MULTIPLICITY OF NONTRIVIAL SOLUTIONS TO A PROBLEM INVOLVING THE WEIGHTED \( p \)-BIHARMONIC OPERATOR

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Dedicated to Professor J. V. Gonçalves on the occasion of his 60\textsuperscript{th} birthday

Abstract

In this paper we prove the existence of three solutions to a problem involving the weighted \( p \)-biharmonic operator. The first and second solutions are obtained as local minima using the Ekeland's Variational Principle and the third one is obtained by a variant of the Mountain Pass Theorem.

1 Introduction

In this paper we study the following class of quasilinear elliptic problems involving the \( p \)-biharmonic operator

\[
\begin{aligned}
\Delta (\rho(x)|\Delta u|^{p-2}\Delta u) + g(x, u) &= \lambda_1 h(x)|u|^{p-2}u & \text{in } \Omega, \\
u = 0 &= \Delta u & \text{on } \partial \Omega,
\end{aligned}
\]

where \( 1 < p < \infty \), \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) is a bounded domain with smooth boundary, \( \rho \in \mathcal{C} (\overline{\Omega}, \mathbb{R}) \) with \( \inf_{\overline{\Omega}} \rho(x) > 0 \). We also use the assumptions

\((G_1)\) \quad \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is bounded continuous function satisfying \( g(x, 0) = 0 \),

and its primitive denoted by

\[(G_2)\] \quad \( G(x, s) = \int_0^s g(x, t)dt \) is assumed to be bounded.
Let $X \equiv W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ be a Sobolev space endowed with the norm given by

$$\|u\| \equiv \left\{ \int_{\Omega} \rho |\Delta u|^p \, dx \right\}^{\frac{1}{p}}.$$ 

We define

$$\lambda_1 = \inf_{N} \left\{ \int_{\Omega} \rho |\Delta u|^p \, dx \right\},$$

where

$$N = \left\{ u \in X : \int_{\Omega} h |u|^p \, dx = 1 \right\},$$

the first eigenvalue of the following weighted eigenvalue problem

$$\begin{cases}
\Delta(\rho(x)|\Delta u|^{p-2}\Delta u) = \lambda_1 h(x)|u|^{p-2}u & \text{in } \Omega, \\
u = 0 = \Delta u & \text{on } \partial\Omega,
\end{cases}$$

(1.2)

where

$$(h) \quad h \in C(\bar{\Omega}, \mathbb{R}), \ h \geq 0 \text{ and } h > 0 \text{ on a subset of } \Omega \text{ with positive measure.}$$

We recall that by using a result by Talbi and Tsouli [18] (see also Drábek and Čotani [8]), we know that the first eigenvalue $\lambda_1$ is simple, isolated and positive. Moreover every eigenfunction $\phi_1$ associated with $\lambda_1$ can be chosen positive.

Here $\Delta(\rho(x)|\Delta u|^{p-2}\Delta)$ denotes the operator of fourth order called the $p$-biharmonic operator with weight. For $p = 2$ and $\rho = 1$, the operator becomes the iterated Laplacian which have been studied by many authors. For example, Lazer and McKenna [13] have pointed out that this type of nonlinearity furnishes a model for studying travelling waves in suspension bridges. Since then, more nonlinear biharmonic equations, including the $p$-biharmonic equations, have been studied. (See [14, 19].)

More exactly, this type of problem appears, for instance, in the study of Hooke’s law of nonlinear elasticity. (See [4, 6] and references therein.) While the $p$-biharmonic operator can be used to study a semilinear hamiltonian system of the form

$$\begin{cases}
-\Delta u = v^p & \text{in } \Omega, \\
u = 0 & \text{in } \partial\Omega,
\end{cases}$$

$$\begin{cases}
-\Delta v = u^q & \text{in } \Omega, \\
u = 0 & \text{in } \partial\Omega,
\end{cases}$$

$$u, v > 0 \text{ in } \Omega, \quad u, v = 0 \text{ on } \partial\Omega,$$
where Ω is smooth bounded domain and p, q ≥ 1.

