Matemática Contemporânea, Vol 36, 01-10

COCOMPACTNESS PROPERTIES OF MOSER-TRUDINGER FUNCTIONAL IN CONNECTION TO SEMILINEAR BIHARMONIC EQUATIONS IN FOUR DIMENSIONS

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Dedicated to Professor J. V. Gonçalves on the occasion of his 60th birthday

Abstract

The paper considers compactness properties in the borderline imbedding of the Sobolev space $H_0^2(\Omega)$ where Ω is the unit ball in $\mathbb{R}^4$. While the Trudinger-Moser-Adams functional $\int_\Omega e^{32\pi^2 u^2} dx$ is bounded on the unit ball B of $H_0^2(\Omega)$, continuous in the whole $H_0^2(\Omega)$ and weakly continuous in $B \setminus \{0\}$, it is not weakly continuous at zero. We show that the functional is continuous at zero if the sequence satisfies a modified weak convergence requirement. Such behavior of the sequence is inspired by cocompactness/concentration compactness reasoning used in the study of other semilinear elliptic problems.

1 Introduction

This paper is concerned with weak continuity properties of functionals with critical growth in the Sobolev spaces $H_0^m(\Omega)$, where $\Omega \subset \mathbb{R}^{2m}$ is the unit ball, $m \in \mathbb{N}$. The critical growth in this case is understood in the sense of the following generalization of Pohozhaev-Trudinger-Moser inequality (case $m = 1$, [5],[6],[10]) for $W_0^{N/m,m}$-spaces by D. R. Adams [1].

Let the norm in the Sobolev space $H_0^m(\Omega)$ over an open bounded set $\Omega \subset \mathbb{R}^{2m}$, $m \in \mathbb{N}$, be specified as

$$\|u\|^2 \stackrel{\text{def}}{=} \int_\Omega \mathcal{L}(u),$$

2000 *Mathematics Subject Classification.* Primary 35J20, 35J35, 35J60; Secondary 46E35, 47J30, 58E05.

Keywords. Trudinger-Moser-Adams inequality, borderline Sobolev imbeddings, concentration compactness, cocompactness, weak convergence, semilinear elliptic equations, biharmonic operator.

*Research supported by TIFR-CAM in Bangalore and by a travel grant from the Swedish Research Council

where $\mathcal{L}(u) = |\Delta^k u|^2$ if $m = 2k$ and $\mathcal{L}(u) = |\nabla\Delta^k u|^2$ if $m = 2k+1$, $k \in \mathbb{N}$. Then

$$\sup_{u\in H_0^m(\Omega),\|u\|\le 1} \int_\Omega e^{\beta_m u^2} < \infty, \tag{1.1}$$

where

$$\beta_m \overset{def}{=} (4\pi)^m m!$$

is the largest constant for which (1.1) holds.

Imbedding of the Sobolev space into the correspondent Orlicz space defined by the exponential nonlinearity in (1.1) is not compact, as it is the case for the limit imbedding of $H^m(\mathbb{R}^N)$ when $N > 2m$. The analogy is, however, not complete, since $\int |u|^{\frac{2N}{N-2m}}$ lacks weak continuity at any point, while $\int e^{\beta_m u^2}$ on the closed unit ball of $H_0^m(\Omega)$ is weakly continuous at every point but the origin (see e.g. [4]). Lack of weak continuity in the case $N > 2m$ is easy to demonstrate by perturbing a given function u by a sequence $t_k^{(N-2)/2} w(t_k(x - x_0))$, $t_k \to +0$ that weakly converges to zero, where the "profile" w, subjected to the scaling, is arbitrary. In fact, presence of such scaling profiles is, roughly speaking, the only way the imbedding of $H^m(\Omega)$ into $L^{2N/(N-2m)}$ loses compactness (see Lions, [3, 4]) and Struwe [7]). A general functional-analytic theorem [8, Theorem 3.1] demonstrated that in Hilbert spaces, equipped with noncompact gauges, one can always improve convergence of a bounded sequence (on an appropriate subsequence) by subtracting from it a series of gauged profiles, and the degree of improvement in the convergence depends on the robustness of the gauge group. In particular, the use of the product group of shifts and of dilation actions, leads to the $L^{2N/(N-2m)}(\mathbb{R}^N)$-convergence. In the terminology introduced in [9] (an imbedding of a Banach space X into a topological vector space Y is cocompact relative to a group D of automorphisms of X if for any $g_k \in D$ $g_k u_k \rightharpoonup 0$ in X implies $u_k \to 0$ in Y), one says that the imbedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2N/(N-2m)}(\mathbb{R}^N)$ is cocompact (relative to the action of translations and dilations on $\mathbb{R}^N$).

