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VIBRATIONS OF BEAMS BY TORSION OR IMPACT (Mathematical Analysis)

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Dedicated to Professor J. V. Gonçalves on the occasion of his 60th birthday

Abstract

This article contains a mathematical analysis of the initial boundary value problem:

$$\left|\begin{array}{l} u''(x,t) - \Delta u(x,t) + \delta(x)u'(x,t) = 0 \text{ in } \Omega \times (0,\infty) \\ u = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} + \alpha(x)u''(x,t) + \beta(x)u'(x,t) = 0 \text{ on } \Gamma_1 \times (0,\infty) \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x) \text{ in } \Omega \end{array}\right. \tag{P}$$

It was motivated by a torsion or impact of cylindrical beams. With restrictions on δ, α, β, u^0, u^1 we prove existence and uniqueness of solutions for (P) and asymptotic behavior of the energy. We employ Faedo-Galerkin method with a special basis idealized by the two last authors, cf. [13].

1 Introduction

The objective of this article is to investigate an initial boundary value problem for the wave operator $\partial^2/\partial t^2 - \Delta + \delta$ in a cylinder $Q = \Omega \times (0,T)$, $T > 0$, of $\mathbb{R}^{n+1}$, with Ω a bounded open set of $\mathbb{R}^n$ with C^2 boundary Γ. The lateral boundary of Q is represented by $\Sigma = \Gamma \times (0,T)$. We consider in our model one boundary condition on a part of Σ and on the complement, a condition containing the second derivative u''. In fact, there exist examples of Mathematicaal Physics with boundary conditions of this type. We mention two cases of this type of problem, cf. Koshlyakov Smirnov-Gliner [7] and [12] for details.

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• When we look for a mathematical model for small deformations of cylindrical beams, one boundary condition is:

$$C^2\theta_x(L,t) = -\theta_{tt}(L,t) \tag{1.1}$$

for $0 < t < T$. To observe that $\theta = \theta(x,t)$ is the angle of torsion of the beam, $0 \leq x \leq L$, fixed at $x = 0$. Look [7], op.cit., page 176. Thus the other boundary condition is $\theta(0,t) = 0$.

• For another example, let us consider a cylindrical beam $[0,L]$, fixed at $x = 0$ and submitted to an impact by a mass m at the extremity L, in the direction of the axis of the beam. The longitudinal vibration of the beam is represented by $u = u(x,t)$. One boundary condition is $u(0,t) = 0$ and the other is:

$$a^2\, u_x(L,t) = -m\, L\, u_{tt}(L,t), \tag{1.2}$$

cf. [7], op.cit., page 64.

Thus, motivated by the above examples (1.1) and (1.2) we will study a general initial boundary value problem as follows.

Let us consider a bounded open set Ω of $\mathbb{R}^n$ with C^2 boundary Γ. Suppose that we have a partition Γ_0, Γ_1 of Γ, both with positive measure such that the intersection of its closures $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is empty. We represent by $Q = \Omega \times (0,T)$, $T > 0$, the cylinder of $\mathbb{R}^{n+1}$ with boundary $\Sigma = \Gamma \times (0,T)$, decomposed in the parts $\Sigma_0 = \Gamma_0 \times (0,T)$ and $\Sigma_1 = \Gamma_1 \times (0,T)$.

Thus, we formulate the initial boundary value problem: to find a function $u\colon Q \to \mathbb{R}$ solution of the initial boundary value problem:

$$\left|\begin{array}{l} u''(x,t) - \Delta u(x,t) + \delta u'(x,t) = 0, \quad (x,t) \in Q \\ u(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) \\ \dfrac{\partial u}{\partial \nu} + \alpha u''(x,t) + \beta u'(x,t) = 0 \text{ on } \Gamma_1 \times (0,T) \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x) \text{ in } \Omega \end{array}\right. \tag{P}$$

In (P) we represent by $\nu = \nu(x)$ the unit exterior normal vector to Γ at x and by $\partial/\partial\nu$ the normal derivative. With δ and α, β we represent positive real functions defined, respectively, in Ω and Γ_1 ; by u^0, u^1 the initial conditions of problem (P).

Remark 1.1. When $\delta = \beta = 0$ and $n = 1$, $\Omega = (0, L)$ we have the case studied in [7]. They solved by D'Alembert method what cannot be done in the present case (P).

All the derivatives, in the present paper, are in the sense of the theory of distributions of Laurent Schwartz, cf. Lions-Magenes [9] or Lions [8], Tartar [15].

It is opportune to mention the following references related to the present paper.

- In M. Cavalcanti-N. Larkin-J. Soriano [2], they considered a problem similar to (P), but with boundary condition:

$$\frac{\partial u}{\partial \nu} + k(u)\, u_{tt} + |u_t|^\rho\, u_t = 0 \text{ on } \Gamma_1 \times (0, T). \tag{1.3}$$

The method employed is different from ours.

- In Doronin and Larkin [4], it is investigated the one dimensional case $u'' - a(u)u_{xx} + g(u_t) = f$, with the boundary condition:

$$u_x + k(u)\, u_{tt} + h(u_t) = 0 \text{ for } x = L. \tag{1.4}$$

Remark 1.2. It is interesting to note that the boundary conditions (1.1) and (1.2) come from application of a linear Hooke's law, that is, the tension τ is a linear function of the deformation $u_x(x, t)$ (cf. [7]). If we adopt a non linear Hooke's law we have infinitely possibilities for non linear boundary condition (1.1) and (1.2) or, in general, for $(\text{P})_3$.

In the papers M. Cavalcanti-N. Larkin-J. Soriano [2], Doronin-Larkin [4], they considered a change of variables and obtained an equivalent problem, but with zero initial data, and for this equivalent problem, with zero initial data, the Faedo-Galerkin method works. In our linear case our initial data u^0, u^1

for (P) are in a weak class and the method does not work. For example, u^1 does not belongs to the domain of $-\Delta$. For this reason we idealized a special basis cf. Milla Miranda-Medeiros [13], which permits to apply Faedo-Galerkin argument with u^0, u^1 non null. This type of basis was employed in a nonlinear problem in M. Cavalcanti-V. Cavalcanti-P. Martinez [3].

Remark 1.3. In problem (P), when $\delta = 0$ and $\beta = 0$, we have that the energy of the system is conserved. The introduction of the damping terms $\delta u'$ and $\beta u'$ permit us to obtain the decay of this energy.