Formally, from the first equation we have
\[ \nu = (-\Delta u)^{1/p} \]
and substituting on the second equation, we get
\[ -\Delta (|\Delta u|^{1/p-1}(-\Delta u)) = -\Delta (-\Delta u)^{1/p} = u^q, \quad x \in \Omega \]
\[ u = \Delta u = 0, \quad x \in \partial \Omega. \]

In this case, we are looking for solution in the Sobolev space \( W^{2,(p+1)}(\Omega) \). (See [7, 11]).

We define the energy functional \( I : X \rightarrow \mathbb{R} \) associated to problem (1.1) by
\[
I(u) \equiv \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx + \int_{\Omega} G(x, u) dx - \frac{\lambda_1}{p} \int_{\Omega} h|u|^p dx. \quad (1.3)
\]

Under assumptions \( G_1 \) and \( G_2 \), the functional \( I \in C^1(\Omega, \mathbb{R}) \) and its Fréchet derivative is given by
\[
I'(u) \cdot v = \int_{\Omega} \rho |\Delta u|^{p-2} \Delta u \Delta v dx + \int_{\Omega} g(x, u)v dx - \lambda_1 \int_{\Omega} h|u|^{p-2} uv dx. \quad (1.4)
\]

The main goal of this paper is to show the existence of multiple solutions for problem (1.1). We were inspired by Gonçalves and Miyagaki [10] and also by Alves, Carrião and Miyagaki [3], in which problems involving the laplacian and \( p \)-laplacian operators are studied, respectively. See also Ma and Sanches [15].

We define
\[
V = \langle \phi_1 \rangle \quad \text{and} \quad Z = \left\{ u \in X : \int_{\mathbb{R}} h u |\phi_1|^{p-2} \phi_1 = 0 \right\}.
\]

Note that \( Z \) is a closed complementary subspace of \( V \) and therefore we have the direct sum
\[ X = V \oplus Z. \]
We define
\[ \lambda_2 = \inf_Z \left\{ \int_{\Omega} \rho|\Delta u|^p dx : \int_{\Omega} h|u|^p dx = 1 \right\}, \tag{1.5} \]
which satisfies \(0 < \lambda_1 < \lambda_2\), and it follows that
\[ \int_{\Omega} h|w|^p dx \leq \frac{1}{\lambda_2} \int_{\Omega} \rho|\Delta w|^p dx, \quad \text{for all } w \in Z. \tag{1.6} \]

We impose the following

\((G_3)\) \quad \(g(x, t) \to 0\) as \(|t| \to \infty\), for all \(x \in \Omega\).

\((G_4)\) \quad \(G(x, t) \geq \frac{\lambda_1 - \lambda_2}{p} h(x)|t|^p\), for all \(x \in \Omega\) and for all \(t \in \mathbb{R}\).

\((G_5)\) \quad There exist \(\delta > 0\) and \(0 < m < \lambda_1\) such that
\[ G(x, t) \geq \frac{m}{p} h(x)|t|^p, \quad \text{for all } x \in \Omega \text{ and for all } |t| < \delta. \]

We define
\[ T(x) = \liminf_{|t| \to \infty} G(x, t) \text{ and } S(x) = \limsup_{|t| \to \infty} G(x, t) \text{ for all } x \in \Omega. \]

\((G_6)\) \quad There exist \(t^-, t^+ \in \mathbb{R}\) with \(t^- < 0 < t^+\) such that
\[ \int_{\Omega} G(x, t^+) \phi_1 dx \leq \int_{\Omega} T(x) dx < 0 \]
and
\[ \int_{\Omega} S(x) dx \leq 0. \tag{G_7} \]

Define the following subsets
\[ C^+ = \{t \phi_1 + z : t \geq 0 \text{ and } z \in Z\} \text{ and } C^- = \{t \phi_1 + z : t \leq 0 \text{ and } z \in Z\}. \]

We remark that \(\partial C^+ = \partial C^- = Z\).

Now we state our main result.
Theorem 1

(i) Under assumptions \((h), (G_1), (G_2), (G_4)\) and \((G_6)\), there exist \(u \in C^+\) and \(v \in C^-\) solutions of problem (1.1) such that \(I(u) < 0\) and \(I(v) < 0\).

(ii) Under assumptions \((h), (G_1)-(G_3), (G_5)-(G_7)\), problem (1.1) has a solution \(w\) such that \(I(w) > 0\).