In the case $N = 2m$, a recent paper [2], dealing with the case $m = 1$ had introduced a group of unitary operators on $H_0^1(\Omega)$ that play a role similar to dilations in the case of higher dimensions, namely the transformations

$$u(x) \mapsto s^{-\frac{1}{2}} u(|x|^{s-1}x), \tag{1.2}$$

and proved cocompactness of imbedding of the subspace of radial functions in $H_0^1(\Omega)$ into the Orlicz space associated with the Pohozhaev-Trudinger-Moser inequality. The paper also proves the following continuity property of the Pohozhaev-Trudinger-Moser functional $\int_\Omega e^{4\pi u^2} dx$: if u_k is a sequence of radial functions in $H_0^1(\Omega)$ such that $(u_k, \mu_{t_k}) \to 0$ for any sequence $t_k \in (0,1)$, and the function $F \in C(\mathbb{R})$ satisfies $0 \le F(s) \le Ce^{pu^2}$ for some $C, p > 0$, then

$$\int_\Omega (F(u_k) - F(0))\, dx \to 0,$$

where μ_t are Moser functions known from evaluation of the best constant in (1.1) (for $m = 1$ see the definition in (2.1) below). This property can be also regarded as a cocompactness property, in the sense that convergence in the target space follows from a weak convergence enhanced by a set of operators. The transformations involved here are, however, not automorphisms of $H_0^2(\Omega)$ but linear functionals on $H_0^2(\Omega)$. Notably, operators (1.2) map Moser functions into Moser functions.

In this paper we prove a similar continuity property for the case $m = 2$. It remains, however, an open problem to find automorphisms for $H_0^m(\Omega)$ for $m >$ similar to (1.2). The case of general m is addressed in Conjecture 2.3.

2 Adams-Moser functions

Let us recall the family of functions $\mu_t \in H_0^m(B_1(0))$, $t \in (0,1)$, introduced by Moser [5] in the case $m = 1$ and by [1] in the case $m \ge 2$, that verify that $beta_m$ is the best exponent in (1.1), that is, for any $\beta > \beta_m$, $\lim_{t\to 0} \int e^{\beta\mu_t} = +\infty$), while $\|\mu_t\|_{H_0^m} = 1$.

For $m = 1$

$$\mu_t(x) \overset{def}{=} (2\pi)^{-\frac{1}{2}} \left(\log \frac{1}{t}\right)^{\frac{1}{2}} h_0 \left(\log \frac{1}{|x|} / \log \frac{1}{t}\right), t \in (0,1), \tag{2.1}$$

where

$$h_0(s) \stackrel{\text{def}}{=} \min\{s, 1\}.$$

The corresponding function family for $m \ge 2$ differs from the family for $m = 1$ only by a normalization constant and the function h_0 replaced by h_ϵ with mollified corners $s = 0, t$ so it remains in $H_0^m(B_1(0))$.

$$\mu_t(x) = \left(\omega_{2m-1} \log \frac{1}{t}\right)^{\frac{1}{2}} \alpha_m^{-1} h_{\epsilon(t)}(\log \frac{1}{|x|} / \log \frac{1}{t}) \tag{2.2}$$

where

$$\alpha_m = \omega_{2m-1} 2^{m/2-1} (\frac{m}{2} - 1)! m(m+2) \cdots (2m-2), m \text{ even},$$

$$\alpha_m = \omega_{2m-1} 2^{(m-1)/2} (\frac{m-1}{2})! (m+1)(m+3) \cdots (2m-2), m \text{ odd},$$

$h_\epsilon(s) = \epsilon \Phi(\frac{s}{\epsilon})$ for $s \in (0, \epsilon)$, $h_\epsilon(s) = s$ for $s \in (\epsilon, 1-\epsilon]$, $h_\epsilon(s) = 1 - \epsilon \Phi(\frac{1-s}{\epsilon})$ for $s \in (1-\epsilon, 1]$ and $h_\epsilon(s) = 1$ for $s > 1$,
with $\Phi \in C^\infty([0,1])$ such that $\Phi(0) = \Phi'(0) = \cdots = \Phi^{(m-1)}(0) = 0$, $\Phi(1) = \Phi'(1) = 1, \Phi''(1) = \cdots = \Phi^{(m-1)}(1) = 0$, and $\epsilon(t)$ is an arbitrarily fixed continuous function such that

$$\lim_{t \to 0} \epsilon(t) = 0, \lim_{t \to 0} \epsilon(t)(\log \frac{1}{t}) = \infty. \tag{2.3}$$