The paper is divided in sections. In Section 2 we fix the notations. We prove Proposition 2.3 which permits the construction of a special basis. The results on trace and Sobolev spaces follows references: Brezis [1]; Lions [8], [10]; Lions-Magenes [9]; Medeiros-Milla Miranda [11]; Milla Miranda [14], Tartar [15]. The Section 3 is dedicated to the proof of the existence and uniqueness of strong solutions for (P). In this section we employ a special basis following the method of Milla-Miranda-Medeiros [13]. In Section 4 we prove the exponential decay for the quadratic form:

$$2E(t) = |u(t)|^2_{L^2(\Omega)} + ||u(t)||^2_V + |\alpha^{1/2}\, u'(t)|^2_{L^2(\Gamma_1)}\,,$$

with $u = u(x,t)$ the strong solution of (P). In this point we employ an argument of Komornik-Zuazua [6].

2 Notations and Preliminaires Results

We denote by $H^m(\Omega)$ the Sobolev space of order $m \in \mathbb{N}$ on a open set Ω of $\mathbb{R}^n$, with inner product and norms $((\,,\,))$ and $||\cdot||$. By $L^2(\Omega)$ we represent the Lebesgue space of reals square integrable function on Ω with inner product $(\,,\,)_{L^2(\Omega)}$ and norm $|\cdot|_{L^2(\Omega)}$. The spaces $H^m(\Omega)$ and $L^2(\Omega)$ are Hilbert spaces. In certain point of this paper we consider Sobolev spaces of order m fractional.

Let us suppose the boundary Γ of Ω of class C^2. Then the trace γ_0 is well defined on $H^1(\Omega)$, cf. Lions [8]. Thus we define the subspace V of $H^1(\Omega)$ by:

$$V = \{v \in H^1(\Omega);\ \gamma_0 v = 0 \text{ on } \Gamma_0\}$$

Γ_0 part of Γ defined in Section 1.

In $H^1(\Omega)$ we have the inner product

$$((u,v)) = \int_\Omega u(x)\cdot v(x)\,dx + \int_\Omega \nabla u(x)\cdot \nabla v(x)\,dx,$$

∇ the gradient operator and $x = (x_1, \ldots, x_n)$ a vector of $\mathbb{R}^n$.

We have Poincaré inequality in V, then the norm

$$||v||_V^2 = \int_\Omega |\nabla v(x)|^2\,dx \tag{2.1}$$

is equivalent in V to the norm of $H^1(\Omega)$. The induced inner product in V is

$$((u,v))_V = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx. \tag{2.2}$$

Thus V is a Hilbert space.

For Sobolev spaces of order s, s a real number, can be seen, among others, references [8], [9], [10], [11], [15].

Proposition 2.1. Let f be in $L^2(\Omega)$ and g in $H^{1/2}(\Gamma_1)$. Then the solution u of the boundary value problem:

$$\left|\begin{array}{rll} -\Delta u = f & \text{in} & \Omega \\ u = 0 & \text{on} & \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = g & \text{on} & \Gamma_1 \end{array}\right. \tag{2.3}$$

belongs to $V \cap H^2(\Omega)$ and

$$||u||_{H^2(\Omega)} \leq c\left[|f|_{L^2(\Omega)} + ||g||_{H^{1/2}(\Gamma)}\right]. \tag{2.4}$$

Remark 2.1. The trace γ_j , of order j, on $H^2(\Omega)$ is $\gamma_j\colon H^2(\Omega) \to H^{2-j-\frac{1}{2}}(\Gamma)$, for $j = 0, 1$, cf. Lions [8]. Thus

$$\gamma_0\colon H^2(\Omega) \to H^{3/2}(\Gamma) \quad \text{and} \quad \gamma_1\colon H^2(\Omega) \to H^{1/2}(\Gamma).$$

Roughly speaking, γ_0 is the restriction of u to Γ and γ_1 the restriction of $\dfrac{\partial u}{\partial \nu}$ to Γ. We suppose Γ of class C^2.

Proof. Let us consider $\{0, \tilde{g}\}$ in $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, with $\tilde{g} = 0$ on Γ_0 and $\tilde{g} = g$ on Γ_1. By Remark 2.1, there exists $h \in H^2(\Omega)$ such that

$$\gamma_0 h = 0 \quad \text{and} \quad \gamma_1 h = \tilde{g} = g \quad \text{in} \quad \Gamma_1 .$$

Still by trace theorem, continuity of γ_j,

$$||h||_{H^2(\Omega)} \leq C||g||_{H^{1/2}(\Gamma_1)} . \tag{2.5}$$

Let w be the weak solution of the boundary value problem:

$$\left| \begin{array}{lll} -\Delta w = f - \Delta h & \text{in} & \Omega \\ w = 0 & \text{on} & \Gamma_0 \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on} & \Gamma_1 \end{array} \right. \tag{2.6}$$

We define weak solution of (2.6) as a function $w \colon \Omega \to \mathbb{R}$, $w \in V$, such that

$$\int_\Omega \nabla w \cdot \nabla v \, dx = \int_\Omega f v \, dx - \int_\Omega \Delta h \cdot v \, dx \tag{2.7}$$

for all $v \in V$. Since $f - \Delta h \in L^2(\Omega)$, it follows, by regularity of weak solutions for elliptic boundary value problems, that the solution w of (2.6) defined by (2.7) belongs to $V \cap H^2(\Omega)$ and

$$||w||_{H^2(\Omega)} \leq C \left[|f|_{L^2(\Omega)} + |\Delta h|_{L^2(\Omega)} \right] , \tag{2.8}$$

and, as solution of (2.6), we have:

$$w = 0 \text{ on } \Gamma_0 \quad \text{and} \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_1 .$$

Remark 2.2. In fact, multiplying both sides of $(2.6)_1$ by $v \in V$ and integrating on Ω, we get:

$$- \int_\Omega \Delta w \cdot v \, dx = \int_\Omega f \cdot v \, dx - \int_\Omega \Delta h \cdot v \, dx.$$

Applying Green's formula, we obtain:

$$\int_\Omega \nabla w \cdot \nabla v \, dx - \int_{\Gamma_1} \frac{\partial w}{\partial \nu} v \, d\Gamma = \int_\Omega f \cdot v \, dx - \int_\Omega \Delta h \cdot v \, dx.$$

But w is weak solution of (2.6), then (2.7) and the last equality implies

$$\int_{\Gamma_1} \frac{\partial w}{\partial \nu} \cdot v \, d\Gamma = 0$$

for all $v \in V$ which implies $\dfrac{\partial w}{\partial \nu} = 0$ on Γ_1.

