Matemática Contemporânea, Vol 35, 143-150

RECOLLECTIONS ON A CONJECTURE IN MATHEMATICS

M. Miranda

A homage to Manfredo do Carmo, in occasion of his 80th birthday

Caro Manfredo,

mi ha fatto molto piacere l'invito dei professori Harold Rosenberg e Hilario Alencar a scrivere alcune pagine del tuo volume ottuagenario. Ho pochi anni meno dei tuoi, anch'io quindi sono nell'etá dei nonni, piú pronto a raccontare una favola che a descrivere una nuova scoperta. Spero di farti cosa gradita,
Mario

Trento, Ottobre 2008

Abstract

In this work we have reconsidered the famous paper of Bombieri, De Giorgi and Giusti [4] and, thanks to the software Mathematica® we made it possible for anybody to control the difficult computations.

I. Ennio De Giorgi (1928-1996) and Wendell H. Fleming (1928-) met each other, the first time, in Genova (Italy), before participating in the International Congress of Mathematicians at Stockholm (Sweden), in August 1962.
Each one had a great interest in the work of the other, especially for the two papers of the last year De Giorgi, *Frontiere orientate di misura minima* [6] and Fleming, *On the oriented Plateau Problem* [10].
Wendell invited Ennio to spend a sabbatical year in the States. Ennio accepted the offer and arrived in New York on February 1964, with a mathematical gift for his friend. De Giorgi had added another result to the Fleming: *Una estensione del Teorema di Bernstein* [7].

Since the methods of Fleming and De Giorgi could be repeated, they hoped to have found a good Conjecture: the full regularities of "Frontiere orientate di misura minima", together with the Bernstein's Theorem in all dimensions.

II. A confirmation by Frederic J. Almgren jr. (1933-1997). Almgren studied mathematics at Brown University with Herbert Federer (1920-). In the same school, Wendell H. Fleming was a colleague of Federer.
In June 1965, Almgren travelled across Europe to meet some mathematicians. He stayed in Pisa to see De Giorgi and show his proof of the De Giorgi–Fleming's Conjecture for the three-dimensional cones in $\mathbb{R}^4$ (see Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's Theorem* [1]).
I was there and learned that Almgren had used a complex function of one complex variable to describe the intersection of the cone with the sphere around the vertex.
That use could not be applied to higher dimensions. Some time later, De Giorgi told me that we could discover minimal surfaces with singularities and non trivial entire functions solutions of the Bernstein's Problem. In other words De Giorgi–Fleming's Conjecture could be stopped at some point.

III. James H. Simons (1938-) proved that there were no singular minimal cones till $\mathbb{R}^7$. This meant also that there were no singular minimal cones in $\mathbb{R}^2$ (easy), $\mathbb{R}^3$ (Fleming), $\mathbb{R}^4$ (Almgren), plus (Simons) $\mathbb{R}^5$, $\mathbb{R}^6$ and $\mathbb{R}^7$ (see Simons, *Minimal varieties in Riemannian manyfolds* [17]).
Moreover, Simons conjectured that the cone

$$C = \left\{ (x,y)\,, x \in \mathbb{R}^4, y \in \mathbb{R}^4, x^2 - y^2 = 0 \right\},$$

singular in the vertex, was a minimal one too.
A second Conjecture was launched by Simons about the Bernstein's Problem: it will be solved in all dimension, without the help of the cones.

It was Spring 1968, when the news from Simons arrived to Pisa and excited Enrico Bombieri (1940-). Enrico could appreciate very well the paper of Simons, but he was not yet so expert with Geometric Measure Theory. Bombieri, De Giorgi and me, in those days, were working at the paper; Bombieri, De Giorgi, Miranda: *Una maggiorazione a priori relativa alle superficie minimali non parametriche* [5]. So, De Giorgi decided to work at the Conjectures of Simons after the Summer holidays.

IV. My sabbatical year in Minneapolis.

I left Pisa in July, after 13 years, 9 of those with De Giorgi, and moved to Minneapolis in September. In November, Bombieri sent me a message: "The first Simons' Conjecture is right, the second is wrong".

Ennio De Giorgi, Enrico Bombieri and Enrico Giusti (1940-) had proven [4] the minimality of Simons' cone and the existence of a solution of the Bernstein's equation in $\mathbb{R}^8$, with zeros only in the cone.

I was happy, because my previous work had been helpful, and in the future I could give talks about the Story, and make it easier to read the original paper.

At Christmastime, I flew to Pisa and talked with Bombieri. Back to Minnesota in January, I found an invitation to Berkeley and, in February, I gave a talk in the S.S. Chern's Seminar. Manfredo was there.