At the end of Section 3 in [1] it is shown that the functions μ_t have $H_0^m(B_1(0))$-norm equal $1 + o_{t \to 0}(1)$. For $m = 1$ the norm equals 1.

Let us in what follows, for the sake of convenience, use the notation $\eta_t \stackrel{\text{def}}{=} \log \frac{1}{t}$ and let $H_{0,r}^m(B_1(0))$ denote the subspace of radially symmetric functions in $H_0^m(B_1(0))$. We prove the following asymptotic orthogonality property of Adams-Moser functions.

Lemma 2.1. *If $\eta_t / \eta_s \to 0$, $s, t \in (0,1)$, then*

$$(\mu_t, \mu_s) \to 0, \tag{2.4}$$

where the scalar product is that of $H_0^2(B_1(0))$.

Proof: Assume without loss of generality that $t > s$. Then

$$(\mu_t, \mu_s) = C\eta_t^{1/2}\eta_s^{1/2}\int_t^1 \Delta h_{\epsilon(t)}(\eta_r/\eta_t)\Delta h_{\epsilon(s)}(\eta_r/\eta_s) =$$

$$C\eta_t^{-1/2}\eta_s^{-1/2}\int_t^1 \frac{1}{r^3}\partial_r[r^2 h'_{\epsilon(t)}(\eta_r/\eta_t)]\frac{1}{r^3}\partial_r[r^2 h'_{\epsilon(s)}(\eta_r/\eta_s)]r^3 dr =$$

$$C\eta_t^{-1/2}\eta_s^{-1/2}\int_t^1 \frac{4}{r^4}h'_{\epsilon(t)}(\eta_r/\eta_t)h'_{\epsilon(s)}(\eta_r/\eta_s)r^3 dr -$$

$$C\eta_t^{-1/2}\eta_s^{-3/2}\int_t^1 \frac{2}{r^4}h'_{\epsilon(t)}(\eta_r/\eta_t)h''_{\epsilon(s)}(\eta_r/\eta_s)r^3 dr -$$

$$C\eta_t^{-3/2}\eta_s^{-1/2}\int_t^1 \frac{2}{r^4}h''_{\epsilon(t)}(\eta_r/\eta_t)h'_{\epsilon(s)}(\eta_r/\eta_s)r^3 dr +$$

$$C\eta_t^{-3/2}\eta_s^{-3/2}\int_t^1 \frac{1}{r^4}h''_{\epsilon(t)}(\eta_r/\eta_t)h''_{\epsilon(s)}(\eta_r/\eta_s)r^3 dr. \quad (2.5)$$

The principal asymptotic term in this expansion is the first term with h'_ϵ replaced with $1_{[0,1]}$. Evaluation of the principal term gives

$$\eta_t^{-1/2}\eta_s^{-1/2}\int_t^1 \frac{1}{r}dr = \eta_t^{1/2}\eta_s^{-1/2} \to 0. \quad (2.6)$$

We now have to estimate the error in the first term in (2.5) due to the difference between h_ϵ and 1, as well as the remaining three terms. We give only typical calculations near the origin (the quantities near $r = 1$ are smaller). For the error in the first term the largest quantity we have

$$\eta_t^{-1/2}\eta_s^{-1/2}\int_t^{t^{1-\epsilon(t)}} \frac{1}{r}dr = \eta_t^{1/2}\eta_s^{-1/2}\epsilon(t)\eta_t = \epsilon(t)\eta_t^{1/2}/\eta_s^{1/2} \to 0 \quad (2.7)$$

as $s \le t \to 0$. For the second term:

$$\eta_t^{-1/2}\eta_s^{-3/2}\int_t^{t^{1-\epsilon(t)}} \frac{1}{r}1/\epsilon(t)dr = \eta_t^{-1/2}\eta_s^{-1/2} \to 0. \quad (2.8)$$

Evaluation for the third term is similar. For the last term we have

$$\eta_t^{-3/2}\eta_s^{-3/2}\int_t^{t^{1-\epsilon(t)}} \frac{1}{r}\frac{1}{\epsilon(t)\epsilon(s)}dr =$$

$$\eta_t^{-3/2}\eta_s^{-3/2}\frac{1}{\epsilon(t)\epsilon(s)}\epsilon(t)\eta_t =$$

$$\eta_t^{-3/2}\eta_s^{-1/2}\frac{1}{\epsilon(s)\eta_s} \to 0 \quad (2.9)$$

as $s \le t \to 0$.