To complete the proof we need to verify inequality (2.4).

We already have (2.8). Set $u = w - h$, which is in $V \cap H^2(\Omega)$ and is solution of (2.3) because $\gamma_0 h = 0$ on Γ_0 and $\gamma_1 h = g$ on Γ_1. Thus, we have:

$$\begin{aligned} ||u||_{H^2(\Omega)} = ||w-h||_{H^2(\Omega)} &\le ||w||_{H^2(\Omega)} + ||h||_{H^2(\Omega)} \le \\ &\le C\left[|f|_{L^2(\Omega)} + |\Delta h|_{L^2(\Omega)}\right] + ||h||_{H^2(\Omega)} \le \\ &\le C\left[|f|_{L^2(\Omega)} + ||g||_{H^{1/2}(\Gamma_1)}\right] \end{aligned}$$

by (2.8) and (2.5). It proves Proposition 2.1.

Proposition 2.2. In $V \cap H^2(\Omega)$, the norm $H^2(\Omega)$ and the norm

$$u \to \left[|\Delta u|^2_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial \nu} \right\|_{H^{1/2}(\Gamma_1)} \right]^{\frac{1}{2}}$$

are equivalent.

Proof. Let us consider $u \in V \cap H^2(\Omega)$. By Proposition 2.1 we have:

$$||u||_{H^2(\Omega)} \le C \left[|\Delta u|^2_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{H^{1/2}(\Gamma_1)} \right].$$

We have $|\Delta u|_{L^2(\Omega)} \le ||u||_{H^2(\Omega)}$ and by trace theorem

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{H^{1/2}(\Gamma)} \le C ||u||_{H^2(\Omega)} .$$

Thus, we consider $V \cap H^2(\Omega)$ equiped with the norm:

$$\left(|\Delta u|^2_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{H^{1/2}(\Gamma_1)} \right)^{1/2}.$$

Proposition 2.3. Suppose Γ_1 of class C^k, with $k \geq r > \dfrac{n}{2}$, k an integer, r a real number, $\beta \in H^r(\Gamma_1)$, $u^0 \in V \cap H^2(\Omega)$, $u^1 \in V$ and $\dfrac{\partial u^0}{\partial \nu} + \beta u^1 = 0$ on Γ_1. Then, for each $\varepsilon > 0$, there exist w and z in $V \cap H^2(\Omega)$ such that:

$$||w - u^0||_{V \cap H^2(\Omega)} < \varepsilon, \quad ||z - u^1||_V < \varepsilon$$

and

$$\frac{\partial w}{\partial \nu} + \beta z = 0 \quad \text{on} \quad \Gamma_1 .$$

Proof. We know that $V \cap H^2(\Omega)$ is dense in V. Thus, if $u^1 \in V$, for each $\varepsilon > 0$ there exists $z \in V \cap H^2(\Omega)$ such that $||z - u^1||_V < \varepsilon$.

Consider $w \in V \cap H^2(\Omega)$ solution of

$$\left| \begin{aligned} \Delta w &= \Delta u^0 \quad \text{in} \quad \Omega \\ w &= 0 \quad \text{on} \quad \Gamma_0 \\ \frac{\partial w}{\partial \nu} &= -\beta z \quad \text{on} \quad \Gamma_1 \end{aligned} \right. \tag{2.9}$$

By Proposition 2.1, the solution w of (2.9) belongs to $V \cap H^2(\Omega)$ and by Proposition 2.2 we have:

$$\begin{aligned} ||w - u^0||_{V \cap H^2(\Omega)} &= \left|\Delta w - \Delta u^0\right|^2_{L^2(\Omega)} + \left\| \frac{\partial w}{\partial \nu} - \frac{\partial u^0}{\partial \nu} \right\|^2_{H^{1/2}(\Gamma_1)} = \\ &= \left\|\beta z - \beta u^1\right\|^2_{H^{1/2}(\Gamma_1)} \leq C \|\beta\|^2_{H^r(\Gamma_1)} \|z - u^1\|^2_{H^{1/2}(\Gamma_1)} \leq \\ &\leq C_1 \left\|z - u^1\right\|^2_V < C_1 \varepsilon^2, \quad C_1 = C \left\|\beta\right\|^2_{H^r(\Gamma_1)} , \end{aligned}$$

where the first inequality is obtained by [11], p. 91 and 92, and local charts. This and $(2.9)_3$ prove Proposition 2.3.

3 Strong Solutions

In this section we fix hypothesis on u^0, u^1, α, β, δ in order to obtain strong solution for the initial boundary value problem:

$$\left| \begin{array}{l} u'' - \Delta u + \delta u' = 0 \quad \text{in} \quad Q = \Omega \times (0,\infty) \\ u = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} + \alpha u'' + \beta u' = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty) \\ u(0) = u^0,\ u'(0) = u^1 \quad \text{in} \quad \Omega \end{array} \right. \tag{3.1}$$

Hypothesis 3.1

We suppose

- Γ_1 of class C^k with $k \geq r > \dfrac{n}{2}$, k an integer, r a real number,
- $\alpha \in L^\infty(\Gamma_1)$, $\beta \in H^r(\Gamma_1)$, $\delta \in L^\infty(\Omega)$, $\alpha(x) \geq 0$, $\beta(x) \geq 0$ a.e. on Γ_1 and $\delta(x) \geq 0$ a.e. in Ω.

Theorem 3.1. Let us consider Γ_1, α, β, δ as in Hypothesis 3.1 and

$$u^0 \in V \cap H^2(\Omega), \quad u^1 \in V, \quad \frac{\partial u^0}{\partial \nu} + \beta u^1 = 0 \text{ on } \Gamma_1\,. \tag{3.2}$$

Then, there exists only one function $u\colon \Omega \times (0,\infty) \to \mathbb{R}$ satisfying:

$$\left| \begin{array}{l} u \in L^\infty(0,\infty; V) \\ u' \in L^\infty(0,\infty; V) \\ u'' \in L^\infty(0,\infty; L^2(\Omega)) \end{array} \right. \tag{3.3}$$

$$\left| \begin{array}{l} \beta^{1/2}\, u' \in L^2(0,\infty; L^2(\Gamma_1) \\ \delta^{1/2}\, u' \in L^2(0,\infty; L^2(\Omega) \end{array} \right. \tag{3.4}$$

$$\left| \begin{array}{l} \alpha^{1/2}\, u'' \in L^\infty(0,\infty; L^2(\Gamma_1)) \\ \delta^{1/2}\, u'' \in L^2(0,\infty; L^2(\Gamma_1)) \end{array} \right. \tag{3.5}$$

and u is solution of (3.1) in the following sense:

$$\left| \begin{array}{l} u'' - \Delta u + \delta u' = 0 \text{ in } L^\infty(0,\infty; L^2(\Omega)) \\ u = 0 \quad \text{on} \quad \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} + \alpha u'' + \beta u' = 0 \text{ in } L^\infty(0,\infty; L^2(\Gamma_1)) \\ u(0) = u^0,\ u'(0) = u^1 \text{ in } \Omega \end{array} \right. \tag{3.6}$$