The first and second solutions are obtained as local minima of the energy functional \(I\). To do this, we use the Ekeland’s variational principle in each of the subsets \(C^+\) and \(C^-\). The third solution is obtained by using a variant of the Mountain Pass Theorem. In the last section we give an example for Theorem 1.

2 Preliminary results

We begin by recalling that \(I : X \to \mathbb{R}\) is said to satisfy the Palais-Smale condition at the level \(c \in \mathbb{R}\) \(((PS)_c\) in short), if any sequence \({u_n} \subset X\) such that

\[
I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to \infty,
\]

has a convergent subsequence in \(X\).

Our first lemma is proved by adapting some arguments used by Anane and Gossez [1] and by Alves, Carrião and Miyagaki [3].

Lemma 2 Assume the conditions \((h), (G_1)\) and \((G_2)\). Then the functional \(I\) satisfies the \((PS)_c\) condition for all \(c < \int_\Omega T(x)dx\).

Proof. We will prove that the sequence \({u_n} \subset X\) is bounded. Suppose, on the contrary, that it is unbounded. Then, up to subsequence, we have

\[
\|u_n\| \to \infty \quad \text{as} \quad n \to \infty.
\]

Define

\[
v_n = \frac{u_n}{\|u_n\|}
\] (2.1)
Clearly $\|v_n\| = 1$ and the sequence $\{v_n\} \subset X$ is bounded. Taking a subsequence if necessary (still denoted in the same way) we obtain

$$v_n \rightharpoonup v \text{ weakly in } X \text{ as } n \to \infty$$

and

$$v_n \rightarrow v \text{ in } L^s(\mathbb{R}), \text{ as } n \to \infty, \text{ for } 1 \leq s < p^* = \frac{np}{n - 2p}, \text{ (2.2)}$$

and $p^* = +\infty$, if $n \leq 2p$.

We will show that $v \neq 0$ and that there exists $\mu \in \mathbb{R}$ such that

$$v(x) = \mu \phi_1(x) \text{ for all } x \in \Omega.$$

We are going to consider only the case $n > 2p$, the other case is easier. By definition of $I$ and by the fact that $\Delta u_n = \Delta v_n \|u_n\|$ we have

$$I'(u_n) \cdot u_n = \int_{\Omega} \rho|\Delta u_n|^p dx + \int_{\Omega} g(x, u_n)u_n dx - \lambda_1 \int_{\Omega} h|u_n|^p dx$$

$$= \|u_n\|^p \int_{\Omega} \rho|\Delta v_n|^p dx + \int_{\Omega} g(x, u_n)u_n dx - \lambda_1 \|u_n\|^p \int_{\Omega} h|v_n|^p dx.$$

Choosing $t_n = \|u_n\|$, it follows that

$$\frac{I'(u_n) \cdot u_n}{t_n^{p_n}} = \int_{\Omega} \rho|\Delta v_n|^p dx + \frac{1}{t_n^{p_n}} \int_{\Omega} g(x, u_n)u_n dx - \lambda_1 \int_{\Omega} h|v_n|^p dx. \text{ (2.3)}$$

We will denote the terms of the equality (2.3) by $I_j$ ($j = 1, 2, 3, 4$), respectively.

Claim 3

(a) $\lim_{n \to \infty} I_1 = 0$,

(b) $\lim_{n \to \infty} I_3 = 0$,

(c) $\lim_{n \to \infty} I_4 = \lambda_1 \int_{\Omega} h|v|^p dx$.

Proof. (a) From the fact that $\lim_{n \to \infty} I'(u_n) = 0$ and since $\{u_n\} \subset X$ is unbounded we have the inequality

$$\left| \frac{I'(u_n) \cdot u_n}{t_n^{p_n}} \right| \leq \epsilon \|u_n\|_p = \epsilon \|u_n\|^{1-p}.$$
This implies that $\lim_{n \to \infty} I_1 = 0$.

(b) By the condition $(G_1)$, the Hölder’s inequality, and (2.2) we get
\[
\left| \frac{1}{t^n} \int_\Omega g(x, u_n) u_n \, dx \right| \leq \frac{C}{t^n} \int_\Omega |u_n| \, dx \leq \frac{C}{t^n} \left[ \int_\Omega |u_n| \, dx \right]^\frac{1}{p} \left[ \int_\Omega 1 \, dx \right]^\frac{p-1}{p},
\]
where $C$ and $M$ are positive constants. This implies that $\lim_{n \to \infty} I_3 = 0$.