$\square$

Lemma 2.2. *Let $u_k \in H^2_{0,r}(\Omega)$. If for every sequence $t_k \in (0,1)$, $(u_k, \mu_{t_k}) \to 0$, then*

$$\sup_{0<r<1} |u_k(r)| \left(\log \frac{1}{r}\right)^{-\frac{1}{2}} \to 0. \tag{2.10}$$

Proof: We have

$$\begin{aligned}(u, \mu_t) = C(\eta_t)^{\frac{1}{2}} \int_t^1 \Delta h_{\epsilon(t)}(\eta_r/\eta_t) \Delta u r^3 dr = \\ C(\eta_t)^{-\frac{1}{2}} \int_t^1 \frac{1}{r^3} \partial_r [r^2 h'_{\epsilon(t)}] \Delta u r^3 dr = \\ C(\eta_t)^{-\frac{1}{2}} \int_t^1 \frac{2}{r^2} h'_{\epsilon(t)} \Delta u r^3 dr - \\ - C(\eta_t)^{-\frac{3}{2}} \int_t^1 \frac{1}{r^2} h''_{\epsilon(t)} \Delta u r^3 dr\end{aligned} \tag{2.11}$$

We give first a heuristic sketch of the evaluation of (u, μ_t). Replace h'_ϵ with the value 1 which it approximates and take note that $1/r^2$ is a biharmonic function in the annulus of integration $t < r < 1$. Thus simplified, we would have, using Gauss formula on the annulus $t < r < 1$,

$$\begin{aligned}(u, \mu_t) \sim C(\eta_t)^{-\frac{1}{2}} \int_t^1 \frac{2}{r^2} \Delta u r^3 dr = \\ C(\eta_t)^{-\frac{1}{2}} (\frac{1}{t^2} u'(t) + \frac{2}{t^3} u) t^3 = \\ C(\eta_t)^{-\frac{1}{2}} \frac{1}{t} (t^2 u(t))'.\end{aligned} \tag{2.12}$$

Then if $(u_k, \mu_{t_k}) \to 0$, then, in the heuristic approximation,

$$(\eta_{t_k})^{-\frac{1}{2}} \frac{1}{t_k} (t_k^2 u_k(t_k))' \to 0,$$

and using compactness of the Sobolev imbedding for radial functions, we would have

$$\sup_{0 \le t \le 1} (\eta_t)^{-\frac{1}{2}} \frac{1}{t} (t_k^2 u_k(t))' \to 0$$

Integrating the (uniform in t) relation

$$(t^2u_k(t))' = o(t(\eta_t)^{-\frac{1}{2}})$$

we obtain (2.10). It remains, however, to evaluate all the errors produced by the heuristic approximation. It suffices to show that the error terms where

$$E_1(u,t) \stackrel{\mathrm{def}}{=} (\eta_t)^{-\frac{1}{2}} \int_t^1 \frac{1}{r^2}(h'_{\epsilon(t)} - 1)\Delta u r^3 dr$$

and

$$E_2(u,t) \stackrel{\mathrm{def}}{=} (\eta_t)^{-\frac{3}{2}} \int_t^1 \frac{1}{r^2} h''_{\epsilon(t)} \Delta u r^3 dr$$

satisfy $E_i(u_k,t_k) \to 0$, $i = 1,2$ for every sequence $t_k \to 0$. Indeed, applying the Gauss formula on the annulus $t < r < 1$, noting that all the boundary terms vanish and that $1/r^2$ is a harmonic function, we have

$$\begin{aligned} E_1(u,t) = (\eta_t)^{-\frac{1}{2}} \int_t^1 \frac{1}{r^2}(\Delta u r^3 dr = \\ (\eta_t)^{-\frac{1}{2}} \int_t^1 \frac{1}{r^2} \Delta h'_{\epsilon(t)} u r^3 dr + \\ 2(\eta_t)^{-\frac{1}{2}} \int_t^1 \partial_r \frac{1}{r^2} \partial_r h'_{\epsilon(t)} u r^3 dr. \end{aligned} \tag{2.13}$$