Proof. We plan to employ the approximated method of Faedo-Galerkin. We have difficulty which is the condition (3.2), that is, $\dfrac{\partial u^0}{\partial \nu} + \beta(x)\, u^1 = 0$ on Γ_1 for the initial data. The method does not work for an arbitrary Hilbert basis, cf. Brezis [1] or Lions [8]. Thus we need idealize a special basis for $V \cap H^2(\Omega)$ which works well for the case $\dfrac{\partial u^0}{\partial \nu} + \beta(x)\, u^1 = 0$ on Γ_1 .

By the hypothesis (3.2) of Theorem 3.1, u^0 and u^1 are in the conditions of the Proposition 2.3, Section 2. It then implies the existence of two sequences $(u_k^0)_{k\in\mathbb{N}}$, $(u_k^1)_{k\in\mathbb{N}}$ of vectors in $V \cap H^2(\Omega)$ satisfying:

$$\left| \begin{array}{l} \lim\limits_{k\to\infty} u_k^0 = u^0 \text{ in } V \cap H^2(\Omega);\ \lim\limits_{k\to\infty} u_k^1 = u^1 \text{ in } V \\ \dfrac{\partial u_k^0}{\partial \nu} + \beta(x)\, u_k^1 = 0 \text{ in } \Gamma_1\,, \text{ for all } k \in \mathbb{N} \end{array} \right. \tag{3.7}$$

To construct the basis we fix $k \in \mathbb{N}$. Let

$$\{w_1^k,, w_2^k, \dots, w_j^k, \dots\}, \tag{3.8}$$

be a basis of $V \cap H^2(\Omega)$ such that u_k^0 and u_k^1 belong to the subspace generated by w_k^1 and w_2^k.

For $m \in \mathbb{N}$, we consider the subspace

$$V_m^k = [w_1^k, w_2^k, \dots, w_m^k]$$

of $V \cap H^2(\Omega)$, generated by the m first vectors $w_1^k, \dots, w_m^k$ of (3.8). If $u_{km}(t) \in V_m^k$, it has the representation:

$$u_{km}(t) = \sum_{j=1}^{m} g_{jkm}(t)\, w_j^k\,. \tag{3.9}$$

Approximate System

The approximate system consists in find $u_{km}(t)$ defined by (3.9) belonging to V_m^k , solution of the following system of linear ordinary differential equations:

$$\left|\begin{array}{l}\left(u''_{km}(t),w\right)_{L^2(\Omega)}+\left((u_{km}(t),w)\right)_V+\\ +\displaystyle\int_{\Gamma_1}\alpha(x)u''_{km}(x,t)w(x)d\Gamma+\int_{\Gamma_1}\beta(x)u'_{km}(x,t)w(x)d\Gamma+\\ +\left(\delta u'_{km}(t),w\right)_{L^2(\Omega)}=0,\quad t>0,\ \text{for all } w\in V_m^k\\ u_{km}(0)=u_k^0\,,\ \ u'_{km}(0)=u_k^1\end{array}\right. \tag{3.10}$$

To observe that if we set $w = w_k^j$ in (3.10) we obtain a system of linear ordinary differential equations in $g_{jm}(t)$, k fixed, which has a solution permiting to define the approximate solution $u_{km}(x,t)$ for $x\in\Omega$ and $t\in[0,+\infty)$.

The next steps are to obtain estimates for $u_{km}(t)\in V_m^k$ permiting to pass to the limit as $m\to\infty$ in (3.10).

Estimate 1. Set $w = 2u'_{km}(t)$ in (2.10). We obtain:

$$\begin{aligned}&\frac{d}{dt}\left[|u'_{km}(t)|^2_{L^2(\Omega)}+||u_{km}(t)||^2_V+|\alpha^{1/2}\,u'_{km}(t)|^2_{L^2(\Gamma_1)}\right]+\\ &\quad+2\left|\beta^{1/2}\,u'_{km}(t)\right|^2_{L^2(\Gamma_1)}+2\left|\delta^{1/2}\,u'_{km}(t)\right|^2_{L^2(\Omega)}=0.\end{aligned}$$

Integrating on $[0,t]$, $0\le t<\infty$, we obtain:

$$\begin{aligned}&|u'_{km}(t)|^2_{L^2(\Omega)}+||u_{km}(t)||^2_V+|\alpha^{1/2}\,u'_{km}(t)|^2_{L^2(\Gamma_1)}+\\ +2&\int_0^t|\beta^{1/2}\,u'_{km}(s)|^2_{L^2(\Gamma_1)}\,ds+2\int_0^t|\delta^{1/2}\,u'_{km}(s)|^2_{L^2(\Omega)}\,ds=\\ &=|u_k^1|^2_{L^2(\Omega)}+||u_k^0||^2_V+|\alpha^{1/2}\,u_k^1|^2_{L^2(\Gamma_1)}\,.\end{aligned}$$

Remark 3.1. By (3.7) we obtain $|u'_k|^2_{L^2(\Omega)}$, $||u_k^0||^2_V$ bounded by constant independent of m, k and $t\in[0,\infty)$. By trace theorem we obtain $|\alpha^{1/2}\,u_k^1|^2_{L^2(\Gamma_1)}$ is also uniformly bounded independent of m and k. We obtain these bounds when $t\to\infty$.

It follows the first estimate:

$$\left| \begin{array}{l} |u'_{km}(t)|^2_{L^2(\Omega)} + ||u_{km}(t)||^2_V + |\alpha^{1/2}\, u'_{km}(t)|^2_{L^2(\Gamma_1)} + \\ + 2\displaystyle\int_0^t |\beta^{1/2}\, u'_{km}(s)|^2_{L^2(\Gamma_1)}\, ds + 2\int_0^t |\delta^{1/2}\, u'_{km}(s)|^2_{L^2(\Omega)}\, ds < C \end{array} \right. \tag{3.11}$$

C independent of m, k for all $t \geq 0$.

