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NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS INVOLVING SINGULAR ϕ-LAPLACIANS

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Dedicated to Professor J. V. Gonçalves on the occasion of his 60th birthday

1 Introduction

In the flat Minkowski space $\mathbb{L}^{N+1} = \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$ with metric $\sum_{j=1}^N (dx_j)^2 - (dt)^2$, let us consider hypersurfaces, i.e. space-like submanifolds of codimension one. Their mean extrinsic curvature is the trace of the second fundamental form. Maximal hypersurfaces have mean extrinsic curvature zero, and hypersurfaces with constant mean extrinsic curvature are also of interest.

More specifically, let Ω be a bounded domain in $\{(x,t) \in \mathbb{L}^{N+1} : t = 0\} \simeq \mathbb{R}^N$, and let us restrict our attention to hypersurfaces which are the graph of functions $v : \Omega \to \mathbb{R}$. An extension of the problem of maximal hypersurfaces consists, given $f : \Omega \times \mathbb{R} \to \mathbb{R}$ continuous and bounded, to maximize the functional I defined by

$$I(v) := \int_\Omega \left[\sqrt{1 - \|\nabla v(x)\|^2} - \int_0^{v(x)} f(x,t)\,dt \right] dx \qquad (1)$$

over

$$\mathcal{C}(\varphi, \Omega) := \{v \in C^{0,1}(\Omega) : v = \varphi \textit{ on } \partial\Omega,\ \|\nabla v\| \le 1 \textit{ a.e. in } \Omega\}. \qquad (2)$$

It has been proved by Bartnik and Simon [2] that the maximum exists if and only if $\mathcal{C}(\varphi, \Omega) \neq \emptyset$, and is unique if, furthermore, $f(x,\cdot)$ is nondecreasing for any $x \in \Omega$. The existence of a maximum for I follows from the fact that I is bounded on $\mathcal{C}(\varphi, \Omega)$, that $\mathcal{C}(\varphi, \Omega)$ is equicontinuous, $\sqrt{1 - \|p\|^2}$ concave, and, using Serrin's semicontinuity theorem [9], I is upper semi-continuous. If one introduces the differential operator

$$\mathcal{M}(v) := \nabla \cdot \left(\frac{\nabla v}{\sqrt{1 - \|\nabla v\|^2}} \right) \tag{3}$$

the problem is then to show that the Euler-Lagrange equation for a maximum of I over $\mathcal{C}(\varphi, \Omega)$ is given by

$$\mathcal{M}(v) = f(x, v) \quad in \quad \Omega, \quad u = \varphi \quad on \quad \partial\Omega \tag{4}$$

A long argument in [2], which essentially consists in showing that a maximizing function u for I is such that $\|\nabla u\| \leq \eta < 1$, gives conditions which imply the existence of a solution $u \in C^1(\Omega) \cap W^{2,2}(\Omega)$ such that $\|\nabla u\| < 1$ for f and φ bounded, when φ has an extension ψ to $\overline{\Omega}$ such that $\|\nabla \psi\| \leq 1$.

More generally, given $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ continuous, one can raise the question of the solvability of the equation

$$\mathcal{M}(v) = f(x, v, \nabla v) \quad in \quad \Omega \tag{5}$$

submitted to Dirichlet or Neumann boundary conditions. When $N = 1$, more general results that the one mentioned above have been obtained in [6, 7], and we describe them in Section 2. A natural question is then to see if corresponding results hold for the radial solutions of (4) when Ω is a ball or an annulus in $\mathbb{R}^N$. This problem has been considered, for homogeneous boundary conditions, in [3] and [4] (also see [5]). Here we extend it to non-homogeneous boundary conditions. In Section 3 we deal with Dirichlet problems, while Section 4 is devoted to Neumann problems. The method of lower and upper solutions for Neumann boundary conditions is developed in Section 5.

