) - \widetilde{f}_2(s - 2T)) & \text{if} \quad s \in [2T, 3T] \\ (4\widetilde{f}_1(T) - \widetilde{f}_1(4T - s), \widetilde{f}_2(4T - s)) & \text{if} \quad s \in [3T, 4T]. \end{cases}$$

As in the case $C > -1$, we can use similar arguments and prove that f is the analytic extension of $\widehat{f}$, and therefore f is similar to the sine function.

Remark: Notice that we can get the reversal ($s \mapsto -s$) of the parameter s making a reflection on $\mathbb{R}^2$ due to the symmetries of $f(M)$. We use this fact in Proposition 5.1.

The following proposition characterizes the solution of (13) and its composition with some symmetries of $\mathbb{R}^2$ as a solution of a system of ordinary differential equations.

Proposition 5.1 *Let $C \neq -1$ and $f = (f_1, f_2) : M \to \mathbb{R}^2$ be the solution of*

$$\left\{ \begin{array}{rcl} f_1'(s) & = & \sigma_1(s) \\ f_2'(s) & = & \sigma_2(s) \\ \sigma_1'(s) & = & -\sigma_2(s)\kappa(s) \\ \sigma_2'(s) & = & \sigma_1(s)\kappa(s) \\ \kappa'(s) & = & C\sigma_1(s).\sigma_2(s) \\ f_1(0) & = & 0 \\ f_2(0) & = & 0 \\ \sigma_1(0) & = & \cos\theta_0 \\ \sigma_2(0) & = & \sin\theta_0 \\ \kappa(0) & = & \kappa_0, \end{array} \right. \tag{17}$$

where θ_0 and κ_0 satisfy $C.\sin^2\theta_0 = \kappa_0^2 - 1$. Denote the solution of (13) by $(\widetilde{f}_1, \widetilde{f}_2)$. Then there exist a constant $D \in \mathbb{R}$ such that f is given by $s \mapsto (\widetilde{f}_1(s+D) - \widetilde{f}_1(D), \widetilde{f}_2(s+D) - \widetilde{f}_2(D))$ composed with isometries of $\mathbb{R}^2$.

Proof: The condition $C\sin^2\theta_0 = \kappa_0^2 - 1$ is necessary due to Equation (16). First of all, let us see how the solutions of (17) are influenced by some isometries of $\mathbb{R}^2$.

1. *Reflection with respect to the x-axis and reversal of the parameter s:* If $s \mapsto (f_1(s+D) - f_1(D), f_2(s+D) - f_2(D))$ is the solution of (17), then $s \mapsto (f_1(-s+D) - f_1(D), -f_2(-s+D) + f_2(D))$ is the solution of (17) with the initial condition $\sigma_1(0) = \cos\theta_0$ replaced by $\sigma_1(0) = -\cos\theta_0$.

2. *Reflection with respect to the y-axis and reversal of the parameter s:* If $s \mapsto (f_1(s+D) - f_1(D), f_2(s+D) - f_2(D))$ is the solution of (17), then $s \mapsto (-f_1(-s+D) + f_1(D), f_2(-s+D) - f_2(D))$ is the solution of (17) with the initial condition $\sigma_2(0) = \sin\theta_0$ replaced by $\sigma_2(0) = -\sin\theta_0$.

3. *Reflection with respect to $(0,0) \in \mathbb{R}^2$ and reversal of the parameter s:* If $s \mapsto (f_1(s+D) - f_1(D), f_2(s+D) - f_2(D))$ is the solution of (17), then $s \mapsto (-f_1(-s+D)+f_1(D), -f_2(-s+D)+f_2(D))$ is the solution of (17) the initial condition $\kappa(0) = \kappa_0$ replaced by $\kappa(0) = -\kappa_0$.

Therefore we can combine the isometries given above in order to make $\cos\theta_0$, $\sin\theta_0$ and κ_0 non-negative. But (17) with non-negative values of $\cos\theta_0$, $\sin\theta_0$ and κ_0 are solved by a translation of the solution of (13). Therefore the general solution of (17) is given by $s \mapsto (\widetilde{f_1}(s+D) - \widetilde{f_1}(D), \widetilde{f_2}(s+D) - \widetilde{f_2}(D))$ composed with isometries of $\mathbb{R}^2$, where $D \in \mathbb{R}$ and $(\widetilde{f_1}, \widetilde{f_2})$ is the solution of (13).

$\square$

6 The classification theorem for curves

In this section we classify the regular smooth curves $f = (f_1, f_2) : M \to \mathbb{R}^2$ such that $\dim \mathfrak{gr}(f) = 3$.

Theorem 6.1 *Let $f = (f_1, f_2) : M \to \mathbb{R}^2$ be a smooth regular curve such that $\dim \mathfrak{gr}(f) = 3$. Then there exist an arclength system of coordinates and a constant $C \neq -1$ such that (f_1, f_2) is a piece of the solution of the system of* *ordinary differential equations*

$$\left\{\begin{array}{rcl} f_1'(s) & = & \sigma_1(s) \\ f_2'(s) & = & \sigma_2(s) \\ \sigma_1'(s) & = & -\sigma_2(s)\kappa(s) \\ \sigma_2'(s) & = & \sigma_1(s).\kappa(s) \\ \kappa'(s) & = & C\sigma_1(s)\sigma_2(s) \\ f_1(0) & = & 0 \\ f_2(0) & = & 0 \\ \sigma_1(0) & = & 1 \\ \sigma_2(0) & = & 0 \\ \kappa(0) & = & 1 \end{array}\right. \tag{18}$$

eventually composed with isometries and dilations of $\mathbb{R}^2$*. Conversely every solution of (18) with* $C \neq -1$ *is a curve such that* $\dim \mathfrak{gr}(f) = 3$*.*

Proof: There exist a non-flat point $p \in M$. Without loss of generality, we can consider that $f(p)$ is the origin of the plane, that the curve is tangent to the x-axis at p and that the curve is locally placed above the x-axis. In addition we can make an dilation in such a way that the curvature of the curve at p is equal to one. Let $s : M \to I \subset \mathbb{R}$ be a arclength coordinate system such that $f(0) = (0,0)$ and $f'(0) = e_1$. Then the Taylor series of σ_1 and σ_2 are given respectively by

$$\sigma_1(s) = 1 - \frac{1}{2}.s^2 + O(s^3)$$

and

$$\sigma_2(s) = s + \frac{\sigma_2''(0)}{2}.s^2 + O(s^3).$$

Moreover we have that the Taylor series of $\kappa = [\sigma_1, \sigma_2]$ is given by

$$\kappa(s) = 1 + \sigma_2''(0)s + O(s^2).$$

We have that $\dim \mathfrak{gr}(f) = 3$, what implies that $\{\sigma_1 d/ds, \sigma_2 d/ds, \kappa d/ds\}$ is a basis of $\mathfrak{gr}(f)$. Set $\sigma_1^\theta = \cos\theta\sigma_1 - \sin\theta\sigma_2$ and $\sigma_2^\theta = \sin\theta\sigma_1 + \cos\theta\sigma_2$, where

$\theta \in \mathbb{R}$. Observe that $\{\sigma_1^\theta d/ds, \sigma_2^\theta d/ds, \kappa d/ds\}$ is a basis of $\mathfrak{gr}(f)$ and that $[\sigma_1^\theta, \sigma_2^\theta] = \kappa$ for every $\theta \in \mathbb{R}$. We are looking of an "adequate" value of θ.