Estimate 2. We estimate the second derivative u''_{km} . One method consists to consider the derivative of both sides of $(3.6)_1$ and proceeds as in the Estimate 1. We need to estimate first $u''_{km}(0)$.

- Estimate of $u''_{km}(0)$.

Set $t = 0$ in the approximate equation $(3.10)_1$ and choose $w = u''_{km}(0)$. We obtain:

$$\begin{aligned} &|u''_{km}(0)|^2_{L^2(\Omega)} + \big((u^0_k, u''_{km}(0))\big)_V + |\alpha^{1/2}\, u''_{km}(0)|^2_{L^2(\Gamma_1)} + \\ &\quad + \big(\beta u^1_k, u''_{km}(0)\big)_{L^2(\Gamma_1)} + \big(\delta u^1_k, u''_{km}(0)\big)_{L^2(\Omega)} = 0. \end{aligned}$$

We modify the above equality applying Green's formula to $\big((u^k_0, u''_{km}(0))\big)_V$ obtaining:

$$\begin{aligned} &|u''_{km}(0)|^2_{L^2(\Omega)} + |\alpha^{1/2}\, u''_{km}(0)|^2_{L^2(\Gamma_1)} = -\big(\Delta u^0_k, u''_k(0)\big)_{L^2(\Omega)} - \\ &\left(\frac{\partial u^0_k}{\partial \nu} + \beta u^1_k, u''_{km}(0)\right)_{L^2(\Omega)} - \big(\delta u^1_k, u''_{km}(0)\big)_{L^2(\Omega)}\, . \end{aligned}$$

By condition $\dfrac{\partial u^0_k}{\partial \nu} + \beta u^1_k = 0$ on Γ_1 , cf. (3.7), we obtain:

$$|u''_{km}(0)|^2_{L^2(\Omega)} + |\alpha^{1/2}\, u''_{km}(0)|^2_{L^2(\Gamma_1)} \leq \left|\big(\Delta u^0_k, u''_{km}(0)\big)_{L^2(\Omega)}\right| + \left|\big(\delta u'_k, u''_{km}(0)\big)_{L^2(\Omega)}\right|. \tag{3.12}$$

By Cauchy-Schwarz inequality and the elementary inequality $2ab \leq a^2 + b^2$, we modify (3.12) obtaining:

$$\begin{aligned} &\left|\big(\Delta u^0_k, u''_{km}(0)\big)_{L^2(\Omega)}\right| + \left|\big(\delta u^1_k, u''_{km}(0)\big)_{L^2(\Omega)}\right| \leq \\ &\quad \leq \frac{1}{2\varepsilon}\, |\Delta u^0_k|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\, |u''_{km}(0)|^2_{L^2(\Omega)} + \\ &\quad + \frac{1}{2\varepsilon}\, |\delta u^1_k|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\, |u''_{km}(0)|^2_{L^2(\Omega)}\, . \end{aligned}$$

Set $2\varepsilon = 1$ and substituting in (3.12) we get:

$$\frac{1}{2}\,|u''_{km}(0)|^2_{L^2(\Omega)} + |\alpha^{1/2}\,u''_{km}(0)|^2_{L^2(\Omega)} \le |\Delta u^0_k|^2_{L^2(\Omega)} + |\delta|_{L^\infty(\Omega)}\,|u^1_k|^2_{L^2(\Omega)}\,.$$

From (3.12) and (3.7) we obtain a constant $C > 0$ independent of k and m, such that

$$|u''_{km}(0)|^2_{L^2(\Omega)} + |\alpha^{1/2}\,u''_{km}(0)|_{L^2(\Gamma_1)} < C. \tag{3.13}$$

Now, consider the derivative with respect to t of both sides of (3.10) and set $w = 2u''_{km}(t)$. Integrate on $[0,t]$, $0 \le t < \infty$. We obtain

$$\begin{aligned} &|u''_{km}(t)|^2_{L^2(\Omega)} + ||u'_{km}(t)||^2_V + |\alpha^{1/2}\,u''_{km}(t)|^2_{L^2(\Gamma_1)} + \\ &+ 2\int_0^t |\beta^{1/2}\,u''_{km}(s)|^2_{L^2(\Omega)}\,ds + 2\int_0^t |\delta^{1/2}\,u''_{km}(s)|^2_{L^2(\Omega)}\,ds \le \\ &\le |u''_{km}(0)|^2_{L^2(\Omega)} + ||u^1_k||^2_V + |\alpha^{1/2}\,u''_{km}(0)|^2_{L^2(\Gamma_1)}\,. \end{aligned}$$

From (3.13) and convergences of $(u^1_k)_{k\in\mathbb{N}}$ to u^1 in V, cf. (3.7), we obtain from the above inequality:

$$\begin{aligned} &|u''_{km}(t)|^2_{L^2(\Omega)} + ||u'_{km}(t)||^2_V + |\alpha^{1/2}\,u''_{km}(t)|^2_{L^2(\Gamma_1)} + \\ &+ 2\int_0^t |\beta^{1/2}\,u''_{km}(s)|^2_{L^2(\Gamma_1)}\,ds + \int_0^t |\delta^{1/2}\,u''_{km}(s)|^2_{L^2(\Omega)}\,ds \le C \end{aligned} \tag{3.14}$$

for all $0 \le t < \infty$, including when $t \to \infty$, for all $k, m \in \mathbb{N}$.

From estimates (3.11) and (3.14) we obtain:

$$\left|\begin{array}{lll} u_{km} \text{ and } u'_{km} \text{ bounded in } L^\infty(0,\infty;V) & & \\ u''_{km} \text{ bounded in } L^\infty(0,\infty;L^2(\Omega)) & & \\ \alpha^{1/2}\,u''_{km} & \text{bounded in} & L^\infty(0,\infty;L^2(\Gamma_1)) \\ \beta^{1/2}\,u'_{km} & \text{bounded in} & L^2(0,\infty;L^2(\Gamma_1)) \\ \delta^{1/2}\,u'_{km} & \text{bounded in} & L^2(0,\infty;L^2(\Omega)) \end{array}\right. \tag{3.15}$$

From $(3.15)_3$ we extract a subsequence, still denoted by $\alpha^{1/2}\,u''_{km}$, such that

$$\alpha^{1/2}\,u''_{km} \rightharpoonup \chi_k \quad \text{weak star in} \quad L^\infty(0,\infty;L^2(\Gamma_1)).$$

