Matemática Contemporânea, Vol 33, 199-213
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ENNEPER REPRESENTATION AND THE GAUSS MAP OF MINIMAL SURFACES IN THE PRODUCT $\mathbb{H}^2 \times \mathbb{R}$

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Dedicated to Professor Francesco Mercuri on his 60th birthday

Abstract

We give an Enneper-type representation for minimal surfaces in the product of the hyperbolic plane with the real line. We apply this representation to study the Gauss map of minimal surfaces in this space.

1 Introduction

In [1] P. Andrade introduced a new method to describe minimal surfaces in the Euclidean three-dimensional space which is equivalent to the classical Weierstrass representation. The method described by Andrade allows to construct a local conformal minimal immersion $\chi : \Omega \subset \mathbb{C} \to \mathbb{C} \times \mathbb{R}$, from a harmonic function $h : \Omega \to \mathbb{R}$, provided that one chooses two holomorphic functions $L, H : \Omega \to \mathbb{C}$ satisfying:

$$L_z\, H_z = (h_z)^2 \quad \text{and} \quad |L_z(z)| + |H_z(z)| \neq 0, \quad \forall z \in \Omega.$$

The desired conformal minimal immersion terns out to be $\chi(z) = (L(z) - \overline{H}(z), h(z))$ and it is now called an *Enneper immersion* associated to h. The image $\chi(\Omega)$ is called an *Enneper graph* of h and it is known that any immersed

2000 *Mathematics Subject Classification.* 53C42, 53A10
Key words and phrases: Minimal surfaces, Enneper immersions, hyperbolic space, Gauss map.
The first author was supported by Prin05: Metriche Riemanniane e Varietà Differenziabili (Italy)
The second author was supported by FAPESP and CNPq (Brazil)

minimal surface in $\mathbb{R}^3$ is, locally, the Enneper graph of some harmonic function [1].

In the first part of this note we shall give an Enneper-type representation for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ by using the Weierstrass-type formula for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ described in [9].

The second part is devoted to the Gauss map of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. We shall apply the Enneper-type representation to prove that given a non-holomorphic complex function $g : \mathcal{M} \to \mathbb{C}$, satisfying a certain condition, there exists a unique minimal immersion $\chi : \mathcal{M} \to \mathbb{H}^2 \times \mathbb{R}$ with Gauss map g.

We would like to point out that the theory of minimal surfaces in the product $\mathbb{H}^2 \times \mathbb{R}$ is nowadays a very rich subject, which is growing very rapidly, with major contributions from P. Collin, B. Daniel, L. Hauswirth, B. Nelli, R. Sa Earp, H. Rosenberg, E. Toubiana (see, for example, [2, 6, 10, 11, 12]). Other authors have considered the Gauss map for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and we refer the reader to [4, 8, 7] and the references therein.

2 Preliminaries

We shall use the upper half-plane model $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 > 0\}$ for the hyperbolic space $\mathbb{H}^2$ endowed with the metric $g_{\mathbb{H}} = (dx_1^2 + dx_2^2)/x_2^2$. The space $\mathbb{H}^2 \times \mathbb{R}$, with the group structure given by

$$(x_1, x_2, x_3) * (x_1', x_2', x_3') = (x_1' x_2 + x_1, x_2 x_2', x_3 + x_3'), \tag{2.1}$$

is a three-dimensional Lie group and the product metric $g_{\mathbb{H}^2\times\mathbb{R}} = g_{\mathbb{H}} + dx_3^2$ is left invariant. With respect to the metric $g_{\mathbb{H}^2\times\mathbb{R}}$ an orthonormal basis of left invariant vector fields is given by

$$E_1 = x_2 \frac{\partial}{\partial x_1}, \qquad E_2 = x_2 \frac{\partial}{\partial x_2}, \qquad E_3 = \frac{\partial}{\partial x_3}.$$