Direct calculations show that

$$\begin{cases} \sigma_2^\theta(s) & = & \sin\theta + \cos\theta s + O(s^2) \\ \left[\kappa(s), \sigma_1^\theta(s)\right] & = & -(\cos\theta.\sigma_2''(0) + \sin\theta) - (\cos\theta(\sigma_2^{(3)}(0) + 2) \\ & & + \sin\theta\sigma_2''(0))s + O(s^2) \\ \sigma_1^\theta(s) & = & \cos\theta - \sin\theta s + O(s^2) \\ \left[\sigma_2^\theta(s), \kappa(s)\right] & = & (\sin\theta\sigma_2''(0) - \cos\theta) + (\sin\theta(\sigma_2^{(3)}(0) + 2) \\ & & - \cos\theta\sigma_2''(0)).s + O(s^2) \end{cases} \tag{19}$$

Observe that there exist η, A_1 and A_2 such that the first two terms of the Taylor series of $[\sigma_1^\eta, \kappa] - A_2\sigma_2^\eta$ and $[\sigma_2^\eta, \kappa] - A_1.\sigma_1^\eta$ are simultaneously zero. This happens because the first two terms of the Taylor series of $[\sigma_1^\eta, \kappa]$ is proportional to the first two terms of the Taylor series of σ_2^η if and only if the same happens with the pair $[\sigma_2^\eta, \kappa]$ and σ_1^η. Then

$$\begin{cases} [\sigma_1^\eta, \sigma_2^\eta] & = & \kappa \\ [\sigma_2^\eta, \kappa] - A_1\sigma_1^\eta & = & B_1\sigma_1^\eta + B_2\sigma_2^\eta + B_3\kappa = O(s^2) \\ [\kappa, \sigma_1^\eta] - A_2\sigma_2^\eta & = & C_1\sigma_1^\eta + C_2\sigma_2^\eta + C_3\kappa = O(s^2). \end{cases} \tag{20}$$

Claim: A_1 and A_2 are non-zero constants and $B_i = C_i = 0$ for $i = 1, 2, 3$.

Notice that $[\sigma_2^\eta, \kappa] - A_1\sigma_1^\eta$ and $[\kappa, \sigma_1^\eta] - A_2\sigma_2^\eta$ are linearly dependent (See Theorem 3.2). Thus the vectors (B_1, B_2, B_3) and (C_1, C_2, C_3) are linearly dependent. Taking these facts into consideration and considering the Jacobi identity

$$[[\sigma_1^\eta, \sigma_2^\eta], \kappa] + [[\sigma_2^\eta, \kappa], \sigma_1^\eta] + [[\kappa, \sigma_1^\eta], \sigma_2^\eta] = 0 \tag{21}$$

we have that $A_1.C_3 = A_2.B_3 = C_1 - B_2 = 0$. If $B_3 = 0$, then $B_1\sigma_1^\eta(s) + B_2\sigma_2^\eta(s) = O(s^2)$ what implies that $B_1 = B_2 = 0$. If $C_3 = 0$, we can conclude that $C_1 = C_2 = 0$ analogously. Therefore if A_1 and A_2 are non-zero constants, the claim is proved. We will prove that the other cases do not happen.

If $A_1 = A_2 = 0$, then $[\sigma_1^\eta, \kappa]$ and $[\sigma_2^\eta, \kappa]$ would be linearly dependent and κ would be a linear combination of σ_1^η and σ_2^η, what contradicts $\dim \mathfrak{gr}(f) = 3$.

If $A_1 \neq 0$ and $A_2 = 0$, then $C_1 = C_2 = C_3 = 0$. This would imply that $[\sigma_2^\eta, \kappa] = 0$, that is, σ_2^η and κ are linearly dependent, what gives another contradiction. Therefore the claim is proved.

The system of equations (20) is given by

$$\begin{array}{lcl} [\sigma_1^\eta, \sigma_2^\eta] & = & \kappa \\ [\sigma_2^\eta, \kappa] & = & A_1 \sigma_1^\eta \\ [\kappa, \sigma_1^\eta] & = & A_2 \sigma_2^\eta \end{array} \tag{22}$$

We claim that at least one of the constants A_1 or A_2 is negative. In fact, if $\cos\eta = 0$, then (19) implies that $A_2 = -1$. Analogously, if $\sin\eta = 0$, then (19) implies that $A_1 = -1$. Suppose that $\cos\eta$ and $\sin\eta$ are different from zero. Then A_1 can be written as $\langle (b_2, 1), (\tan\eta, -1)\rangle$ and A_2 can be written as $\langle (b_2, 1), (-\cot\eta, -1)\rangle$ where $\langle\cdot,\cdot\rangle$ is the canonical inner product of $\mathbb{R}^2$ (see (19)). But $(-\cot\eta, -1)$ and $(\tan\eta, -1)$ are orthogonal vectors in the lower half of $\mathbb{R}^2$ and $(b_2, 1)$ is in the upper half of $\mathbb{R}^2$. Therefore by geometrical considerations A_1 or A_2 (or both) must be negative.

We claim that there exist an η such that $A_1 < 0$. In fact if $A_1 > 0$, then we can choose $\widetilde{\eta} = \eta + \pi/2$. This means that $\sigma_1^{\widetilde{\eta}} = \sigma_2^\eta$ and $\sigma_2^{\widetilde{\eta}} = -\sigma_1^\eta$. Then

$$\begin{array}{lcl} \left[\sigma_1^{\widetilde{\eta}}, \sigma_2^{\widetilde{\eta}}\right] & = & \kappa \\ \left[\sigma_2^{\widetilde{\eta}}, \kappa\right] & = & A_2 \sigma_1^{\widetilde{\eta}} \\ \left[\kappa, \sigma_1^{\widetilde{\eta}}\right] & = & A_1 \sigma_2^{\widetilde{\eta}} \end{array} \tag{23}$$

and the claim follows.