From $(3.15)_1$ we extract a subsequence u'_{km} such that

$$u'_{km} \rightharpoonup u'_k \quad \text{weak star in} \quad L^\infty(0,\infty;V).$$

By trace theorem $\gamma_0 \, u'_k \in L^2(\Gamma_1)$ and

$$|\gamma_0 \, u'_{km}|_{L^2(\Gamma_1)} \leq C||u'_{km}||_V \, .$$

so

$$u'_{km} \rightharpoonup u'_k \quad \text{weakly in} \quad L^2_{\text{loc}}(0,\infty;L^2(\Gamma_1)).$$

By Milla Miranda [14], the preceding convergence implies

$$\alpha^{1/2} \, u''_{km} \rightharpoonup \alpha^{1/2} \, u''_k \quad \text{weakly} \quad H^{-1}_{\text{loc}}(0,\infty;L^2(\Gamma_1)).$$

Thus $\chi_k = \alpha^{1/2} \, u''_k$ and

$$\alpha^{1/2} \, u''_{km} \rightharpoonup \alpha^{1/2} \, u''_k \quad \text{weak star} \quad L^\infty(0,\infty;L^2(\Gamma_1)). \tag{3.16}$$

Similarly we have

$$\beta^{1/2} \, u_{km} \rightharpoonup \beta^{1/2} \, u_k \quad \text{weak star in} \quad L^\infty(0,\infty;L^2(\Gamma_1)). \tag{3.17}$$

From (3.15), (3.16) and (3.17) we are able to pass to the limit in approximate equation (3.10). Observe that the estimates are uniform in m and k and the convergences with respect to m and k are correct. Thus, letting m, k go to ∞ in (3.10), we obtain a function u in the class (3.3) satisfying:

$$\begin{aligned} &\int_0^\infty \big(u''(t),\varphi(t)\big)_{L^2(\Omega)}\,dt + \int_0^\infty \big((u(t),\varphi(t))\big)_V\,dt + \\ &+\int_0^\infty \big(\alpha u''(t),\varphi(t)\big)_{L^2(\Gamma_1)}\,dt + \int_0^\infty (\beta u'(t),\varphi)_{L^2(\Gamma_1)}\,dt + \\ &+\int_0^\infty \big(\delta u'(t),\varphi(t)\big)_{L^2(\Omega)}\,dt = 0, \end{aligned} \tag{3.18}$$

for all $\varphi \in L^1(0,\infty;V) \cap L^2(0,\infty;L^2(\Omega))$.

In particular, set $\varphi(t) = \theta(t)v$ where $v \in V$ and $\theta \in \mathcal{D}(0,\infty)$. We obtain, from (3.18),

$$\begin{aligned}
&\int_0^\infty (u''(t),v)_{L^2(\Omega)}\,\theta(t)dt + \int_0^\infty ((u(t),v))_V\,\theta(t)dt+\\
&+\int_0^\infty (\alpha u''(t),v)_{L^2(\Gamma_1)}\,\theta(t)dt + \int_0^\infty (\beta(t)u'(t),v)_{L^2(\Gamma_1)}\,\theta(t)dt+\\
&+\int_0^\infty (\delta u'(t),v)_{L^2(\Omega)}\,\theta(t)dt = 0.
\end{aligned} \tag{3.19}$$

Set $v \in \mathcal{D}(\Omega) \subset V$ in (3.19). We get

$$\int_0^\infty (u''(t),v)_{L^2(\Omega)}\,\theta(t)dt + \int_0^\infty ((u(t),v))_V\,\theta(t)dt + \int_0^\infty (\delta u'(t),v)_{L^2(\Omega)}\,\theta(t)dt = 0 \tag{3.20}$$

Thus (3.20) is true for all $v \in \mathcal{D}(\Omega)$ and $\theta \in \mathcal{D}(0,\infty)$.

We can write (3.20) as

$$\begin{aligned}
&\left(\int_0^\infty u''(t)\theta(t)dt, v\right)_{L^2(\Omega)} + \left\langle \int_0^\infty -\Delta u(t)\theta(t)dt, v\right\rangle_{H^1(\Omega)\times H_0^1(\Omega)} +\\
&\qquad + \left(\int_0^\infty \delta(t)u'(t)\theta(t), v\right)_{L^2(\Omega)} = 0
\end{aligned}$$

for all $v \in \mathcal{D}(\Omega)$, $\theta \in \mathcal{D}(0,\infty)$, what implies

$$\int_0^\infty \big[u''(t) + \delta u'(t)\big]\theta(t)dt = \int_0^\infty \Delta u(t)\theta(t)dt$$

in $H^{-1}(\Omega)$, for all $\theta \in \mathcal{D}(0,\infty)$.

Then it implies:

$$\Delta u = u'' + \delta u' \quad \text{in} \quad \mathcal{D}'(0,\infty; H^{-1}(\Omega)).$$

Since $u'' \in L^\infty(0,\infty;L^2(\Omega))$, $\delta \in L^\infty(\Omega)$, $u' \in L^\infty(0,\infty;V)$, we obtain

$$\Delta u \in L^\infty(0,\infty;L^2(\Omega))$$

and

$$u'' - \Delta u + \delta u' = 0 \quad \text{in} \quad L^\infty(0,\infty.L^2(\Omega)).$$

Otherwise, as $u \in L^\infty(0,\infty;V)$ and $\Delta u \in L^\infty(0,\infty;L^2(\Omega))$, we can evaluate $\dfrac{\partial u}{\partial \nu}$ on Γ_1, that is, the trace $\gamma_1 u$,

$$\frac{\partial u}{\partial \nu} \in L^\infty\left(0,\infty;H^{-\frac{1}{2}}(\Gamma_1)\right),$$

and holds the Green formula

$$\int_0^\infty \big(-\Delta u(t), z(t)\big)_{L^2(\Omega)}\,dt = \int_0^\infty \big((u(t),z(t))\big)_V\,dt - \int_0^\infty \left\langle \frac{\partial u}{\partial \nu}(t), z(t)\right\rangle_{H^{-\frac{1}{2}}(\Gamma_1)\times H^{\frac{1}{2}}(\Gamma_1)} dt,$$

for all $z \in L^1(0,\infty;V)$, cf. Milla Miranda [14].

Interpretation of the boundary condition on Γ_1 ($u = 0$ on Γ_0 because $u \in V$).