(c) Follows immediately from (2.2).

Using Claim 3 and (2.1) we obtain that $v \neq 0$ because
\[
\lim_{n \to \infty} \left[ \int_\Omega \rho |\Delta u_n|^p \, dx - \lambda_1 \int_\Omega h |u_n|^p \, dx \right] = 1 - \lambda_1 \int_\Omega h |v|^p \, dx = 0.
\]
Since $v_n \rightharpoonup v$ weakly in $X$, as $n \to \infty$, we have $\|v\| \leq \liminf_{n \to \infty} \|v_n\| = 1$. Therefore
\[
\|v\| \leq 1 \quad (2.4)
\]
and we conclude that $v$ is an eigenfunction associated to the simple eigenvalue $\lambda_1$. Hence, there exists $\mu \in \mathbb{R}$, $\mu \neq 0$, such that
\[
v(x) = \mu \phi_1(x) \text{ for all } x \in \Omega. \quad (2.5)
\]
In particular, by (2.1) we conclude that
\[
\lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} \frac{u_n}{\|u_n\|} = v(x) = \mu \phi_1(x), \text{ for all } x \in \Omega.
\]
But $\mu \phi_1(x) \neq 0$, then $v_n(x) \neq 0$ and this implies that
\[
\lim_{n \to \infty} |u_n(x)| = \lim_{n \to \infty} \|u_n(x)\|v_n(x) = \infty, \text{ for all } x \in \Omega. \quad (2.6)
\]
Using Fatou’s Lemma, we have that
\[
\liminf_{n \to \infty} \int_\Omega G(x, u_n(x)) \, dx \geq \int_\Omega \liminf_{n \to \infty} G(x, u_n(x)) \, dx \geq \int_\Omega T(x) \, dx. \quad (2.7)
\]
By definition of $\lambda_1$ we conclude that
\[
\int_\Omega \rho |\Delta u_n|^p \, dx - \lambda_1 \int_\Omega h |u_n|^p \, dx \geq 0 \quad (2.8)
\]
and hence
\[ c + o_n(1) = I(u_n) \geq \int_{\Omega} G(x, u_n(x)) dx. \tag{2.9} \]

Since \( \lim_{n \to \infty} |u_n(x)| = \infty \), by \((G_2)\) it follows that
\[ c \geq \int_{\Omega} T(x) dx, \]
which contradicts the hypothesis of the Lemma. Hence the sequence \( \{u_n\} \subset X \) is bounded.

We claim that \( \lim_{n \to \infty} u_n = u \in X \). In fact, consider
\[
I'(u_n) \cdot (u_n - u) = \int_{\Omega} \rho |\Delta u_n|^{p-2} \Delta u_n \Delta (u_n - u) dx + \int_{\Omega} g(x, u_n)(u_n - u) dx \\
- \lambda_1 \int_{\Omega} h|u_n|^{p-2} u_n (u_n - u) dx.
\]

Since the sequence \( \{u_n - u\} \subset X \) is bounded and \( \lim_{n \to \infty} I'(u_n) = 0 \), we have
\[
\lim_{n \to \infty} I'(u_n) \cdot (u_n - u) = 0. \tag{2.10}
\]

Using \((G_2)\), the facts that \( u_n \to u \) in \( L^s(\mathbb{R}) \) (for \( 1 \leq s < p^* \)) and that \( u_n \to u \) a.e. on \( \Omega \) as \( n \to \infty \), as well as the Dominated Convergence Theorem we obtain
\[
\lim_{n \to \infty} \int_{\Omega} g(x, u_n)(u_n - u) dx = 0. \tag{2.11}
\]
and
\[
\lim_{n \to \infty} \lambda_1 \int_{\Omega} h|u_n|^{p-2} u_n (u_n - u) dx = 0. \tag{2.12}
\]

It follows from (2.10), (2.11) and (2.12) that
\[
0 = \lim_{n \to \infty} \left[ \int_{\Omega} \rho |\Delta u_n|^{p-2} \Delta u_n \Delta (u_n - u) dx \right]. \tag{2.13}
\]