Suppose that η was chosen in (22) such that $A_1 < 0$. We can make a dilation in order to make $A_1 = -1$. In fact, consider the curve defined by $\widetilde{f}(s) = c.f(c^{-1}.s)$, where c is the positive square root of $-A_1^{-1}$. Observe that $\widetilde{\sigma}_1(s)\frac{d}{ds} = \sigma_1(c^{-1}.s)\frac{d}{ds}$ and $\widetilde{\sigma}_2(s)\frac{d}{ds} = \sigma_2(c^{-1}.s)\frac{d}{ds}$ are the gradient of the height functions with respect to e_1 and e_2 respectively. Direct calculations show that

$$\left\{\begin{array}{lcl}[\widetilde{\sigma}_1^\eta,\widetilde{\sigma}_2^\eta] & = & \widetilde{\kappa}\\ [\widetilde{\sigma}_2^\eta,\widetilde{\kappa}] & = & -\widetilde{\sigma}_1^\eta\\ [\widetilde{\kappa},\widetilde{\sigma}_1^\eta] & = & \widetilde{A}_2\widetilde{\sigma}_2^\eta\end{array}\right. \tag{24}$$

and hence $A_1 = -1$. Therefore there exist a curve f^η, which is the composition of f with isometries and dilations of $\mathbb{R}^2$ such that

$$\left\{\begin{array}{lcl}[\sigma_1^\eta,\sigma_2^\eta] & = & \kappa\\ [\sigma_2^\eta,\kappa] & = & -\sigma_1^\eta\\ [\kappa,\sigma_1^\eta] & = & A\sigma_2^\eta\end{array}\right. \tag{25}$$

holds for f^η.

Let us prove that (25) is equivalent to the following system of differential equations

$$\left\{\begin{array}{lcl}\sigma_1^{\eta\prime} & = & -\sigma_2^\eta.\kappa\\ \sigma_2^{\eta\prime} & = & \sigma_1^\eta.\kappa\\ \kappa' & = & -(A+1).\sigma_1^\eta.\sigma_2^\eta.\end{array}\right. \tag{26}$$

Take $[\sigma_1^\eta,\sigma_2^\eta] = \sigma_1^\eta.(\sigma_2^\eta)' - (\sigma_1^\eta)'.\sigma_2^\eta = \kappa$ and multiply it by $-\sigma_2^\eta$. Take the derivative of $(\sigma_1^\eta)^2 + (\sigma_2^\eta)^2 = 1$ and multiply it by $\sigma_1^\eta/2$. Sum these equations and it follows that $\sigma_1^{\eta\prime} = -\sigma_2^\eta.\kappa$.

Take $[\sigma_1^\eta,\sigma_2^\eta] = \sigma_1^\eta.(\sigma_2^\eta)' - (\sigma_1^\eta)'.\sigma_2^\eta = \kappa$ again and multiply it by σ_1^η. Take the derivative of $(\sigma_1^\eta)^2+(\sigma_2^\eta)^2 = 1$ and multiply it by $\sigma_1^\eta/2$. Sum these equations and it follows that $\sigma_2^{\eta\prime} = \sigma_1^\eta.\kappa$.

Take $[\sigma_2^\eta,\kappa] = \sigma_2^\eta.\kappa' - (\sigma_2^\eta)'.\kappa = -\sigma_1^\eta$ and multiply by σ_2^η. Take $[\kappa,\sigma_1^\eta] = \kappa.(\sigma_1^\eta)' - \kappa'.\sigma_1^\eta = A_2\sigma_2^\eta$ and multiply it by $-\sigma_1^\eta$. Sum these equations and it follows that $\kappa' = -(A+1).\sigma_1^\eta.\sigma_2^\eta$. Then (25) implies (26).

In order to prove that (26) implies (25) it is enough to notice that $\kappa^2 = (\sigma_1')^2 + (\sigma_2')^2$ and make direct calculations.

Set $e_1^\eta = \cos\eta.e_1 - \sin\eta.e_2$ and $e_2^\eta = \sin\eta.e_1 + \cos\eta.e_2$ and fix $\{e_1^\eta, e_2^\eta\}$ as a basis of $\mathbb{R}^2$. Therefore $f^\eta = (f_1^\eta, f_2^\eta) : M \to \mathbb{R}^2$ with respect to this new basis satisfies

$$\left\{\begin{array}{rcl} f_1^{\eta\prime}(s) & = & \sigma_1^\eta(s) \\ f_2^{\eta\prime}(s) & = & \sigma_2^\eta(s) \\ \sigma_1^{\eta\prime}(s) & = & -\sigma_2^\eta(s).\kappa(s) \\ \sigma_2^{\eta\prime}(s) & = & \sigma_1^\eta(s).\kappa(s) \\ \kappa'(s) & = & C.\sigma_1^\eta(s).\sigma_2^\eta(s) \\ f_1^\eta(0) & = & 0 \\ f_2^\eta(0) & = & 0 \\ \sigma_1^\eta(0) & = & \cos\eta \\ \sigma_2^\eta(0) & = & \sin\eta \\ \kappa(0) & = & \kappa_0. \end{array}\right. \tag{27}$$

Moreover we have that $C\sin^2\eta = \kappa_0^2 - 1$ due to the same reason that Equation (13) holds. Notice that (27) coincides with (17). Proposition 5.1 settles the Theorem.

□

Therefore we arrive at the following classification.

Theorem 6.2 *Every regular smooth curve $f : M \to \mathbb{R}^n$ such that $\dim \mathfrak{gr}(f) < \infty$ is contained in an affine two dimensional subspace $A^2 \subset \mathbb{R}^n$. Moreover f is a piece of straight line ($\dim \mathfrak{gr}(f) = 1$) or there exist an arclength coordinate system s such that f is a piece of the solution of (13) composed with isometries and dilations of A^2.*

7 Holonomy and linear isotropy

In this section we relate the holonomy group of the induced metric on M with the linear part of the isotropy group of the action of $\mathrm{gr}(f)$.

Let $\pi : BM \to M$ be the bundle of frames of M and write $OM \subset BM$ for the subbundle of orthonormal frames. As in Kobayashi-Nomizu [10] an element of BM is a linear isomorphism $p : \mathbb{R}^N \to T_xM$, $x \in M$, while $p \in OM$ if it is an isometry between the standard inner product in $\mathbb{R}^N$ and the Riemannian metric in T_xM. Also the right action of $\mathrm{Gl}\,(n,\mathbb{R})$ on BM is given by $pa = p \circ a$, $p \in BM$, $a \in \mathrm{Gl}\,(n,\mathbb{R})$.