If $\varphi \in L^1(0,\infty;V)\cap L^2(0,\infty;L^2(\Omega))$ we have by the last three results

$$\int_0^\infty \big(u''(t),\varphi(t)\big)_{L^2(\Omega)}\,dt + \int_0^\infty \big((u(t),\varphi(t))\big)_V\,dt -$$
$$- \int_0^\infty \left\langle \frac{\partial u}{\partial \nu}(t), \varphi(t)\right\rangle_{H^{-\frac{1}{2}}(\Gamma_1)\times H^{1/2}(\Gamma_1)} dt + + \int_0^\infty \big(\delta u(t),\varphi(t)\big)_{L^2(\Omega)}\,dt = 0. \tag{3.21}$$

From (3.18) and(3.21) we obtain:

$$\int_0^T \left\langle \frac{\partial u}{\partial \nu} + \alpha u'' + \beta u', \varphi\right\rangle_{H^{-\frac{1}{2}}(\Gamma_1)\times H^{1/2}(\Gamma_1)} dt = 0 \tag{3.22}$$

for each φ above choosed.

We know that

$$\alpha u'' \in L^\infty\big(0,\infty;L^2(\Gamma_1)\big) \tag{3.23}$$

and

$$\beta u' \in L^\infty\big(0,\infty;L^2(\Gamma_1)\big) \tag{3.24}$$

Thus, from (3.22), (3.23)) and (3.24) we have:

$$\frac{\partial u}{\partial \nu} \in L^\infty\big(0,\infty;L^2(\Gamma_1)\big)$$

and

$$\frac{\partial u}{\partial \nu} + \alpha u'' + \beta u' = 0 \quad \text{in} \quad L^\infty\big(0,\infty;L^2(\Gamma_1)\big).$$

To complete the proof of the Theorem 3.1 we need to verify initial data and uniqueness. It is not difficult to do.

4 Asymptotic Behavior

If $u = u(x,t)$ is the solution given by Theorem 3.1 we define the quadratic form $E(t)$, called Energy, by

$$2E(t) = |u'(t)|^2_{L^2(\Omega)} + ||u(t)||^2_V + |\alpha^{1/2}\, u'(t)|^2_{L^2(\Gamma_1)}\,.$$

Set k_0 the constant of Poincaré's inequality $|v|^2_{L^2(\Omega)} \leq k_0 ||v||^2_V$, for $v \in V$, k_1 the constant of trace γ_0 , $|v|^2_{L^2(\Gamma_1)} \leq k_1 ||v||^2_V$, $v \in V$.

Theorem 4.1. Assume hypothesis (3.1) with the suplementary conditions:

$$\alpha \neq 0, \quad \beta(x) \geq \beta_0 > 0 \text{ a.e. on } \Gamma_1\,, \quad \delta(x) \geq \delta_0 > 0 \text{ a.e. in } \Omega.$$

Then the solution u of Theorem 3.1 satisfies

$$E(t) \leq 3E(0)\, e^{-\frac{2}{3}\eta t}, \quad \text{for all} \quad t \geq 0,$$

with

$$\eta = \min\left\{\frac{1}{2C_0}, \frac{2}{3}\,\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}}, \frac{2}{3}\,\delta_0\right\},$$

$$C_0 = 1 + k_0^2 + k_1^2\,|\alpha|_{L^\infty(\Gamma_1)} + k_1^2\,|\beta|_{L^\infty(\Gamma_1)} + k_0^2\,|\delta|_{L^\infty(\Omega)}\,.$$

Proof. The solution $u = u(x,t)$ given by Theorem 3.1 satisfies:

$$\left|\begin{array}{l} u'' - \Delta u + \delta u' = 0 \quad \text{in} \quad L^\infty\big(0,\infty; L^2(\Omega)\big) \\ u = 0 \quad \text{on} \quad \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} + \alpha u'' + \beta u' = 0 \quad \text{in} \quad L^\infty\big(0,\infty; L^2(\Gamma_1)\big) \\ u(0) = u^0,\ u'(0) = u^1 \quad \text{in} \quad \Omega \end{array}\right. \tag{4.1}$$

By Theorem 3.1, $u' \in L^\infty(0,\infty;V)$. Multiply both sides of $(4.1)_1$ by u' and integrate on Ω. We obtain

$$E'(t) = -\left|\beta^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)} - \left|\delta^{1/2}\, u'(t)\right|^2_{L^2(\Omega)} \tag{4.2}$$

Multiplying both sides of $(4.1)_1$ by u and integrating on Ω we get

$$\big(u''(t), u(t)\big)_{L^2(\Omega)} - \big(\Delta u(t), u(t)\big)_{L^2(\Omega)} + \big(\delta u'(t), u(t)\big)_{L^2(\Omega)} = 0. \tag{4.3}$$

We have

$$-\big(\Delta u(t), u(t)\big) = ||u(t)||_V^2 - \int_{\Gamma_1} \frac{\partial u}{\partial \nu}\, u(t) d\Gamma$$

and

$$\big(u''(t), u(t)\big) = \frac{d}{dt}\,\big(u'(t), u(t)\big)_{L^2(\Omega)} - |u'(t)|^2_{L^2(\Omega)}\,.$$

Substituting in (4.3) we obtain:

$$\begin{aligned} &\frac{d}{dt}\,\big(u'(t), u(t)\big)_{L^2(\Omega)} - |u'(t)|^2_{L^2(\Omega)} + ||u(t)||_V^2 - \\ &- \int_{\Gamma_1} \frac{\partial u}{\partial \nu}\, u(t) d\Gamma + \frac{1}{2}\,\frac{d}{dt}\,\Big|\delta^{1/2}\, u(t)\Big|^2_{L^2(\Omega)} = 0. \end{aligned} \tag{4.4}$$

Substituting $(4.1)_3$ in (4.4) we get:

$$\begin{aligned} &\frac{d}{dt}\,\big(u'(t), u(t)\big)_{L^2(\Omega)} - |u'(t)|^2_{L^2(\Omega)} + ||u(t)||_V^2 + \\ &+ \int_{\Gamma_1} \big[\alpha u''(t) + \beta u'(t)\big] u(t) d\Gamma + \frac{1}{2}\,\frac{d}{dt}\,[\delta^{1/2}\, u(t)|_{L^2(\Omega)} = 0 \end{aligned} \tag{4.5}$$

We have

$$\int_{\Gamma_1} \alpha u''(t) u(t) dt = \frac{d}{dt} \int_{\Gamma_1} \alpha u'\, u\, d\Gamma - \Big|\alpha^{1/2}\, u'(t)\Big|^2_{L^2(\Gamma_1)}$$

$$\int_{\Gamma_1} \beta u'\, u\, d\Gamma = \frac{1}{2}\,\frac{d}{dt}\,\Big|\beta^{1/2}\, u(t)\Big|^2_{L^2(\Gamma_1)}\,.$$