Since \( |\Delta u|^{p-2} \Delta u \in L^{p^*}(\mathbb{R}) \), \( \rho \Delta (u_n - u) \in L^p(\mathbb{R}) \), by a result in [12, Theorem 13.44] we conclude that
\[
\lim_{n \to \infty} \int_{\Omega} \rho |\Delta u|^{p-2} \Delta u \Delta (u_n - u) dx = 0, \tag{2.14}
\]
where we are assuming that
\[
\Delta u_n \longrightarrow \Delta u, \text{ a.e., as } n \to \infty.
\]
The above affirmative can be proved arguing as in [5] (see also Alves, Carrião and Miyagaki in [2] for the case in dimension 1), together with the inequalities

\[
[|x|^{p-2}x - |y|^{p-2}y](x-y) \geq \begin{cases} 
C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2 \\
C_p |x-y|^p & \text{if } p \geq 2, \forall x, y \in \mathbb{R}^N,
\end{cases}
\]

(for the proof, see [16, 17]).

Now, by using again the above inequality, we obtain by (2.13) and (2.14)

\[
0 = \lim_{n \to \infty} \int_{\Omega} \left[ |\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u \right] \rho (u_n - u) dx \\
\geq \begin{cases} 
C_p \lim_{n \to \infty} \int_{\Omega} \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx & \text{if } 1 < p < 2 \\
C_p \lim_{n \to \infty} \int_{\Omega} \rho |\Delta u_n - \Delta u|^p dx & \text{if } p \geq 2.
\end{cases} \tag{2.15}
\]

If \( p \geq 2 \), we have that

\[
\lim_{n \to \infty} \int_{\Omega} \rho |\Delta u_n - \Delta u|^p dx \leq 0.
\]

If \( 1 < p < 2 \), by Hölder’s inequality it follows that

\[
\int_{\Omega} \rho |\Delta u_n - \Delta u|^p dx \\
\leq \left[ \int_{\Omega} \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx \right]^{\frac{p}{2}} \left[ \int_{\Omega} \rho (|\Delta u_n| + |\Delta u|)^p dx \right]^{\frac{2-p}{2}} \\
\leq C \left[ \int_{\Omega} \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx \right]^{\frac{p}{2}}.
\]

By (2.15) and the previous inequality it follows that

\[
0 \geq C_p \lim_{n \to \infty} \int_{\Omega} \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx \geq C_p C^{-1} \left[ \lim_{n \to \infty} \int_{\Omega} \rho |\Delta u_n - \Delta u|^p dx \right]^{\frac{2}{p}}.
\]

Therefore, in both cases we have

\[
\lim_{n \to \infty} \|u_n - u\| = 0 \text{ in } X
\]

and this concludes the proof of the Lemma.
Lemma 4 Assume the conditions (h), (G_2) and (G_6). Then the functional I is bounded from below on X and \( \inf_{C \pm} I \) is negative on \( C^+ \) and on \( C^- \).

**Proof.** Let \( u \in X \); by condition \( G_2 \), we have \( |\int_{\Omega} G(x, u)dx| \leq C \). Hence, by the definition of \( \lambda_1 \) we get
\[
|I(u)| \geq \int_{\Omega} G(x, u)dx \geq -C
\]
and I is bounded from below on X.

Using condition \( (G_6) \) and the eigenfunction \( \phi_1 \) associated to the eigenvalue \( \lambda_1 \) we obtain
\[
I(t\pm\phi_1) = \int_{\Omega} G(x, t\pm\phi_1)dx \leq \int_{\Omega} T(x) < 0. \quad (2.16)
\]
If \( u \in C^+ \), we have that \( I(u) = I(t\phi_1 + z) \). In particular, consider \( t = t^+ \) and \( z = 0 \); by inequality (2.16), we obtain that \( I(t^+\phi_1) < 0 \). Similarly, we have \( I(t^-\phi_1) < 0 \). Hence \( \inf_{C^\pm} I(t^\pm\phi_1) < 0 \). This concludes the proof of the lemma.

Now we show that the energy functional I verifies the geometry of the Mountain Pass Theorem.

**Lemma 5** The energy functional I verifies the following properties.

(a) \( I(0) = 0 \).

(b) There exist positive constants \( \rho \) and \( R \) such that \( I(u) \geq \rho > 0 \) if \( \|u\| = R \).