Any $\phi \in \mathrm{gr}\,(f)$ is a local diffeomorphism so that we can lift it to the local diffeomorphism $\overline{\phi}$ of BM by $\overline{\phi}\,(p) = d\phi \circ p$, which commutes with the right action of the structural group $\mathrm{Gl}\,(n,\mathbb{R})$. This lifting defines a local action of $\mathrm{gr}\,(f)$ on BM.

On the other hand each gradient vector field X^v admits a lift $\overline{X}^v$ to BM by linearization, that is, the flow of $\overline{X}^v_t$ is the lifting of the flow of X^v, namely $\overline{X}^v_t\,(p) = dX_t \circ p$ for $p \in BM$. The lifted vector fields $\overline{X}^v$ are invariant under the right action of $\mathrm{Gl}\,(n,\mathbb{R})$ of BM.

Now, let $\mathrm{gr}\,(f)\,(p)$ be the orbit of $p \in BM$ under $\mathrm{gr}\,(f)$. It is clear that any $q \in \mathrm{gr}\,(f)\,(p)$ is the end point of a curve obtained by concatenation of trajectories of the vector fields $\overline{X}^v$, $v \in \mathbb{R}^N$. This means that $\mathrm{gr}\,(f)\,(p)$ is an orbit of a family of vector fields in the sense of Stefan [12] and Sussmann [13]. By the results of [13] it follows that the orbits $\mathrm{gr}\,(f)\,(p)$, $p \in M$, are submanifolds of BM. The right invariance of $\overline{X}^v$, $v \in \mathbb{R}^N$, imply that $\mathrm{gr}\,(f)\,(p)\,a = \mathrm{gr}\,(f)\,(pa)$ and that $\mathrm{gr}\,(f)\,(p)$ projects onto M. In particular all the orbits have the same dimension. But by Stefan [12] the orbits are the leaves of a foliation hence they are maximal integral manifolds of a distribution without singularities.

For $p \in BM$ put

$$\begin{aligned} G_p &= \{a \in \mathrm{Gl}\,(n,\mathbb{R}) : \mathrm{gr}\,(f)\,(pa) = \mathrm{gr}\,(f)\,(p)\} \\ &= \{a \in \mathrm{Gl}\,(n,\mathbb{R}) : \mathrm{gr}\,(f)\,(p)\,a = \mathrm{gr}\,(f)\,(p)\} \end{aligned}$$

for the subgroup leaving invariant $\mathrm{gr}\,(f)\,(p)$. Another way of defining G_p is by taking the intersection I of $\mathrm{gr}\,(f)\,(p)$ with the fiber through p. Then $a \in G_p$ if and only if $pa \in I$. We call G_p the linear isotropy group at p.

(Although we do not need this below we mention that it is possible to prove that G_p is a Lie subgroup of $\mathrm{Gl}(n, \mathbb{R})$ and $\mathrm{gr}(f)(p)$ is a subbundle of BM having G_p as structural group.)

Now recall that the holonomy group $\mathrm{Hol}(p)$ of a connection based at p is the set of $a \in \mathrm{Gl}(n, \mathbb{R})$ such that pa is the end point of a horizontal curve starting at p. We consider here the Levi-Civita connection, so that if $p \in OM$ then $\mathrm{Hol}(p)$ is subgroup of the orthogonal group $\mathrm{O}(n)$. The following statement relates G_p with $\mathrm{Hol}(p)$.

Proposition 7.1 *Take a frame $p \in BM$ and let $\mathrm{Hol}(p)$ be the holonomy group at p. Then $\mathrm{Hol}(p) \subset G_p$ and the holonomy bundle through p is contained in the orbit $\mathrm{gr}(f)(p)$.*

Proof: Given $x \in M$, let $\{\xi_i\}_{i=1,\ldots,m}$, $m = N - n$, be an orthonormal basis of vector fields orthogonal to M around x. Since X^v is the orthogonal projection we have $X^v = v - \sum_{i=1}^m \langle v, \xi_i \rangle \xi_i$. Hence an easy computation shows that

$$\nabla_Y X^v = \sum_{i=1}^m \langle v, \xi_i \rangle A_{\xi_i}(Y)$$

where A_{ξ_i} is the second fundamental form of M in the direction of ξ_i. It follows that $\nabla_Y X^v = 0$ if $v \in T_x M$, that is, $\nabla X^v = 0$, as a linear map of $T_x M$. But if ω stands for the connection form in BM then for any frame p above x we have

$$\nabla X^v = \frac{1}{2} p \circ \omega_p \left(\overline{X}^v \right) \circ p^{-1}.$$

Therefore $\overline{X}^v$ is horizontal above x if v is tangent to M at x. This shows that the horizontal spaces H_p of the connection are tangent to the orbits. Hence a horizontal curve starting in $\mathrm{gr}(f)(p)$ stays in this orbit, so that if

$a \in \mathrm{Hol}\,(p)$ then pa is the end point of a horizontal curve starting at p showing $pa \in \mathrm{gr}\,(f)\,(p)$, that is, $a \in G_p$.

□

The above proposition is the basic tool to be used below in the discussion of the finite dimensional gradient groups. As an easy consequence of Proposition 7.1 we mention that in case M is a hypersurface then $\mathrm{SO}\,(n) \subset G_p$. This is because there exists $x \in M$ such that the second fundamental form A at x is nondegenerate. If $\{f_i\}_{i=1,\ldots,n}$ is an orthonormal basis diagonalizing A then the curvature $R\,(f_i, f_j)$ is given in the basis $\{f_i\}$ by a skew-symmetric matrix with nonzero i,j and j,i entries, so that the holonomy group contains $\mathrm{SO}\,(n)$.

8 $\mathrm{gr}\,(f)$ is semi-simple

From now on we assume that $\mathfrak{gr}\,(f)$ is finite dimensional, so that $\mathrm{gr}\,(f)$ is a (connected) Lie group if the vector fields in $\mathfrak{gr}\,(f)$ are complete. Also we assume without loss of generality that M is not contained in an affine subspace of $\mathbb{R}^n$. In this case the linear map $v \in \mathbb{R}^n \mapsto X^v \in \mathfrak{gr}\,(f)$ is injective, so that if $V = \{X^v \in \mathfrak{gr}\,(f) : v \in \mathbb{R}^n\}$ is its image then V has an inner product and we can view M as an immersed submanifold of $V \subset \mathfrak{gr}\,(f)$.