Substituting in (4.5) we obtain:

$$\begin{aligned} &\frac{d}{dt}\,(u'(t), u(t))_{L^2(\Omega)} - |u'(t)|^2_{L^2(\Omega)} + ||u(t)||_V^2 + \frac{d}{dt}\,\big(\alpha u'(t), u(t)\big)_{L^2(\Gamma_1)} - \\ &- \Big|\alpha^{1/2}\, u'(t)\Big|^2_{L^2(\Gamma_1)} + \frac{1}{2}\,\frac{d}{dt}\,\Big|\beta^{1/2}\, u(t)\Big|^2_{L^2(\Gamma_1)} + \frac{1}{2}\,\frac{d}{dt}\,\Big|\delta^{1/2}\, u(t)\Big|^2_{L^2(\Omega)} = 0 \end{aligned} \tag{4.6}$$

If we define

$$\begin{aligned} \rho(t) = &\big(u'(t), u(t)\big)_{L^2(\Omega)} + \big(\alpha u'(t), u(t)\big)_{L^2(\Gamma_1)} + \\ &+ \frac{1}{2}\,\Big|\beta^{1/2}\, u(t)\Big|^2_{L^2(\Gamma_1)} + \frac{1}{2}\,\Big|\delta^{1/2}\, u(t)\Big|^2_{L^2(\Omega)}\,, \end{aligned} \tag{4.7}$$

we obtain from (4.6):

$$\rho'(t) = |u'(t)|^2_{L^2(\Omega)} - ||u(t)||_V^2 + |\alpha^{1/2}\, u'(t)|^2_{L^2(\Gamma_1)} \tag{4.8}$$

For $\varepsilon > 0$ we define the perturbed energy $E_\varepsilon(t)$ by

$$E_\varepsilon(t) = E(t) + \varepsilon\rho(t). \tag{4.9}$$

If we consider $|\rho(t)|$, elementary inequality $2ab \le a^2 + b^2$, Poincaré's inequality in V and trace theorem in $H^1(\Omega)$, we get:

$$|\rho(t)| \le C_0\, E(t), \tag{4.10}$$

C_0 is the constant defined above.

Then, by (4.9) and (4.10) we obtain:

$$|E_\varepsilon(t) - E(t)| < \varepsilon|\rho(t)| < \varepsilon\, C_0\, E(t).$$

Thus,

$$(1 - \varepsilon\, C_0)E(t) \le E_\varepsilon(t) \le (1 + \varepsilon\, C_0)E(t).$$

Choose $\varepsilon > 0$ such that $1 - \varepsilon\, C_0 \ge \dfrac{1}{2}$, that is, $0 < \varepsilon \le \dfrac{1}{2C_0}$ and $1 < 1 + \varepsilon\, C_0 \le \dfrac{3}{2}\cdot$ Then,

$$\frac{1}{2}\, E(t) \le E_{\varepsilon_0}(t) \le \frac{3}{2}\, E(t), \tag{4.11}$$

for all $t \ge 0$ and for $0 < \varepsilon_0 \le \dfrac{1}{2C_0}\cdot$

Since $E'_\varepsilon(t) = E'(t) + \varepsilon\rho'(t)$, by (4.2), (4.8) and (4.9), it follows:

$$\begin{aligned} E'_\varepsilon(t) = &- \left|\beta^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)} - \left|\delta^{1/2}\, u'(t)\right|^2_{L^2(\Omega)} + \\ &+ \varepsilon\left(|u'(t)|^2_{L^2(\Omega)} - ||u(t)||^2_V + |\alpha^{1/2}\, u'(t)|^2_{L^2(\Gamma_1)}\right) \end{aligned} \tag{4.12}$$

We have

$$\left|\beta^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)} \ge \beta_0\, |u'(t)|^2_{L^2(\Gamma_1)} \ge \frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}} \left|\alpha^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)},$$

$$\left|\delta^{1/2}\, u'(t)\right|^2_{L^2(\Omega)} \ge \delta_0\; |u'(t)|^2_{L^2(\Omega)}\, .$$

Substituting in (4.12) we get:

$$\begin{aligned} E'_\varepsilon(t) \le &- \left(\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}} - \varepsilon\right) \left|\alpha^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)} - \\ &- (\delta_0 - \varepsilon)|u'(t)|^2_{L^2(\Omega)} - \varepsilon||u(t)||^2_V\, . \end{aligned} \tag{4.13}$$

Choose $0 < \varepsilon_1 \leq \min\left(\frac{2}{3}\,\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}}\,,\frac{2}{3}\,\delta_0\right)$ set $\varepsilon = \varepsilon_1$ in (4.13). We have $\varepsilon_1 < \frac{2}{3}\,\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}}$ that implies

$$-\left(\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}} - \varepsilon_1\right) \leq -\frac{\varepsilon_1}{2}$$

or

$$-\left(\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}} - \varepsilon_1\right) \left|\alpha^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)} \leq -\frac{\varepsilon_1}{2}\, \left|\alpha^{1/2}\, u'(t)\right|^2_{L^2(\Gamma_1)}\,.$$

Similar argument proves that

$$-(\delta_0 - \varepsilon_1)|u'(t)|^2_{L^2(\Omega)} \leq -\frac{\varepsilon_1}{2}\,|u'(t)|^2_{L^2(\Omega)}\,.$$

Thus from (4.13) we get

$$E'_{\varepsilon_1}(t) \leq -\varepsilon_1\, E(t). \tag{4.14}$$

If we consider $\eta = \min\left\{\frac{1}{2C_0}, \frac{2}{3}\,\frac{\beta_0}{|\alpha|_{L^\infty(\Gamma_1)}}, \frac{2}{3}\,\delta_0\right\}$ we have (4.11) and (4.14) for this $\eta > 0$, that is,

$$E'_\eta(t) \leq -\frac{2}{3}\,E_\eta(t), \quad t \geq 0.$$

Integrating on $[0,t]$ this differential inequality, we obtain:

$$E_\eta(t) \leq E_\eta(0)\,e^{-\frac{2}{3}\eta t}, \quad t \geq 0.$$

From (4.11) it follows:

$$E(t) \leq 3\,E(0)\,e^{-\frac{2}{3}\eta t}, \quad \text{for all} \quad t \geq 0.$$

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