2 The case where $N = 1$

For $N = 1$ and, say $\Omega = (0,1)$, it is shown in [7] that the non-homogeneous Dirichlet boundary value problem

$$\left(\frac{v'}{\sqrt{1-|v'|^2}}\right)' = f(x,v,v'), \quad v(0) = A,\ v(1) = B \tag{6}$$

is solvable for all $f \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and $|B-A| < 1$. The proof is based upon the application of Schauder fixed point theorem to an equivalent fixed point problem that we describe now. The following elementary lemma, proved in [7], is required to obtain the fixed point operator in the case of Dirichlet conditions.

Lemma 1 *Let $\phi : (-a,a) \to \mathbb{R}$ be an increasing homeomorphism such that $\phi(0) = 0$. For any $(h,d) \in C[0,1]\times(-a,a)$, there exists a unique $\alpha := Q_\phi(h,d)$ such that $\int_0^1 \phi^{-1}(h(s)-\alpha)\,ds = d$. Furthermore $Q_\phi : C[0,1]\times(-a,a) \to \mathbb{R}$ is completely continuous.*

Set $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$, and define $N_f : C^1[0,1] \to C[0,1]$ by

$$N_f(v)(t) = f(t,v(t),v'(t)),$$

and $H : C[0,1] \to C^1[0,1]$, by

$$Hw(t) = \int_0^t w(s)\,ds.$$

It is proved in [7] that v is a solution to (6) if and only if

$$v = A + H\circ\phi^{-1}\circ(HN_f(v) - Q_\phi[HN_f(v), B-A]),$$

and the operator defined by the right-hand side is well defined and completely continuous on $C^1[0,1]$ when $|A-B| < 1$, and maps $C^1[0,1]$ into $B_0(|A|+2) \subset C^1[0,1]$. With respect to the general case treated by Bartnik and Simon [2], the boundedness condition upon f has been removed. Condition $|B-A| < 1$

corresponds to the possibility of extending the boundary values to a function $\psi : [0,1] \to \mathbb{R}$ such that $|\psi'(x)| < 1$.

In the same way, the non-homogeneous Neuman boundary value problem

$$\left(\frac{v'}{\sqrt{1-|v'|^2}}\right)' = f(x,v,v'), \quad v'(0) = \frac{C}{\sqrt{1+C^2}},\ v'(1) = \frac{D}{\sqrt{1+D^2}} \tag{7}$$

is shown in [7] to be solvable for all $C, D \in \mathbb{R}$ and all $f \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ for which there exists $R > 0$ and $\varepsilon \in \{-1,1\}$ such that

$$\varepsilon\ (sgn\ v)\left[\int_0^1 f(x,v(x),v'(x))\,dx - (D-C))\right] \geq 0$$

when $\min_{[0,1]}|v| \geq R$, and $\|v'\|_\infty < 1$. The proof of this result is again based upon a reduction to a fixed point problem. Define $P, Q : C[0,1] \to \mathbb{R}$ by

$$Pv = v(0), \quad Qv = \int_0^1 v(x)\,dx,$$

and $g \in C[0,1]$ by $g(t) = (1-t)C + tD$. Then, v is a solution to (7) if and only if

$$v = Pv + QN_f(v) - (D-C) + H\circ\phi^{-1}\circ[H(I-Q)N_f(v)+g].$$

3 Radial solutions of Dirichlet extrinsic mean curvature problem in a ball

We consider now the problem of the existence of radial solutions to the Dirichlet problem in $B_0(1)$

$$\begin{aligned} \nabla\cdot\left(\frac{\nabla v(x)}{\sqrt{1-\|\nabla v(x)\|^2}}\right) &= f(\|x\|, v(x), \frac{\partial v}{\partial \nu}(x)) \quad in \quad B_0(1),\\ v &= A \quad on \quad \partial B_0(1), \end{aligned} \tag{8}$$

where $f \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and $A \in \mathbb{R}$. Here and below, $\dfrac{\partial v}{\partial \nu}(x)$ stands for the directional derivative of v at $x \neq 0$ in the direction $\dfrac{x}{\|x\|}$. A radial solution of