2.1 The Gauss map

Lets G be a $(n+1)$-dimensional Lie group endowed with a left invariant metric and $\mathcal{M}^n$ an orientable hypersurface of G. Denoting by e the identity of G and by $\mathbb{S}^n$ the unit sphere centred at the origin of the tangent space T_eG, we define the Gauss map

$$\gamma : \mathcal{M} \to \mathbb{S}^n \subset T_eG$$

by

$$\gamma(p) = (dL_{p^{-1}})_p(\xi(p)), \qquad p \in \mathcal{M}.$$

Here $L_p : G \to G$ is the left translation and ξ is a unit differentiable vector field of G normal to $\mathcal{M}$. Since L_p is an isometry we have

$$\begin{aligned} d\gamma_p(T_p\mathcal{M}) &\subseteq T_{\gamma(p)}\mathbb{S}^n \\ &= \{\gamma(p)\}^{\perp} = (dL_{p^{-1}})_p(\xi(p)^{\perp}) \\ &= (dL_{p^{-1}})_p(T_p\mathcal{M}). \end{aligned} \tag{2.2}$$

Consequently,

$$(dL_p)_e(d\gamma_p(T_p\mathcal{M})) \subseteq T_p\mathcal{M}.$$

Let now $\{E_1, \ldots, E_{n+1}\}$ be an orthonormal basis of left invariant vector fields of G. Writing

$$\xi = \sum_{i=1}^{n+1} \xi_i \, E_i,$$

it results that

$$\gamma(p) = \sum_{i=1}^{n+1} \xi_i(p) \, E_i(e).$$

2.2 The Weierstrass representation

From now on, Ω denotes a simply connected domain in the complex plane $\mathbb{C}$, $z = u + iv$ (with $u, v \in \mathbb{R}$) the complex coordinate and, as usual, we will adopt the following notation for the complex derivatives:

$$\frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\Big);\qquad \frac{\partial}{\partial \bar z}=\frac{1}{2}\Big(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\Big).$$

As a convention we shall write, for a complex function f, $\frac{\partial f}{\partial z}=f_z$ and so on. With $\mathcal{R}e(f)$ and $\mathcal{I}m(f)$ we shall denote the real and imaginary parts of f respectively.

With the above notations we have the following representation of minimal surfaces in $\mathbb{H}^2\times\mathbb{R}$.

Theorem 2.1 ([9])**.** *Let ψ_i, $i=1,2,3$, three complex valued functions defined in the domain Ω, such that the following conditions hold:*

$$\sum_{i=1}^{3}\psi_i^2=0,\qquad \sum_{i=1}^{3}|\psi_i|^2\neq 0,$$

$$\begin{cases}(\psi_1)_{\bar z}-\overline{\psi_1}\psi_2=0\\ (\psi_2)_{\bar z}+|\psi_1|^2=0\\ (\psi_3)_{\bar z}=0.\end{cases}\tag{2.3}$$

Then, for a fixed point $z_0\in\Omega$, the map $\chi:\Omega\to\mathbb{H}^2\times\mathbb{R}$ given by

$$\chi(z)=\left(2\,\mathcal{R}e\int_{z_o}^{z}\chi_2\;\psi_1\,dz,\ \chi_2,\ 2\,\mathcal{R}e\int_{z_o}^{z}\psi_3\,dz\right),$$

where $\ln(\chi_2)=2\,\mathcal{R}e\int_{z_o}^{z}\psi_2\,dz$, defines a conformal minimal immersion.

As in the Euclidean case, rewriting the equation $\sum_{i=1}^{3}\psi_i^2=0$ like

$$\psi_3^2=-(\psi_1+i\psi_2)(\psi_1-i\psi_2)$$

and assuming that $\psi_1-i\psi_2$ is not identically zero, we can introduce the following complex functions:

$$f=\psi_3\qquad\text{and}\qquad g=\frac{\psi_3}{\psi_1-i\psi_2}=-\frac{\psi_1+i\psi_2}{\psi_3}.\tag{2.4}$$

From g and f we can reconstruct the ψ_i setting

$$\begin{cases} \psi_1 = \dfrac{f}{2g}(1-g^2), \\ \psi_2 = \dfrac{if}{2g}(1+g^2), \\ \psi_3 = f. \end{cases} \tag{2.5}$$

Moreover, with respect to f and g, (2.3) becomes

$$\begin{cases} f_{\bar z} = 0 \\ g_{\bar z} + \dfrac{i}{2}\dfrac{\bar f\, g}{\bar g}\,(1-\bar g^2) = 0. \end{cases} \tag{2.6}$$

The triple (Ω, g, f) is called the *Weierstrass data* of χ.