(c) There exists \( z \in X \) such that \( I(z) < 0 = I(0) \) if \( \|z\| > R \).

**Proof.** The proof of the item (a) is immediate.

Since \( G \) is bounded and continuous, there exist \( \theta \in \mathbb{R} \) (with \( p < \theta < p^* \)) and a constant \( C \) such that
\[
G(x, t) \geq \frac{m}{p}h(x)|t|^p - C|t|^\theta, \text{ for all } |t| > \delta,
\]
where \( \delta \) is given by \( (G_5) \). Therefore, by \( (G_5) \) we conclude that
\[
G(x, t) \geq \frac{m}{p}h(x)|t|^p - C|t|^\theta, \text{ for all } |t| \in R, p < \theta < p^* \text{ and for all } x \in \Omega.
\]
\[ (2.17) \]
By the previous inequality we have
\[ I(u) \geq \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx + \int_{\Omega} \left[ \frac{m}{p} h(x)|u|^p - C|u|^\theta \right] dx - \frac{\lambda_1}{p} \int_{\Omega} h|u|^p dx. \]

We recall that the embedding \( W^{1,p}(\mathbb{R}) \hookrightarrow L^s(\mathbb{R}) \) is continuous for \( 1 < p < s \leq p^* \) and compact for \( s < p^* \) and
\[ \lambda_1 \leq \frac{\int_{\Omega} \rho |\Delta u|^p dx}{\int_{\Omega} h|u|^p dx}. \]

Then, for \( p < \theta < p^* \) we have
\[ I(u) \geq \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx - \left[ \frac{\lambda_1 - m}{p} \right] \frac{1}{\lambda_1} \int_{\Omega} \rho |\Delta u|^p dx - C \int_{\Omega} |u|^p dx \]
\[ \geq \frac{m}{p \lambda_1} \|u\| - \mu \|u\|^\theta. \]

Since
\[ I(u) \geq \frac{m}{p \lambda_1} \|u\| + o(\|u\|), \text{ as } \|u\| \to \infty, \]
we can find \( R > 0 \) small enough and \( \rho > 0 \) such that if \( \|u\| \leq R \), then \( I(u) \geq 0 \) and if \( \|u\| = R \), then \( I(u) \geq \rho > 0 \). As a result, item \( (b) \) is proved.

To prove item \( (c) \), it is sufficient to remark that by \((G_6)\) we conclude that \( I(t^\pm \phi_1) < 0 \). Then we define \( z \equiv t^\pm \phi_1 \) and we get \( \|z\| = \int_{\Omega} \rho |\Delta(t^\pm \phi_1)|^p dx = t^\pm \|\phi_1\| \equiv R_1 \). Note that \( R_1 > R \) and it follows that \( \|z\| > R \) and \( I(z) < 0 \). This concludes the proof of item \( (c) \).

\( \square \)

3 Proof of Theorem 1

To prove item \( (i) \) we use inequality \((2.16)\) to obtain
\[ \inf_{C^\pm} I(u) \leq I(t^\pm \phi_1) = \int_{\Omega} G(x,t^\pm \phi_1) dx \leq \int_{\Omega} T(x) < 0. \]
If \( \inf_{C^\pm} I(u) = I(t^\pm \phi_1) \), then it is enough to take \( u = t^+ \phi_1 \) and \( v = t^- \phi_1 \) to get two solutions such that \( I(u) < 0 \) and \( I(v) < 0 \).
Otherwise, if \( \inf_{C^\pm} I(u) < I(t^\pm \phi_1) \) then we have

\[
\inf_{C^\pm} I(u) < \int_\Omega T(x). \tag{3.1}
\]

By Lemma 4, the functional \( I \) is bounded from below on \( X \) and it is easy to prove that \( I \) is lower semicontinuous in \( X \). Hence, the Ekeland’s Variational Principle guarantees the existence of two sequences \( u_n \subset C^+ \) and \( v_n \subset C^- \) satisfying

\[
I(u_n) \to \inf_{C^+} I(u) \quad \text{and} \quad I'(u_n) \to 0,
\]

and

\[
I(v_n) \to \inf_{C^+} I(v) \quad \text{and} \quad I'(v_n) \to 0.
\]

as \( n \to \infty \). By (3.1) and by Lemma 2, there exist \( u \) and \( v \) such that

\[
u_n \to u \quad \text{and} \quad v_n \to v \quad \text{in} \quad X
\]

as \( n \to \infty \). Therefore, \( u \) and \( v \) are solutions of problem 1.1 verifying

\[
I(u) = \inf_{C^+} I(z) < 0 \quad \text{and} \quad I(v) = \inf_{C^-} I(z) < 0.
\]