(8) is a solution of the form $v(x) = u(\|x\|)$, where, if $r := \|x\|$, u is solution of the one-dimensional boundary value problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' = r^{N-1}f(r,u,u'), \quad u'(0)=0,\ u(1)=A. \tag{9}$$

More generally, if $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, let us consider the problem

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(0)=0,\ u(1)=A. \tag{10}$$

By a (classical) solution of (10), we mean a function $u \in C^1 := C^1[0,1]$ such that $r \mapsto (r^{N-1}\phi(u'(r)) \in C^1$ and which satisfies (10). The Banach space C^1 is considered with the norm $\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty$. If $C^1_D = \{u \in C^1 : u'(0) = 0, u(1) = A\}$, $C_0 = \{u \in C[0,1] : u(0) = 0\}$, let us introduce the operators $S : C[0,1] \to C_0$, $K : C[0,1] \to C^1$ and $M : C^1_D \to C^1_D$ respectively defined by

$$\begin{aligned} Su(r) &= \frac{1}{r^{N-1}}\int_0^r t^{N-1}u(t)dt \quad (r \in (0,1]) \\ Ku(r) &= \int_1^r u(t)dt \quad (r \in [0,1]) \\ Mu &= A + K \circ \phi^{-1} \circ S \circ N_f(u). \end{aligned}$$

The following result is proved in [3] for $A = 0$ but the proof is essentially the same for arbitrary A.

Lemma 2 *M is completely continuous and u is a solution of (10) if and only if u is a fixed point of M.*

An easy consequence is the following existence result, proved (for $A = 0$) in [3].

Theorem 1 *If $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, then problem (10) has at least one solution for all $f \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and all $A \in \mathbb{R}$.*

Proof. By Lemma 2, it suffices to find u such that $u = M(u)$ and the existence of such an u follows from Schauder's fixed point theorem because $M(C^1_D) \subset \{u \in C^1_D : \|u\|_{C^1} < 2a + |A|\}$.

$\square$

Corollary 1 *The problem*

$$\mathcal{M}(v) = f(\|x\|, v, \partial v/\partial\nu) \quad in \quad B_0(1), \quad v = A \quad on \quad \partial B_0(1)$$

has at least one radial solution for any $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$ and any $A \in \mathbb{R}$.

Corollary 2 *If furthermore $f = f(r,u)$ and $f(r,\cdot)$ is nondecreasing, the radial solution is unique.*

4 Radial solutions of Neumann extrinsic mean curvature problem

Let $\rho \in [0,1)$, $A_\rho = B_0(1) \setminus \overline{B}_0(\rho)$ when $\rho > 0$, $A_0 = B_0(1)$. We now consider the Neumann extrinsic mean curvature problem

$$\begin{aligned} \mathcal{M}(v) &= f(\|x\|, v, \frac{\partial v}{\partial \nu}) \quad in \quad A_\rho \\ \frac{\partial v}{\partial \nu} &= \frac{C}{\sqrt{1+C^2}} \quad on \quad \partial B_0(\rho), \quad \frac{\partial v}{\partial \nu} = \frac{D}{\sqrt{1+D^2}} \quad on \quad \partial B_0(1), \end{aligned} \tag{11}$$

where $f \in C([\rho,1] \times \mathbb{R}^2, \mathbb{R})$ and $C, D \in \mathbb{R}$. We assume that $C = 0$ when $\rho = 0$. Letting $v(x) = u(\|x\|)$, this is equivalent to solving the boundary value problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' = r^{N-1} f(r,u,u'), \quad u'(\rho) = \frac{C}{\sqrt{1+C^2}},\ u'(1) = \frac{D}{\sqrt{1+D^2}}.$$

More generally, if $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, we consider the boundary value problem

$$(r^{N-1}\phi(u'))' = r^{N-1} f(r,u,u'), \quad u'(\rho) = \phi^{-1}(C),\ u'(1) = \phi^{-1}(D). \tag{12}$$