Remark 2.2. With respect to the function g, defined in (2.4), the Gauss map of χ is

$$\gamma = \frac{1}{|g|^2+1}[2\,\mathcal{R}e(g)E_1(e) + 2\,\mathcal{I}m(g)E_2(e) + (|g|^2-1)E_3(e)].$$

Thus, using the extended stereographic projection $\pi : \mathbb{S}^2(1) \to \mathbb{C}\cup\{\infty\}$, we have

$$\pi\circ\gamma = g.$$

This means that g can be identified with the Gauss map of χ.

3 Enneper-type immersions in $\mathbb{H}^2\times\mathbb{R}$

We start this section by proving an Enneper-type formula for minimal surfaces in $\mathbb{H}^2\times\mathbb{R}$.

Theorem 3.1. *Let $h : \Omega \to \mathbb{R}$ be a harmonic function and let $F : \Omega \to \mathbb{C}$ be a complex valued function with positive imaginary part such that the following conditions are satisfied:*

$$(h_z)^2 = -\frac{F_z\overline{F}_z}{\mathcal{I}m(F)^2}, \tag{3.1}$$

$$|F_z| + |\overline{F}_z| \neq 0, \tag{3.2}$$

and

$$\frac{\partial}{\partial \bar{z}}\Big(\frac{\mathcal{I}m(F)}{\overline{F}_z}\Big) + i\,\frac{(\mathcal{R}e(F))_{\bar{z}}}{\overline{F}_z} = 0. \tag{3.3}$$

Then the map $\chi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$, defined by $\chi(z) = \big(F(z), h(z)\big)$, is a conformal minimal immersion.

Proof: We shall apply Theorem 2.1. Let define three complex functions

$$\begin{cases} \psi_1 = \dfrac{(\mathcal{R}e(F))_z}{\mathcal{I}m(F)}, \\ \psi_2 = \dfrac{(\mathcal{I}m(F))_z}{\mathcal{I}m(F)}, \\ \psi_3 = h_z. \end{cases} \tag{3.4}$$

Then, the ψ_i satisfy

$$\begin{cases} \psi_1 + i\psi_2 = \dfrac{F_z}{\mathcal{I}m(F)}, \\ \psi_1 - i\psi_2 = \dfrac{\overline{F}_z}{\mathcal{I}m(F)}. \end{cases} \tag{3.5}$$

Now, from (3.2), it results that

$$\sum_{i=1}^{3} |\psi_i|^2 = \frac{(|F_z| + |\overline{F}_z|)^2}{2\,\mathcal{I}m(F)} \neq 0.$$

Moreover, using (3.5) and (3.1), it is straightforward to check that

$$\sum_{i=1}^{3} \psi_i^2 = \frac{F_z \overline{F}_z}{\mathcal{I}m(F)^2} + (h_z)^2 = 0,$$

We now observe that, as h is a harmonic function, $f = \psi_3 = h_z$ is holomorphic and that

$$g = \frac{\psi_3}{\psi_1 - i\psi_2} = h_z\,\frac{\mathcal{I}m(F)}{\overline{F}_z}. \tag{3.6}$$

Consequently, from (3.3) and taking into account (3.1), we have

$$g_{\bar{z}} + \frac{i}{2}\frac{\bar{f}\,g}{\bar{g}}\,(1 - \bar{g}^2) = h_z\,\Big[\frac{\partial}{\partial \bar{z}}\Big(\frac{\mathcal{I}m(F)}{\overline{F}_z}\Big) + i\,\frac{(\mathcal{R}e(F))_{\bar{z}}}{\overline{F}_z}\Big] = 0.$$

Thus f and g satisfy (2.6) and using Theorem 2.1 we can conclude.