Moreover, it follows from assumption \((G_4)\) and from inequality (1.6) that

\[
I(z) \geq \frac{1}{p} \int_\Omega |\Delta u|^p dx - \frac{\lambda_2}{p} \int_\Omega |z|^p dx \geq 0, \quad \text{for all} \quad z \in Z.
\]

Then \( I(z) \geq 0 \) for all \( z \in Z \) and Lemma 2 implies that the infimum of \( I \) on \( C^\pm \) is achieved in \( C^\pm \setminus Z \). Therefore \( u \in C^+ \) and \( v \in C^- \).

To prove item \((ii)\) we use Lemma 5 and a variant of the Mountain Pass Theorem without the Palais-Smale condition. (See [9, Theorem 6].) Then there exists a sequence \( \{w_n\} \subset X \) such that

\[
I(w_n) \to c_1 > \rho > 0 \quad \text{and} \quad \|I'(w_n)\|_{X^*} (1 + \|w_n\|) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty. \tag{3.2}
\]

Arguing as in the proof of Lemma 2, we choose \( t_n = \|w_n\| \) to obtain

\[
\left| \frac{I'(u_n) \cdot w_n}{t_n^p} \right| \leq \frac{\|I'(w_n)\|_{X^*} (1 + \|w_n\|)}{t_n^p} \to 0, \quad \text{as} \quad n \to \infty.
\]
If the sequence \( \{w_n\} \subset X \) is unbounded, then
\[
|w_n(x)| \to 0, \text{ as } n \to \infty, \text{ for all } x \in \Omega.
\]
Since
\[
|I'(w_n) \cdot w_n| \leq \|I'(w_n)\|_X \cdot (1 + \|w_n\|),
\]
by (3.2) we obtain that
\[
|I'(w_n) \cdot w_n| \to 0 \text{ as } n \to \infty
\]
and hence
\[
o(1) = I'(w_n) \cdot w_n = \|w_n\|^p + \int_{\Omega} g(x, w_n) w_n dx - \lambda_1 \int_{\Omega} h|w_n|^p dx.
\]
By (2.8) we conclude that
\[
0 \leq \|w_n\|^p - \lambda_1 \int_{\Omega} h|w_n|^p dx = - \int_{\Omega} g(x, w_n) w_n dx + o(1) \leq \left| \int_{\Omega} g(x, w_n) w_n dx \right| + o(1)
\]
By \((G_1)\) and \((G_3)\), the function \(g(x, w_n(x))w_n(x)\) is bounded for all \(x \in \Omega\) and for all \(n\). By (2.6), \(g(x, w_n(x))w_n(x) \to 0\) as \(n \to \infty\) a.e. on \(\Omega\). Using the Dominated Convergence Theorem we obtain \(\int_{\Omega} g(x, w_n) w_n dx \to 0\), as \(n \to \infty\). Then
\[
\|w_n\|^p - \lambda_1 \int_{\Omega} h|w_n|^p dx \to 0, \text{ as } n \to \infty.
\]
Since
\[
c_1 + o(1) = I(w_n) = \frac{1}{p} \left[ \|w_n\|^p - \frac{\lambda_1}{p} \int_{\Omega} h|w_n|^p dx \right] + \int_{\Omega} G(x, w_n) dx,
\]
using Fatou’s Lemma, together with (2.6) and \((G_7)\), we obtain
\[
c_1 \leq \limsup_{n \to \infty} \int_{\Omega} G(x, w_n) dx \leq \int_{\Omega} S(x) dx \leq 0,
\]
which contradicts (3.2). Hence the sequence \(\{w_n\} \subset X\) is bounded and, passing to a subsequence if necessary (still denoted in the same way), there exists \(w \in X\) such that
\[
w_n \rightharpoonup w \text{ in } X, \text{ as } n \to \infty.
\]
We also have \( \| I'(w_n) \|_{X^*}(1 + \|w_n\|) \to 0 \) as \( n \to \infty \), it follows that \( \| I'(w_n) \|_{X^*} \to 0 \) in \( X^* \) as \( n \to \infty \) and by a similar argument as that of Lemma 2 we conclude that
\[
w_n \to w, \quad \text{in } X \quad \text{as } n \to \infty
\]
and the Theorem is proved. \( \square \)

4 Example

In this section, inspired by [3], we will define a function \( g \) that satisfies the assumptions \((G_1) - (G_7)\).