We note that $\mathrm{gr}\,(f)$ is not compact since the infinitesimal action of $\mathfrak{gr}\,(f)$ on M contains nonzero gradient vector fields. For $x \in M$ let $\mathrm{gr}_x(f)$ be the isotropy subgroup at x. The isotropy representation of $\mathrm{gr}_x(f)$ on T_xM is given by $g \in \mathrm{gr}_x(f) \mapsto dg_x \in \mathrm{Gl}\,(T_xM)$. If $p \in B_xM$ then $a \in G_p$ if and only if there exists $g \in \mathrm{gr}_x(f)$ such that $dg_x \circ p = pa$, that is $a = p^{-1} \circ dg_x \circ p$ so that the linear isotropy G_p at p is identified to the image of the isotropy representation, implying that G_p is a Lie subgroup. We write $\mathfrak{g}_p$ for its Lie algebra.

For $p \in OM$ the holonomy group $\mathrm{Hol}\,(p)$ is compact. We write $\mathfrak{hol}\,(p)$ for its Lie algebra. The representation of the Lie algebra $\mathfrak{hol}\,(p)$ on T_xM decomposes into irreducible representations

$$T_xM = V_1 \oplus \cdots \oplus V_s.$$

Suppose that none of these representations is trivial, that is, $\dim V_i \geq 2$, $i = 1, \ldots, s$. Then for every $0 \neq v \in T_xM$ there exists $A \in \mathfrak{hol}\,(p)$ such that $Av \neq 0$. But it follows from Proposition 7.1 that $\mathfrak{hol}\,(p) \subset \mathfrak{g}_p$. Hence the same assertion holds for $\mathfrak{g}_p$ instead of $\mathfrak{hol}\,(p)$.

This has the following consequence on the Lie algebra $\mathfrak{gr}_x(f)$ of the isotropy group $\mathrm{gr}_x(f)$.

Proposition 8.1 *Suppose that the representation of $\mathfrak{hol}\,(p)$ on T_xM has no trivial components. Let $\mathfrak{g}_x$ be the isotropy subalgebra and denote by $\mathfrak{n}$ its normalizer in $\mathfrak{gr}(f)$. Then $\mathfrak{n} = \mathfrak{g}_x$.*

Proof: The representation of $\mathfrak{g}_x$ on T_xM is equivalent to its representation on $\mathfrak{gr}\,(f)\,/\mathfrak{g}_x$. The image of the representation on T_xM is $\mathfrak{g}_p$, and as mentioned above the assumption implies that for all $0 \neq v \in T_xM$ there exists $A \in \mathfrak{g}_p$ such that $Av \neq 0$. Therefore for every $0 \neq w \in \mathfrak{gr}(f)/\mathfrak{g}_x$ there exists $Y \in \mathfrak{g}_x$ such that $Yw \neq 0$. But this means that for every $X \notin \mathfrak{g}_x$ there exists $Y \in \mathfrak{g}_x$ such that $[Y, X] \notin \mathfrak{g}_x$, so that $\mathfrak{g}_x$ is its own normalizer.

□

Next we apply Lie group theory to prove that if a compact homogeneous space G/H is such that the isotropy subalgebra coincides with its normalizer then G is semi-simple. We do this in two steps. First we prove that the Lie algebra $\mathfrak{g}$ of G decomposes as the direct sum of semi-simple subalgebra and an abelian ideal. In a second turn we prove that the abelian ideal is $\{0\}$.

Proposition 8.2 *Let G/H be a compact homogeneous space and denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Suppose that the normalizer $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$, and that the action of G on G/H is effective. Then $\mathfrak{g}$ decomposes as*

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$$

where $\mathfrak{r}$ is an abelian ideal and $\mathfrak{l}$ is a semi-simple subalgebra.

Proof: Let $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$. By the decomposition theorem of Levi it is enough to prove that $\mathfrak{r}$ is abelian or, equivalently, that its derived algebra $\mathfrak{r}' = \{0\}$. This will be achieved if we prove that $\mathfrak{r}'$ is contained in some isotropy subalgebra $\mathfrak{g}_x$ because the action is assumed to be effective so that $\mathfrak{g}_x$ does not contain non trivial ideals.

Let R' be the connected subgroup whose Lie algebra is $\mathfrak{r}'$. We shall prove that R' has a fixed point in G/H. It is known that $\mathfrak{r}'$ is a nilpotent ideal of $\mathfrak{g}$, so that by Engels' theorem there exists a basis of $\mathfrak{g}$ such that $\mathrm{ad}(X)$ is upper triangular, with $0'$s on the diagonal, for all $X \in \mathfrak{r}'$. In this basis $\mathrm{Ad}(g)$, $g \in R'$, is upper triangular with 1's on the diagonal.

Now, let

$$N(\mathfrak{h}) = \{g \in G : \mathrm{Ad}(g)\,\mathfrak{h} = \mathfrak{h}\}$$

be the normalizer of $\mathfrak{h}$ in G. This is a closed subgroup whose Lie algebra is $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$. Clearly $H \subset N(\mathfrak{h})$ and since these groups have the same Lie algebra, it follows that the fibration $G/H \to G/N(\mathfrak{h})$ is a covering with fiber $N(\mathfrak{h})/H$. By assumption G/H is compact so that $G/N(\mathfrak{h})$ is compact as well.

Now let $k = \dim \mathfrak{h}$ and denote by $\mathrm{Gr}_k(\mathfrak{g})$ the Grassmannian of k dimensional subspaces of $\mathfrak{g}$. The group G acts on $\mathrm{Gr}_k(\mathfrak{g})$ by the adjoint representation. Let $G \cdot \mathfrak{h}$ be the orbit of $\mathfrak{h}$. Then $G \cdot \mathfrak{h} = G/N(\mathfrak{h})$ because $N(\mathfrak{h})$ is the isotropy at $\mathfrak{h}$.

By a theorem of Vinberg [14] a triangular linear group acting on a Grassmannian has a fixed point in any compact invariant subset. Clearly $G \cdot \mathfrak{h}$ is compact and R'-invariant. Therefore R' has a fixed point, say $x_0 \in G/N(\mathfrak{h})$. Since R' is connected and the fibration $G/H \to G/N(\mathfrak{h})$ is a covering, it follows that any point in G/H above x_0 is also fixed by R'. Hence R' has fixed points in G/H, concluding the proof.