□

Remark 3.2. Condition (3.1) ensures that χ is conformal while (3.2) that it is an immersion. Equation (3.3), which guaranties minimality, can be rewritten as:

$$F_{z\bar{z}} + i\,\frac{F_{\bar{z}}F_z}{\mathcal{I}m(F)} = 0, \tag{3.7}$$

which means that F is a harmonic map from $(\Omega, |dz|^2)$ to $(\mathbb{H}^2, g_{\mathbb{H}})$.

The Weierstrass data of an Enneper immersion χ are given by

$$(\Omega, g, f) = \left(\Omega, h_z\,\frac{\mathcal{I}m(F)}{\overline{F}_z}, h_z\right).$$

Also, in analogy to the Euclidean case, we shall call

$$\mathcal{D}_\chi = \left(\frac{F_z}{\mathcal{I}m(F)}, -\frac{\overline{F}_z}{\mathcal{I}m(F)}, h_z\right) \tag{3.8}$$

the *Enneper data* of χ.

We now construct some examples of minimal immersions starting from the Enneper data. The simplest case is when $h_z = 1$. In this case we must have

$$\mathcal{D}_\chi = \left(\frac{1}{A}, A, 1\right)$$

where A is a complex valued function such that

$$\begin{cases} \dfrac{F_z}{\mathcal{I}m(F)} = \dfrac{1}{A} \\ \dfrac{\overline{F}_z}{\mathcal{I}m(F)} = -A. \end{cases} \tag{3.9}$$

Example 3.3. We assume that A is a real valued function. Substituting (3.9) in (3.3) it results

$$2A_{\bar{z}} = i\,(1 - A^2)$$

which implies that

$$\begin{cases} A_u = 0 \\ A_v = 1 - A^2. \end{cases} \tag{3.10}$$

Denoting the real and imaginary parts of F by L and H respectively and substituting (3.9) in (3.7) we get

$$F_{z\bar{z}} = i\,\mathcal{I}m(F),$$

which gives

$$\begin{cases} \Delta L = 0 \\ \Delta H = 4H. \end{cases} \tag{3.11}$$

Now, the first equation of (3.9) is equivalent to

$$\begin{cases} L_u + H_v = \dfrac{2H}{A} \\ H_u - L_v = 0. \end{cases} \tag{3.12}$$

Differentiating (3.12) and using (3.11) gives immediately

$$L_v = H_u = 0.$$

Finally, form the first equation of (3.12), we deduce that $L_u = c =$ costant. The function H can be computed using (3.10) and

$$\frac{2H}{A} - H_v = c.$$

The corresponding minimal immersion is given by

$$\chi(z) = (L(z), H(z), 2\mathcal{R}e(z)) = (cu + c_1, H(z), 2u), \quad c_1 \in \mathbb{R},$$

which is clearly either a part of the plane $x_3 = \frac{2}{c}x_1 - \frac{2c_1}{c}$, when $c \neq 0$, or a part of the plane $x_1 = c_1$ otherwise.

Example 3.4 (The helicoid)**.** We now assume that A is a complex valued function. Denoting by B and C the real and imaginary parts of A, with calculations similar to those in the previous example we find that B and C are solutions of

$$\begin{cases} B_u - C_v = \dfrac{-2BC}{|A|^2} \\ C_u + B_v = \dfrac{B^2 - C^2 - |A|^4}{|A|^2}. \end{cases} \tag{3.13}$$

One can check that

$$\begin{cases} B = \dfrac{(\sin v + \cos v)(\cos(2u) + \sin(2v))}{(\sin v - \cos v)(1 + \cos(2u)\,\sin(2v))} \\ \\ C = \dfrac{\sin(2u)(\sin v + \cos v)^2}{1 + \cos(2u)\,\sin(2v)} \end{cases} \tag{3.14}$$

is a solution of (3.13). The corresponding Enneper immersion is a minimal helicoid in $\mathbb{H}^2 \times \mathbb{R}$ given by:

$$\chi(u, v) = \left(\frac{\sin(2u)\,\sin(2v)}{1 + \cos(2u)\,\sin(2v)}, \frac{\cos(2v)}{1 + \cos(2u)\,\sin(2v)}, 2u\right).$$

From (3.6), we have that $g = -A(u, v)^{-1}$ and, therefore, we conclude that the rank of the Gauss map of the helicoid is two.