V. As a mathematician, I liked to travel abroad, alone or with my family. My preferences, California and Brazil. Twice in Brazil, I decided to start something good for Mathematics.

The first time, I was in Campinas and met Leopoldo Nachbin (1922-1993), who asked me to write a Monograph for "Notas de Matematica series".

I accepted, and gave Leopold my title: *Minimal surfaces of codimension one*, together with the decision of having my student Umberto Massari (1948-) as coauthor.

North Holland Publishing Company distributed our book of 243 pages in 1984

[12]. While writing it, Massari and I stopped in front of the famous Bombieri–De Giorgi–Giusti paper and were lucky to prove the minimality of Simons' cone, modifying the formula:

$$C = \left\{ (x, y), x \in R^4, y \in \mathbb{R}^4, \frac{x^2 - y^2}{2} \cdot \frac{x^2 + y^2}{2} = 0 \right\}.$$

VI. The Bernstein's Problem in higher dimension can be written: find $f \in C^2(\mathbb{R}^n)$, such that

$$Mf := \left(1 + |Df|^2\right) \Delta f - \sum_{i,j=1}^{n} D_i f D_j f D_{ij} f = 0.$$

The operator M is convenient if f is a polynomial. Unfortunately, nobody has discovered a non trivial polynomial solution of Bernstein's Problem.
But, the polynomial

$$P(x, y) = \frac{x^2 - y^2}{2} \cdot \frac{x^2 + y^2}{2}$$

gives

$$MP(x, y) = 3\left(x^2 - y^2\right)\left\{2 + \left(x^2 - y^2\right)^2 \left(x^2 + y^2\right)\right\},$$

and $P(x, y)$ is a subsolution for $x^2 > y^2$ and $P(x, y)$ is a supersolution for $x^2 < y^2$.
And more, the family $\{AP(x, y)\}_{A>0}$ of polynomials, gives

$$M\{AP(x, y)\}_{A>0} = 3A\left(x^2 - y^2\right)\left\{2 + A^2\left(x^2 - y^2\right)^2 \left(x^2 + y^2\right)\right\}.$$

So, $AP(x, y)$ is a subsolution for $x^2 > y^2$, $\forall A > 0$, $AP(x, y)$ is a supersolution for $x^2 < y^2$, $\forall A > 0$. The limits

$$\lim_{A \to +\infty} AP(x, y) = +\infty, \quad \forall x^2 > y^2;$$

$$\lim_{A \to +\infty} AP(x, y) = -\infty, \quad \forall x^2 < y^2;$$

implying

$$E_A = \left\{(x, y, z) : x, y \in \mathbb{R}^4, z < AP(x, y)\right\}$$

$$E_{+\infty} = \left\{(x,y) : x, y \in \mathbb{R}^4, x^2 = y^2\right\} \times \mathbb{R}.$$

$E_{+\infty}$ is a cylinder, minimal, because of the sub– and super–solution of $AP(x,y)$; and

$$\left\{(x,y) : x, y \in \mathbb{R}^4, x^2 = y^2\right\}$$

is a minimal cone!

VII. Manfredo celebrated his 60th Anniversary in Rio, in August 1988. I was there. We met at the IMPA, and one of those days, a young mathematician, Danilo Benarros (1960-), asked me if he could come to Trento for his Ph.D. studies. He had studied in Manaus and IMPA. He had a good knowledge of Differential Calculus for manifolds, and liked to make mathematics by computer.

I remembered Bombieri telling me in December 1968, how hard had been the calculi necessary to transfer De Giorgi's idea into existence of the first solution, non trivial, the Bernstein's Problem.

I thought that Danilo could make that transfer by S. Wolfram Mathematica® book [18].

So, Danilo came to Trento in 1990, stayed, with his family, four years, wrote his Ph.D. thesis: *I coni di Lawson e il teorema di Bernstein* (1994) [2].