□

Proposition 8.3 *In the same situation as in the above proposition we have* $\mathfrak{r} = \{0\}$*, that is,* $\mathfrak{g}$ *is semi-simple.*

Proof: If $X \in \mathfrak{r}$ then $\mathrm{ad}(X)\mathfrak{g} \subset \mathfrak{r}$ because $\mathfrak{r}$ is an ideal. But $\mathfrak{r}$ is abelian so that $\mathrm{ad}(X)^2 = 0$. Therefore by Engels' theorem $\mathrm{ad}(\mathfrak{r})$ is a triangular Lie algebra. The proof then follows as in the above proposition.

□

Summarizing the results obtained so far we get the following theorem about the gradient group being finite dimensional.

Theorem 8.4 *Let* $f : M \to \mathbb{R}^N$ *be a smooth isometric immersion such that*

1. $\mathrm{gr}(f)$ *is finite dimensional.*

2. M *is compact.*

3. *The representation of the holonomy algebra* $\mathfrak{hol}$ *on* $T_x M$ *has no trivial subrepresentations.*

Then $\mathrm{gr}\,(f)$ *is semi-simple noncompact.*

We can go a bit further and look at the properties of M as a homogeneous space of $\mathrm{gr}\,(f)$. Write $M = \mathrm{gr}(f)/H$, with a $H \subset \mathrm{gr}(f)$ a closed subgroup with Lie algebra $\mathfrak{h}$. Take an Iwasawa decomposition $\mathrm{gr}(f) = KAN$ of $\mathrm{gr}(f)$.

With the assumptions of the theorem the Lie algebra $\mathfrak{h}$ is its own normalizer. Hence the fibration $M = \mathrm{gr}(f)/H \to \mathrm{gr}(f)/N\,(\mathfrak{h})$ is a covering. Now as in the proof of Proposition 8.2 the space $\mathrm{gr}(f)/N\,(\mathfrak{h})$ is a compact orbit in a Grassmannian. Clearly it is AN-invariant. But AN is a triangular group. Hence another application of Vinberg's theorem shows that AN has a fixed point in $\mathrm{gr}(f)/N\,(\mathfrak{h})$. Since AN is connected we conclude that this group has fixed points in $M = \mathrm{gr}(f)/H$. Hence up to a conjugation we can suppose that $AN \subset H$. This yields the following statement.

Proposition 8.5 *Let the assumptions be as in Theorem 8.4. Then* $M = \mathrm{gr}\,(f)\,/H$ *and the isotropy subgroup contains* AN *for an Iwasawa decomposition* $\mathrm{gr}\,(f) = KAN$ *of* $\mathrm{gr}\,(f)$*. This implies that* K *acts transitively on* M*, and* $M = K/H \cap K$*. Moreover the center* $Z\,(\mathrm{gr}\,(f))$ *of* $\mathrm{gr}\,(f)$ *is finite.*

Proof: We need to prove only the last statement. For this consider the fibration $\pi : \mathrm{gr}\,(f)\,/H \to \mathrm{gr}\,(f)\,/N\,(\mathfrak{h})$ and let x_0 be the origin of $\mathrm{gr}\,(f)\,/N\,(\mathfrak{h})$. Clearly $Z\,(\mathrm{gr}\,(f)) \subset N\,(\mathfrak{h})$, so that $gx_0 = x_0$ for any $g \in Z\,(\mathrm{gr}\,(f))$ and g leaves invariant the fiber above x_0. If $Z\,(\mathrm{gr}\,(f))$ were infinite then there would exist $g \in Z\,(\mathrm{gr}\,(f))$ of infinite order. But then the infinite group $\{g^n : n \in \mathbb{Z}\}$ acts on the finite set $\pi^{-1}\{x_0\}$, so that for some $n \in \mathbb{Z}$, g^n has a fixed point in $\pi^{-1}\{x_0\}$, that is $g^n \in Z(\mathrm{gr}(f))$ belongs to some isotropy subgroup of $M = \mathrm{gr}(f)/H$. But this is a contradiction since the action of $\mathrm{gr}(f)$ on M is effective.

$\square$

Remark: If $\mathfrak{gr}(f) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition at the Lie algebra level then the above proposition says that $\mathfrak{a} \oplus \mathfrak{n}$ is contained in the isotropy subalgebra $\mathfrak{gr}_x(f)$. We shall prove below that the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$ is also contained in $\mathfrak{gr}_x(f)$, so this Lie algebra is actually a parabolic subalgebra of $\mathfrak{gr}(f)$.

9 Parabolic subalgebra

In this section we prove that under the conditions of Theorem 8.4 the isotropy subalgebra is parabolic, so that M is a flag manifold of $\mathrm{gr}(f)$ or a covering of a flag manifold.

The point is that we proved in the last section that the solvable part AN of an Iwasawa decomposition is contained in the isotropy subgroup $\mathrm{gr}_x(f)$ of the action of $\mathrm{gr}(f)$ on M. Hence the isotropy subalgebra $\mathfrak{gr}_x(f)$ contains $\mathfrak{a} \oplus \mathfrak{n}$. It remains to show that $\mathfrak{gr}_x(f)$ contains the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$. This implies that $\mathfrak{gr}_x(f)$ is a parabolic subalgebra since it contains the minimal parabolic subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

To that purpose we prove first a result which has independent interest, namely that if a one-parameter group $\exp tX$ of a semi-simple Lie group G is the flow of a gradient vector field in a homogeneous space of G then the eigenvalues of $\mathrm{ad}(X)$ are real.

We recall the following facts.

1. Let X be a gradient vector field in a Riemannian manifold M and write X_t for its flow. Suppose that x is a singularity of X. Then the eigenvalues of the map $dX_t : T_xM \to T_xM$ are real.

2. Let X and Y be commuting vector fields, $[X, Y] = 0$, and suppose that x is an isolated singularity of X. Then $Y(x) = 0$ and $Y_t(x) = x$ if Y_t is the flow of Y. In fact, the points $Y_t(x)$ are also fixed points of X.

3. Let K be a compact Lie group acting differentiably on a manifold M. Suppose that x is a fixed point of K. Then a theorem of E. Cartan ensures that the linear action of K on T_xM is equivalent to the action of K on a neighborhood of x. This means that there are neighborhoods U of $0 \in T_xM$ and V of x and a diffeomorphism $\phi : U \to V$ such that for every $k \in K$ it holds $\phi \circ dk_x = k \circ \phi$.