Example 3.5 (Horizontal planes)**.** If F is holomorphic and $F_z \neq 0$, then (3.2) and (3.3) of Theorem 3.1 are satisfied and (3.1) implies that $h(z) = \chi_3(z) = c \in \mathbb{R}$. This gives the conformal immersion in $\mathbb{H}^2 \times \mathbb{R}$ of the totally geodesic plane $z = c$.

We now show that any minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ can be rendered as the Enneper graph of a harmonic function. More precisely, we have the following

Theorem 3.6. *Let $\tilde{\rho} : \mathcal{M}^2 \to \mathbb{H}^2 \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$ be a minimal immersion of a surface $\mathcal{M}$ in $\mathbb{H}^2 \times \mathbb{R}$. Then there exists a simply connected domain $\Omega \subset \mathbb{C}$ and a harmonic function $h : \Omega \subset \mathbb{C} \to \mathbb{R}$ such that the immersed minimal surface $\tilde{\rho}(\mathcal{M})$ is an Enneper graph of h.*

Proof: Suppose that the minimal immersion is given by $\tilde{\rho}(z) = (\tilde{\rho}_1(z) + i\tilde{\rho}_2(z), \tilde{\rho}_3(z))$, with $\tilde{\rho}_2(z) > 0$. Since $\mathcal{M}$ is minimal it cannot be compact (if it were, then the third component would be a harmonic function on a compact surface and thus constant) so, from the Uniformization Theorem, it results that its covering space Ω is either the complex plane $\mathbb{C}$ or the open unit complex disc.

We denote by $\pi : \Omega \to \mathcal{M}$ the universal covering of $\mathcal{M}$ and by $\rho : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ the lift of $\tilde{\rho}$, i.e. $\rho = \tilde{\rho} \circ \pi$. Since ρ is a conformal minimal immersion, putting $\phi(z) = \rho_z(z)$ it follows, from [9], that $\sum_{i,j=1}^{3} g_{ij}\phi_i\phi_j = 0$, that is

$$\frac{(\rho_1)_z^2 + (\rho_2)_z^2}{(\rho_2)^2} + (\rho_3)_z^2 = 0. \tag{3.15}$$

Moreover, the minimality of ρ reduces to the following system:

$$\begin{cases} \dfrac{\partial}{\partial \bar{z}} \dfrac{(\rho_1)_z}{\rho_2} - \dfrac{\overline{(\rho_1)_z}(\rho_2)_z}{\rho_2^2} = 0 \\ \dfrac{\partial}{\partial \bar{z}} \dfrac{(\rho_2)_z}{\rho_2} + \dfrac{|(\rho_1)_z|^2}{\rho_2^2} = 0 \\ (\rho_3)_{z\bar{z}} = 0. \end{cases} \tag{3.16}$$

Last equation of (3.16) means that the function $\rho_3(z)$ is a harmonic function. Moreover, (3.15) suggests to define, for a fixed point $z_0 \in \Omega$, the following two complex functions:

$$\begin{cases} h(z) = \rho_3(z) \\ F(z) = \int_{z_0}^{z} [(\rho_1)_z + i\,(\rho_2)_z]\, dz + \int_{z_0}^{z} [(\rho_1)_{\bar{z}} + i\,(\rho_2)_{\bar{z}}]\, d\bar{z}. \end{cases}$$