VIII. De Giorgi's ideas. The first one was not too hard: $\frac{x^2-y^2}{2}\sqrt{\frac{x^2+y^2}{2}}$. The function

$$f(x,y) = \frac{x^2-y^2}{2}\sqrt{\frac{x^2+y^2}{2}},$$

was not a polynomial, but similar to our

$$P(x,y) = \frac{x^2-y^2}{2}\frac{x^2+y^2}{2}.$$

Not too easy, but not too hard, it was

$$Mf(x,y) > 0, \qquad \forall x^2 > y^2.$$

It is convenient to use Danilo's alphabet, for those who want to read his formulas. So, instead of f, I'd use φ, and

$$u = |x|, v = |y|, \quad \text{where } x, y \in \mathbb{R}^4.$$

Therefore, our first idea will be written,

$$\varphi(u,v) = \frac{u^2 - v^2}{2}\sqrt{\frac{u^2+v^2}{2}},$$

The second function had to be $\Phi(u,v) > \varphi(u,v)$, $\forall u > v$, and

$$M\Phi(u,v) < 0, \qquad \forall u > v;$$

$$M\Phi(u,v) > 0, \qquad \forall u < v.$$

The first attempt for Φ is

$$\Phi\left(u,v\right) = \frac{u^2-v^2}{2}\left\{1 + \sqrt{\frac{u^2+v^2}{2}}\left(1 + A\left|\frac{u^2-v^2}{u^2+v^2}\right|^a\right)\right\},$$

with $a = \frac{31}{96}$ and $A > 1$ to be fixed. We get:

$$\Phi(u,v) > \varphi(u,v) > 0, \qquad \text{for } u > v;$$

$$\Phi(u,v) < \varphi(u,v) < 0, \qquad \text{for } u < v.$$

Unfortunately

$$M\left(\Phi\right) < 0, \qquad \text{for } \frac{u^2-v^2}{2} \geq \frac{96}{\sqrt{31}} \text{ only.}$$

At last De Giorgi wrote

$$K\left(\Phi\left(u,v\right)\right) = \int_0^{\Phi} \exp\left(\int_{\tau}^{+\infty} B\frac{1}{w^{1-a}+w^{1+a}}dw\right)d\tau, \quad \forall \Phi > 0$$

with $B > 0$ to be fixed. And hit the target.

Post Scriptum

The computer calculi of Danilo Benarros can be found in *The Bernstein Theorem in higher Dimension* by U. Massari, M. Miranda, M. Miranda,jr, [13].

References

[1] Almgren, Jr., F. J., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2), 84 (1966), 277–292.

[2] Benarros, D., *I coni di Lawson e il Teorema di Bernstein*, University of Trento, 1994. Ph.D. Thesis.

[3] Benarros, D.; Miranda, M., *Lawson cones and the Bernstein Theorem*, Adv. in Geom. An. and Cont. Mech., 1995.

[4] Bombieri, E.; De Giorgi, E.; Giusti, E., *Minimal cones and the Bernstein problem*, Invent. Math., 7 (1969), 243–268.

[5] Bombieri, E.; De Giorgi, E.; Miranda, M., *Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche*, Arch. Rational Mech. Anal., 32 (1969), 255–267.

[6] De Giorgi, E.; Colombini, F.; Piccinini, L. C., *Frontiere orientate di misura minima e questioni collegate*, Scuola Normale Superiore, Pisa, 1972.

[7] De Giorgi, E., *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa (3), 19 (1965), 79–85.

[8] De Giorgi, E., *Selected Papers*, Springer–Verlag, Berlin, 2006.

[9] Do Carmo, M.; Peng, C. K., *Stable complete minimal hypersurfaces*, In Proceedings of 1980 Bejing Symposium on Differential Geometry and Differential Equations.

[10] Fleming, W. H., *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2), 11 (1962), 69–90.

[11] Massari, U.; Miranda, M., *A remark on minimal cones*, Boll. Un. Mat. Ital. A (6), 2(1)(1983), 123–125.

[12] Massari, U.; Miranda, M., *Minimal Surfaces of Codimension One*, North Holland Publishing Co., (1984).

[13] Massari, U.; Miranda, M.; Miranda Jr, M., *The bernstein theorem in higher dimension*, Boll. U. Mat. Ital., (9), I, (2008).

[14] Miranda, M., *Comportamento delle successioni convergenti di frontiere minimali*, Rend. Sem. Mat. Univ. Padova, 38 (1967), 238–257.

[15] Miranda, M., *Un principio di massimo forte per le frontiere minimali e una sua applicazione alla risoluzione del problema al contorno per l'equazione delle superfici di area minima*, Rend. Sem. Mat. Univ. Padova, 45 (1971), 355–366.

[16] Miranda, M., *Un teorema di esistenza e unicità per il problema dell'area minima in n variabili*, Ann. Scuola Norm. Sup. Pisa (3), 19 (1965), 233–249.

[17] Simons, J., *Minimal varieties in riemannian manifolds*, Ann. of Math. (2), 88 (1968), 62–105.

[18] Wolfram, S., *The Mathematica® book*, Fourth edition. Wolfram Media, Inc., Champaign, IL; Cambridge University Press, Cambridge, (1999).

E-mail: mario-miranda@fastwebnet.it