Let $C^1 := C^1[\rho,1]$ with the norm $\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty$, and let us write $u \in C^1$ as $u = \overline{u} + \widetilde{u}$, with $\overline{u} = u(\rho)$, and $\widetilde{u} \in \widetilde{C}^1 := \{u \in C^1 : u(\rho) = 0\}$. Let us define the linear operators $P, Q, L : C[\rho,1] \to C[\rho,1]$ and $H : C[\rho,1] \to C^1$ by

$$\begin{aligned}
Pu &= u(\rho),\\
Qu &= \frac{N}{1-\rho^N}\int_\rho^1 r^{N-1}u(r)dr,\\
Lu(r) &= \frac{1}{r^{N-1}}\int_\rho^r t^{N-1}u(t)dt, \qquad (13)\\
Hu(r) &= \int_\rho^r u(t)dt \quad (r \in [\rho,1]),
\end{aligned}$$

and let $p_N : [\rho,1] \to \mathbb{R}, r \mapsto r^{1-N}$.

Lemma 3 *u is a solution of (12) if and only if u is a fixed point of the operator $R : C^1 \to C^1$ defined by*

$$\begin{aligned}
R(u) &= Pu + [QN_f(u) - \frac{N}{1-\rho^N}(D - \rho^{N-1}C)] \qquad (14)\\
&+ H \circ \phi^{-1} \circ \left\{\rho^{N-1}Cp_N + L[(I-Q)N_f(u) + \frac{N}{1-\rho^N}(D-\rho^{N-1}C)]\right\}.
\end{aligned}$$

Proof. If u is a solution of (12), then

$$D - \rho^{N-1}C = \int_\rho^1 r^{N-1}f(r,u(r),u'(r))\,dr, \qquad (15)$$

i.e.

$$\frac{N}{1-\rho^N}[D - \rho^{N-1}C] = QN_f(u). \qquad (16)$$

Hence

$$(r^{N-1}\phi(u'))' = r^{N-1}[f(r,u,u') - QN_f(u) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]]$$

Consequently

$$\begin{aligned}
&r^{N-1}\phi(u'(r)) = \rho^{N-1}C \qquad (17)\\
&+ \int_\rho^r s^{N-1}\left[f(s,u(s),u'(s)) - QN_f(u) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]\right]ds,
\end{aligned}$$

so that

$$
\begin{aligned}
& u'(r) = \phi^{-1}\Big\{\rho^{N-1}Cp_N(r) & (18)\\
+ \quad & \frac{1}{r^{N-1}}\int_\rho^r s^{N-1}\left[f(s,u(s),u'(s)) - QN_f(u) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]\right] ds\Big\}\\
= \quad & \phi^{-1}\circ\left\{\rho^{N-1}Cp_N(r) + L\left[(I-Q)N_f(u)(r) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]\right]\right\}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& u(r) - u(\rho) & (19)\\
= \quad & H\circ\phi^{-1}\circ\left\{\rho^{N-1}Cp_N(r) + L\left[(I-Q)N_f(u)(r) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]\right]\right\}.
\end{aligned}
$$

As equations (16) and (19) take value in the supplementary spaces $\mathbb{R}$ (constant functions) and $\widetilde{C}^1$, they can be written as the unique equation

$$
u(r) = u(\rho) + QN_f(u) - \frac{N}{1-\rho^N}[D-\rho^{N-1}C] + H\circ\phi^{-1} \qquad (20)
$$
$$
\circ\left\{\rho^{N-1}Cp_N(r) + L\left[(I-Q)N_f(u)(r) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]\right]\right\}.
$$

i.e. as $u = R(u)$. Conversely, if $u = R(u)$, then, taking $r = \rho$ we get (15) and what remains is (19). By differentiating we get (18) and then (17) and the differential equation in (12). The boundary conditions easily follow from (15) and (17).