Now let $\mathfrak{g}$ be a noncompact semi-simple Lie algebra. The Jordan-Schur decomposition of an element $X \in \mathfrak{g}$ says that X decomposes uniquely into commuting elements as $X = Z + H + Y$ in such a way that $\mathrm{ad}(Z)$ and $\mathrm{ad}(H)$ are semi-simple and $\mathrm{ad}(Y)$ is nilpotent and the eigenvalues of $\mathrm{ad}(Z)$ are purely imaginary while those of $\mathrm{ad}(H)$ are real.

Proposition 9.1 *Let G be a connected, noncompact semi-simple Lie group with finite center and Lie algebra $\mathfrak{g}$. Suppose $L \subset G$ is a closed subgroup such that G acts effectively on G/L. Take $X \in \mathfrak{g}$ with Jordan-Schur decomposition $X = Z + H + Y$. Suppose that the vector field induced by X on G/L is gradient with respect to some immersion $f : G/L \to \mathbb{R}^n$ (and the induced Riemannian metric on G/L) and that it has an isolated singularity.*

Then $Z = 0$.

Proof: Assume without loss of generality that the origin x_0 of G/L is an isolated fixed point of $\exp tX$. Then X belongs to the Lie algebra $\mathfrak{l}$ of L as well as its components Z, H and Y in the Jordan-Schur decomposition.

Write $T = \mathrm{cl}\{\exp tZ : t \in \mathbb{R}\}$. Since the eigenvalues of $\mathrm{ad}\,(Z)$ are purely imaginary and G has finite center, it follows that T is a compact subgroup which is contained in L because $Z \in \mathfrak{l}$.

Now, let $\rho : L \to \mathrm{Gl}\,(T_{x_0}G/L)$ be the isotropy representation on the tangent space and denote also by ρ the corresponding representation of $\mathfrak{l}$. This representation is equivalent to the adjoint representation of $\mathfrak{l}$ on the quotient $\mathfrak{g}/l$. This implies that the Jordan-Schur decomposition of $\rho\,(X)$ is $\rho\,(X) = \rho\,(Z) + \rho\,(H) + \rho\,(Y)$. In particular the eigenvalues of $\rho\,(X)$ are those of $\rho\,(Z + H)$. But the eigenvalues of $\rho\,(X)$ are real, so that $\rho\,(Z) = 0$ and hence $\rho\,(T) = \mathrm{cl}\{\exp t\rho\,(Z) : t \in \mathbb{R}\} = \{\mathrm{id}\}$.

By the theorem of Cartan mentioned above, there exists a neighborhood V of x_0 such that the action of T on V reduces to the identity. Now, the action on G/L is analytic and since this space is connected, it follows that every $h \in T$ acts on G/L by the identity map. The assumption that the action is effective then implies that T is the trivial group, so that $Z = 0$.

$\square$

Next we show that in the case of interest here the nilpotent part Y in the decomposition of X also annihilates.. We approach this question via Conley's dynamical concept of chain recurrence for flows. In the appendix to this section we recall the main definitions and results about chain recurrence to be used.

Let us take as before a substantial immersion of M into $\mathbb{R}^N$ and the height functions $h_v\,(x) = \langle v, x\rangle$, $v \in \mathbb{R}^N$. It is well known that there exists a dense subset $D \subset \mathbb{R}^N$ such that h_v is a Morse function if $v \in D$ (see e.g. Bott-Tu [2]). For such v the gradient vector field X^v has isolated singularities, which are the chain recurrent points of X^v.

Note that we can view X^v as an element of $\mathfrak{gr}(f)$ and hence take its Jordan-Schur decomposition $X^v = H^v + Y^v$, whose imaginary part $Z^v = 0$ by the above proposition.

Lemma 9.2 *Let the notation be as above and suppose that h_v is a Morse function. Then the fixed points of $H_t^v = \exp tH^v$, $t \in \mathbb{R}$, on M are isolated.*

Proof: Let AN be the solvable component of an Iwasawa decomposition of $\mathrm{gr}\,(f)$ which is contained in the isotropy subgroup $\mathrm{gr}_x\,(f)$. There is a natural fibration $\pi : \mathrm{gr}\,(f)\,/AN \to M = \mathrm{gr}\,(f)\,/\mathrm{gr}_x\,(f)$. The chain recurrent set $\mathcal{R}$ for the flow on $\mathrm{gr}\,(f)\,/AN$ induced by $X_t^v = \exp tX^v$ is the set of fixed points of H_t^v (see the appendix below). By Proposition 9.4 a connected component C of $\mathcal{R}$ projects into a chain recurrent component of X_t^v in $\mathrm{gr}\,(f)\,/\mathrm{gr}_x\,(f)$. These chain components are isolated fixed points because h_v is a Morse function. On the other hand a fixed point of H_t^v projects to a fixed point. Therefore, the fixed points of H_t^v in M are isolated, proving the lemma.

□

Now we can prove the main result of this section ensuring that the isotropy subalgebra is parabolic.

Theorem 9.3 *Assume that M is compact and the tangent space representation of the holonomy Lie algebra has no trivial subrepresentations. Then the isotropy* *subalgebra $\mathfrak{gr}_x(f)$ is parabolic and M is a finite covering of a flag manifold of* $\mathrm{gr}\,(f)$.

Proof: Take $v \in \mathbb{R}^N$ such that h_v is a Morse function and write $X^v = H^v + Y^v$ as in the above lemma, so that the fixed points of H_t^v are isolated. Now let $Z\,(H^v)$ be the centralizer of H^v in $\mathrm{gr}\,(f)$ and $\mathfrak{z}\,(H^v)$ its centralizer in $\mathfrak{gr}(f)$. The identity component of $Z\,(H^v)$ is $Z\,(H^v)_0 = \exp\langle\mathfrak{z}\,(H^v)\rangle$. Let $x \in M$ be a fixed point of H_t^v. Then gx is also a fixed point if $g \in Z\,(H^v)$. This implies that the orbit $Z\,(H^v)_0\, x$ reduces to x. Hence $\mathfrak{z}\,(H^v)$ is contained in the isotropy subalgebra $\mathfrak{gr}_x(f)$ at x. In particular $\mathfrak{gr}_x(f)$ contains the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$. Since we had proved that $\mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{gr}_x(f)$ it follows that the minimal parabolic subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{gr}_x(f)$, so that $\mathfrak{gr}_x(f)$ is a parabolic subalgebra.

$\square$

Remark: In the situation of the above proof it can be checked that H^v is regular, that is, $\alpha\,(H^v) \neq 0$ for every root α, which implies that $Y^v = 0$.

Remark: We do not know whether M is in fact a flag manifold and not just a covering of it, that is, whether the isotropy subgroup $\mathrm{gr}_x(f)$ is itself parabolic or only contained in a parabolic subgroup.