Since Ω is simply connected and the 1-forms $(\rho_i)_z\, dz$ don't have real periods, the above integrals don't depend on the path from z_0 to z, so there are well defined. We shall prove that $\rho(z) = (F(z), h(z))$. For this, we have

$$\begin{aligned} F(z) &= \int_{z_0}^{z} [(\rho_1)_z + i\,(\rho_2)_z]\, dz + \int_{z_0}^{z} [(\rho_1)_{\bar{z}} + i\,(\rho_2)_{\bar{z}}]\, d\bar{z} \\ &= \int_{z_0}^{z} d\rho_1 + i \int_{z_0}^{z} d\rho_2 = \rho_1(z) + i\rho_2(z), \end{aligned}$$

where, in the last equality, we have assumed, without loss of generality, that $\rho(z_0) = (0, 0, 0)$. Next, from

$$\overline{F}_z = (\rho_1)_z - i(\rho_2)_z, \quad F_z = (\rho_1)_z + i(\rho_2)_z, \quad \mathcal{I}m(F) = \rho_2,$$

and taking into account (3.15), it results that

$$-\frac{F_z \overline{F}_z}{\mathcal{I}m(F)^2} = -\frac{(\rho_1)_z^2 + (\rho_2)_z^2}{(\rho_2)^2} = (\rho_3)_z^2 = (h_z)^2,$$

which is condition (3.1) of Theorem 3.1. In order to verify condition (3.2) of Theorem 3.1, subtract to the first equation of (3.16) the second multiplied by i to get

$$\frac{\partial}{\partial \bar{z}}\left(\frac{(\rho_1)_z - i(\rho_2)_z}{\rho_2}\right) = i\,\overline{(\rho_1)_z}\,\frac{(\rho_1)_z - i(\rho_2)_z}{\rho_2^2}.$$

Using this equality we conclude that

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}\Big(\frac{\mathcal{I}m(F)}{\overline{F}_z}\Big) + i\,\frac{(\mathcal{R}e(F))_{\bar{z}}}{\overline{F}_z} &= \frac{\partial}{\partial \bar{z}}\left(\frac{\rho_2}{(\rho_1)_z - i(\rho_2)_z}\right) + i\,\frac{\overline{(\rho_1)_z}}{(\rho_1)_z - i(\rho_2)_z} \\
= \frac{\partial}{\partial \bar{z}}\left(\frac{\rho_2}{(\rho_1)_z - i(\rho_2)_z}\right) &+ \frac{(\rho_2)^2}{[(\rho_1)_z - i(\rho_2)_z]^2}\,\frac{\partial}{\partial \bar{z}}\left(\frac{(\rho_1)_z - i(\rho_2)_z}{\rho_2}\right) = 0.
\end{aligned}$$

Finally, to prove that ρ is an Enneper immersion associated to the harmonic function h, we must verify (3.2), or equivalently that, for all $z \in \Omega$,

$$|(\rho_1)_z + i(\rho_2)_z| + |(\rho_1)_z - i(\rho_2)_z| \neq 0.$$

Suppose that there exists a point $z_1 \in \Omega$ such that

$$\big((\rho_1)_z + i(\rho_2)_z\big)(z_1) = 0 \quad \text{and} \quad \big((\rho_1)_z - i(\rho_2)_z\big)(z_1) = 0.$$

Then $(\rho_1)_z(z_1) = (\rho_2)_z(z_1) = 0$ and, from (3.15), $(\rho_3)_z(z_1) = 0$. This means that $\rho_u(z_1) = \rho_v(z_1) = 0$ which is in contradiction to the fact that ρ is an immersion.

$\square$

4 The Gauss map

In this section we study the Gauss map g of a minimal surface in $(\mathbb{H}^2\times\mathbb{R}, g_{\mathbb{H}^2\times\mathbb{R}})$. We start showing the following important fact.

Proposition 4.1. *If $\chi:\mathcal{M}^2\to\mathbb{H}^2\times\mathbb{R}$ is a conformal minimal immersion of a surface $\mathcal{M}^2$ in the space $\mathbb{H}^2\times\mathbb{R}$, then the Gauss map g of χ satisfies:*

$$g_{z\bar z}=\left(\frac{g_z}{g}-\frac{(1+\bar g^2)}{\bar g(1-\bar g^2)}\bar g_z\right)g_{\bar z}. \tag{4.1}$$

Proof: We know, from the second equation of (2.6), that

$$g_{\bar z}+\frac{i}{2}\frac{\bar f\, g}{\bar g}\,(1-\bar g^2)=0.$$

Differentiating this equation with respect to $\bar z$ and using the fact that $\bar f_z=0$, we get (4.1).