Consider \( \Omega = (0, 1) \), \( p = 2 \) and \( h = 1 \). In this case, the function \( \phi_1(x) = \sin(\pi x) \) is an eigenfunction associated to the first eigenvalue \( \lambda_1 = \pi^4 \) of problem (1.1). We remark that \( \phi_1 \) is symmetric with respect to \( x = \frac{1}{2} \).

Let \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be defined by
\[
g(x, s) = R(x)g_1(x)
\]
where \( R(x) = 1 \) and \( g_1 : \mathbb{R} \to \mathbb{R} \) is given by
\[
g_1(s) = \begin{cases} 
  s, & \text{for } 0 \leq s \leq 1, \\
  2 - s, & \text{for } 1 < s \leq 5, \\
  s - 8, & \text{for } 5 < s \leq 8 + \frac{\sqrt{30}}{2}, \\
  8 + \sqrt{30} - s, & \text{for } 8 + \frac{\sqrt{30}}{2} < s \leq 8 + \sqrt{30}, \\
  0, & \text{for } s \geq 8 + \sqrt{30}, \\
  -g(-s), & \text{for } s \leq 0,
\end{cases}
\]

Defining \( G_1(s) = \int_0^s g_1(t)dt \) we have
\[
G(x, s) = \int_0^s g(x, t)dt = R(x)G_1(s) \quad \text{and} \quad S(x) = T(x) = -\frac{R(x)}{2}.
\]
Choosing $\delta < 1$ it is easy to see that $g$ verifies the assumptions $(G_1) - (G_5)$ and $(G_7)$.

Now we have to prove that $g$ also verifies $(G_6)$, for $t^+ = 8$ and $t^- = -8$.

Since $\phi_1$ is symmetric with respect to $x = \frac{1}{2}$, the same is true for $G$. Then we have $G(x, 8\phi_1(x)) = G(1 - x, 8\phi_1(1 - x))$ and

$$
\int_0^1 G(x, 8\phi_1(x))dx = 2 \int_0^{\frac{1}{2}} G(x, 8\phi_1(x))dx = \int_0^{\frac{1}{2}} G(8\phi_1(x))dx \\
= 2 \left[ \int_0^{\frac{8}{7}} G_1(8\phi_1(x))dx + \int_{\frac{4}{7}}^{\frac{8}{7}} G_1(8\phi_1(x))dx + \int_{\frac{4}{7}}^{\frac{8}{7}} G_1(8\phi_1(x))dx \right].
$$

Note that for $0 \leq x \leq \frac{1}{6}$ we have that $0 \leq 8 \sin(\pi x) \leq 4$. Then we have

$$
\max_{x \in [0, \frac{1}{6}]} G_1(8\phi_1(x)) = \max_{y \in [0, 4]} G_1(y) = G_1(2).
$$

Similarly,

$$
\max_{x \in [\frac{1}{6}, \frac{1}{4}]} G_1(8\phi_1(x)) = \max_{y \in [\frac{1}{4}, 4\sqrt{3}]} G_1(y) = G_1(4)
$$

and

$$
\max_{x \in [\frac{1}{4}, \frac{1}{2}]} G_1(8\phi_1(x)) = \max_{y \in [4\sqrt{3}, 8]} G_1(y) = G_1(4\sqrt{3}) < G_1(6).
$$

Therefore,

$$
\int_0^1 G(x, 8\phi_1(x))dx \leq 2 \left[ \int_0^{\frac{1}{8}} G_1(2)dx + \int_{\frac{1}{8}}^{\frac{1}{4}} G_1(4)dx + \int_{\frac{1}{4}}^{\frac{1}{2}} G_1(6)dx \right] < \int_0^1 T(x)dx < 0.
$$

Similarly, we can prove that $G$ satisfies $(G_6)$ for $t^- = -8$.

References


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