9.1 Appendix: Chain recurrence

We recall here some concepts and results on chain recurrence and transitivity of flows that were used above. (For the details we refer to Colonius-Kliemann [4], Conley [5], [6], Braga-San Martin [3], Patrão-San Martin [11] and references therein.)

Let $\phi : \mathbb{R} \times Z \to Z$ be a flow on a compact metric space (Z, d). For $x, y \in Z$ and $\varepsilon, T > 0$ an ε, T-chain of ϕ from x to y is given by points $x = x_0, x_1, \ldots, x_n = y \in Z$ and $t_0, \ldots, t_{n-1} \geq T$, for some $n \in \mathbb{N}$, such that $d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon$, $i = 0, 1, \ldots, n-1$. Let $\mathcal{C}_{\varepsilon,T}(x)$ be the set of those $y \in Z$ such that there exists an ε, T-chain from x to y, and put $\mathcal{C}(x) = \bigcap_{\varepsilon,T} \mathcal{C}_{\varepsilon,T}(x)$. The point x is said to be chain recurrent if $x \in \bigcap_{\varepsilon,T} \mathcal{C}_{\varepsilon,T}(x)$. The set $\mathcal{R}$ of chain recurrent points is compact and each of its connected component C is maximal chain transitive in the sense that for every $x, y \in C$ we have $x \in \bigcap_{\varepsilon,T} \mathcal{C}_{\varepsilon,T}(y)$ and $y \in \bigcap_{\varepsilon,T} \mathcal{C}_{\varepsilon,T}(x)$ and C is maximal with this property (see [4], Theorem B.2.22).

There are the following known examples of chain recurrent sets.

1. Suppose ϕ is the flow of a gradient vector field on a compact manifold M. Then $x \in M$ is chain recurrent if and only if x is a fixed point of the flow, $\phi_t(x) = x$, $t \in \mathbb{R}$. (See [6].)

2. Let G be a semi-simple Lie group with finite center and Lie algebra $\mathfrak{g}$. Take $X \in \mathfrak{g}$ with Jordan-Schur decomposition $X = Z + H + Y$ with $\operatorname{ad}(Z + H)$ semi-simple, $\operatorname{ad}(H)$ diagonalizable and $\operatorname{ad}(Y)$ nilpotent. The one-parameter group $\exp tX$ induces flows on the homogeneous spaces of G. In particular take an Iwasawa decomposition $G = KAN$ and the compact homogeneous space G/AN. Then the chain recurrent set of the flow $\exp tX$ is given by the set of fixed points of $\exp tH$, $t \in \mathbb{R}$, in G/AN. (See Ferreira [8]. This result is a generalization of Proposition 5.1.2 of [4], which gives the chain recurrent components on projective space as generalized eigenspaces.)

The following result on projections of chain recurrent components were proved in [3] for general fiber bundles.

Proposition 9.4 *Let G be a Lie group and $H \subset L$ be closed subgroups. There exists and equivariant fibration $\pi : G/H \to G/L$. Let $X_t = \exp tX$ be a one-parameter group of G. Let $C \subset G/H$ be a component of the chain recurrent set for the flow on G/H induced by X_t. Then $\pi(C)$ is contained in a component of the chain recurrent set on G/L.*

10 Examples

The main examples of compact immersed manifolds having finite dimensional gradient groups are given by the immersions of the flag manifolds of semi-simple groups into their Lie algebras.

Let $\mathfrak{g}$ be a noncompact semi-simple Lie algebra an take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Select an abelian subspace (Chevalley subalgebra) $\mathfrak{a} \subset \mathfrak{s}$. Let $K = \langle \exp \operatorname{ad}(\mathfrak{k}) \rangle$ be the maximal compact subgroup. Take $H \in \mathfrak{a}$. Then the K-adjoint orbit $K \cdot H$ identifies with a flag manifold $\mathbb{F}_H$ of a Lie group G with Lie algebra $\mathfrak{g}$. Under this identification we get an action of G on $K \cdot H$. Clearly this adjoint orbit is an embedding of $\mathbb{F}_H$ into $\mathfrak{s}$.

Endow $\mathfrak{s}$ with the Cartan-Killing inner product $\langle \cdot, \cdot \rangle$, so that $\mathbb{F}_H$ is given the induced Riemmannian metric. By the G-action any $X \in \mathfrak{g}$ (in particular any $X \in \mathfrak{s}$) induces a vector field in $\mathbb{F}_H = K \cdot H$.

Now it is known that in case the eigenvalues of $\operatorname{ad}(H)$ are ± 1 and 0 then the vector field induced by $X \in \mathfrak{s}$ is the gradient of the height function $h_X(\cdot) = \langle X, \cdot \rangle$ defined by X itself (see Duistermaat-Kolk-Varadarajan [7]). Therefore the Lie algebra generated by the gradient vector fields is (isomorphic to) a subalgebra of $\mathfrak{g}$ (and is $\mathfrak{g}$ itself if it has no compact factors). This is thus a class of examples of finite dimensional gradient groups.

For instance if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ then $\mathfrak{a}$ can be the subalgebra of diagonal matrices and $\mathrm{ad}(H)$, $H \in \mathfrak{a}$, has eigenvalues ± 1 and 0 if and only if up to a permutation on the diagonal entries H is one of the matrices

$$H_k = \frac{1}{n}\mathrm{diag}\{\underbrace{n-k, \overset{k\times}{\ldots}, n-k}, -k, \ldots, -k\} \qquad k = 1, \ldots, n-1.$$

The flag manifold associated to H_k is the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. Similar remarks hold for $\mathfrak{sl}(n, \mathbb{C})$.

Another particular case is given by the rank one Lie algebra $\mathfrak{so}(1, n)$. There exists just one flag manifold and the adjoint orbits $K \cdot H$ are the spheres in n-dimensional space $\mathfrak{s}$ centered at the origin. This example shows that the gradient group of the round sphere $\mathbb{S}^{n-1}$ embedded in $\mathbb{R}^n$ is finite dimensional and locally isomorphic to $\mathrm{SO}(1, n)$.

Remark: We mention that the vector field induced by $X \in \mathfrak{s}$ on $\mathbb{F}_H = K \cdot H$ is the gradient of the height function $h_X(\cdot) = \langle X, \cdot \rangle$ (taken with respect to the Cartan-Killing form), if the gradient is taken with respect to a metric on $K \cdot H$ introduced by Borel in the fifties. In general the Borel metric does not coincide with the immersion metric on $\mathfrak{s}$, unless the eigenvalues of $\mathrm{ad}(H)$ are ± 1 and 0.

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