□

Remark 4.2. In terms of the Gauss map g the Enneper data (3.8) becomes:

$$D_\chi=\frac{2\,i\,\bar g_z}{\bar g(1-g^2)}(g^2,1,-g).$$

The importance of Proposition 4.1 will become manifest in the following:

Theorem 4.3. *Let $\mathcal{M}^2$ be a simply connected Riemann surface. Let $g:\mathcal{M}\to\mathbb{C}$ be a non-holomorphic solution of (4.1) and fix $z_0\in\mathcal{M}$, $F_0\in\mathbb{C}$ and $h_0\in\mathbb{R}$. Then there exists a unique minimal immersion $\chi:\mathcal{M}\to\mathbb{H}^2\times\mathbb{R}$, $\chi(z)=(F(z),h(z))$, with Gauss map g and $\chi(z_0)=(F_0,h_0)$. Moreover, the immersion χ satisfies the equations:*

$$F_z=2i\frac{g^2\bar g_z}{\bar g(1-g^2)}\,\mathcal{I}m(F),\qquad F_{\bar z}=2i\frac{g_{\bar z}}{g(1-\bar g^2)}\,\mathcal{I}m(F)$$

and

$$h_z = -2i\frac{g\overline{g}_z}{\overline{g}(1-g^2)}.$$

Proof: For the proof we follow the idea given by B. Daniel in [3]. We consider the differential system:

$$\begin{cases} F_z = 2i\dfrac{g^2\overline{g}_z}{\overline{g}(1-g^2)}\,\mathcal{I}m(F) := A\,\mathcal{I}m(F) \\ F_{\overline{z}} = 2i\dfrac{g_{\overline{z}}}{g(1-\overline{g}^2)}\,\mathcal{I}m(F) := B\,\mathcal{I}m(F). \end{cases} \tag{4.2}$$

Using (4.1), we can prove that

$$2i(A_{\overline{z}} - B_z) + |B|^2 - |A|^2 = 0,$$

which is equivalent to

$$F_{z\overline{z}} = F_{\overline{z}z}. \tag{4.3}$$

From Frobenius's Theorem (see, for example, [5, pag. 92]) the integrability condition of (4.2) reduces to (4.3). Thus, as $\mathcal{M}$ is simply connected, we conclude that (4.2) admits a unique solution $F : \mathcal{M} \to \mathbb{C}$ with $F(z_0) = F_0$. We now consider the equation

$$h_z = -2i\frac{g\overline{g}_z}{\overline{g}(1-g^2)}. \tag{4.4}$$

Using again (4.1), it is not hard to check that $h_{z\overline{z}} = 0$ and thus there exists a unique solution $h : \mathcal{M} \to \mathbb{R}$ so that $h(z_0) = h_0$. If we define $\chi(z) = (F(z), h(z))$, it is obvious that $\chi(z_0) = (F_0, h_0)$ and that

$$(h_z)^2 = -\frac{F_z\overline{F}_z}{\mathcal{I}m(F)^2}.$$

This equation implies that χ is conformal (apply (3.1) of Theorem 3.1).

Now, differentiating the first equation in (4.2) with respect to $\bar{z}$ and using (4.1), it follows that $2A_{\bar{z}} = -iA(\overline{A} + B)$, which is equivalent to

$$F_{z\bar{z}} + i\frac{F_z F_{\bar{z}}}{\mathcal{I}m(F)} = 0.$$

Therefore, from Remark 3.2, we conclude that χ is minimal. Also, since $g_{\bar{z}} \neq 0$, we have

$$|F_z|^2 + |F_{\bar{z}}|^2 \neq 0,$$

which guarantees that χ is an immersion.

Finally, from (4.4) and the second equation of (4.2), it results that

$$g = i\frac{\overline{g}(1-g^2)}{2\overline{g}_z}h_z = \frac{\mathcal{I}m(F)}{\overline{F}_z}h_z$$

so, comparing with (3.6), we deduce that g is the Gauss map of the immersion χ.

□

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