After elementary evaluations one may see that all three terms in the right hand side are dominated by the sum of

$$(\eta_t)^{-\frac{1}{2}} \frac{1}{(\epsilon(t)\eta_t)^j} \int_t^{t^{1-\epsilon(t)}} \frac{1}{r^4} u r^3 dr, \tag{2.14}$$

and of

$$(\eta_t)^{-\frac{1}{2}} \frac{1}{(\epsilon(t)\eta_t)^j} \int_{t^{\epsilon(t)}}^t dr, \tag{2.15}$$

with $j = 1$ or $j = 2$. By (2.3) it suffices to consider $j = 1$. The expression (2.15) can be estimated by $C(\epsilon(t)\eta_t^{3/t})^{-1} \int_{B_1(0)} |u| dx$, with $C(\epsilon(t)\eta_t^{3/t})^{-1} \to 0$ as $t \to 0$ by (2.3) and $\int |u_k| \to 0$ since $u_k \rightharpoonup 0$. The term (2.15) can be estimated by use of the Cauchy inequality and then, in the second factor, of the Hardy inequality

$$(\eta_t)^{-\frac{1}{2}}\frac{1}{\epsilon(t)\eta_t}\int_t^{t^{1-\epsilon(t)}}(|u|r^{-\frac{1}{2}})r^{-\frac{1}{2}}dr \leq$$
$$(\eta_t)^{-\frac{1}{2}}\frac{1}{\epsilon(t)\eta_t}\left(\int_t^{t^{1-\epsilon(t)}}\frac{1}{r}dr\right)^{\frac{1}{2}}\cdot$$
$$\left(\int_t^{t^{1-\epsilon(t)}}\frac{u^2}{r^4}r^3dr\right)^{\frac{1}{2}} \leq$$
$$(\eta_t)^{-\frac{1}{2}}\frac{1}{\epsilon(t)\eta_t}(\epsilon(t)\eta_t)^{\frac{1}{2}}\left(\int_t^{t^{1-\epsilon(t)}}\frac{1}{r}dr\right)^{\frac{1}{2}}\|\Delta u\|_2, \quad (2.16)$$

which also converges to zero whenever $t \to 0$. Evaluation of E_2 is analogous: the use of Gauss formula brings only zero boundary terms, $\Delta\frac{1}{r^2} = 0$ and the integrals eventually involved in evaluation are all of the form (2.14) or (2.15) with only different factors that depend on t and vanish as $t \to 0$.

□

Conjecture 2.3. *Orthogonality relation* (2.4) *and relation* (2.10) *are valid for all* $m \in \mathbb{N}$.

The volume of calculations necessary to verify this conjecture is excessive for the scope of this paper. Under the heuristic approximation $h'_\epsilon = 1_{[t,1]}$ the aymptotic orthogonality is easy to illustrate, but (2.10) under the heuristic approximation yields as a counterpart of $(t^r u_k(r))'$ from the proof above, an action of elliptic differential operators of increasing order which is not trivial to integrate. In either the evaluation of the error terms requires very bulky calculations.

3 A cocompactness property

Theorem 3.1. *Assume now that* $u_k \in H^2_{0,r}(\Omega)$ *satisfies*

$$(u_k, \mu_{t_k}) \to 0 \text{ for any sequence } t_k \in (0,1). \quad (3.1)$$

Let $F(x,s)$ be a continuous function on $\Omega \times \mathbb{R}$ satisfying

$$|F(x,s)| \le Ce^{ps^2} \tag{3.2}$$

for some $C, p > 0$. Then

$$\int_\Omega F(x, u_k(x))dx \to \int_\Omega F(x,0)dx. \tag{3.3}$$

Proof: By Lemma 2.2, there exists a sequence ϵ_k such that $|u_k(r)| \le \epsilon_k \eta_r^{\frac{1}{2}}$.

Then for k sufficiently large

$$|F(x, u_k(x))| \le Cr^{-p\epsilon_k} \le Cr^{-1} \tag{3.4}$$

Then (3.3) follows from (2.10) and Lebesgue theorem.

□

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