□

A slight modification of the argument in [4] shows that R is a completely continuous operator on C^1.

Let us now introduce the modified operator $\widetilde{R} : \mathbb{R}\times\widetilde{C}^1 \to \widetilde{C}^1$ defined by

$$
\begin{aligned}
& \widetilde{R}(\overline{u},\widetilde{u})\\
= \quad & H\circ\phi^{-1}\circ\left\{\rho^{N-1}Cp_N + L\left[(I-Q)N_f(\overline{u}+\widetilde{u}) + \frac{N}{1-\rho^N}[D-\rho^{N-1}C]\right]\right\}.
\end{aligned}
$$

The fixed points of $\widetilde{R}$ are the solutions of the modified Neumann problem

$$
\begin{aligned}
(r^{N-1}\phi(\widetilde{u}'))' &= r^{N-1}\{N_f(\overline{u}+\widetilde{u}) - [QN_f(\overline{u}+\widetilde{u}) - \frac{N}{1-\rho^N}(D-\rho^{N-1}C)]\},\\
u'(\rho) &= \phi^{-1}(C),\ u'(1) = \phi^{-1}(D) \qquad (21)
\end{aligned}
$$

as it is easily verified.

Lemma 4 *The set of solutions* $(\overline{u},\widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1$ *of (21) contains a continuum* $\mathcal{C}$ *with* $proj_{\mathbb{R}}\mathcal{C} = \mathbb{R}$ *and* $proj_{\widetilde{C}^1}\mathcal{C} \subset B_0((2-\rho)a)$.

Proof. Given $\overline{u} \in \mathbb{R}$, $\widetilde{u} \in \widetilde{C}^1$ is a solution if and only if

$$\widetilde{u} = \widetilde{R}(\overline{u},\widetilde{u}).$$

The nonlinear operator $\widetilde{R}$ is completely continuous, and such that $\widetilde{R}(\mathbb{R}\times\widetilde{C}^1) \subset B_0((2-\rho)a)$. The result follows from a version of Leray-Schauder theorem with unbounded parameter set.

$\square$

An easy consequence of Lemma 4 is the following theorem.

Theorem 2 *If* $\rho \in [0,1), \phi : (-a,a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(0) = 0$, *and if there exists* $\varepsilon \in \{-1,1\}$ *and* $R > 0$ *such that*

$$\varepsilon(sgn\ u)\left[\int_\rho^1 r^{N-1} f(r,u(r),u'(r))\,dr - \left(D - \rho^{N-1}C\right)\right] \geq 0 \tag{22}$$

for any $u \in C^1$ *such that* $\min_{[\rho,1]}|u| \geq R$ *and* $||u'||_\infty < a$, *then problem (12) has at least one solution.*

Proof. Apply Lemma 4 to get $\mathcal{C}$ and observe that $QN_f - \frac{N}{1-\rho^N}\left(D - \rho^{N-1}C\right)$ takes opposite signs on $\mathcal{C}$. By connectedness $QN_f(\overline{u}+\widetilde{u}) - \frac{N}{1-\rho^N}\left(D-\rho^{N-1}C\right)$ vanishes for some $(\overline{u},\widetilde{u}) \in \mathcal{C}$, and $u = \overline{u} + \widetilde{u}$ is a solution of (12).

$\square$

Corollary 3 *Under the same conditions upon* f, *problem (11) in* A_ρ *has at least one radial solution.*

Corollary 4 *If furthermore $f = f(r,u)$ and $f(r,\cdot)$ is nondecreasing, the radial solution is unique.*

We can deduce from Theorem 2 some surjectivity results.

Corollary 5 *For any $g \in C([\rho,1] \times \mathbb{R}, \mathbb{R})$ such that*

$$\lim_{s \to \pm\infty} g(r,s) = \pm\infty \quad or \quad \mp\infty \tag{23}$$

uniformly in $[\rho,1]$, and for all bounded $h \in C([\rho,1] \times \mathbb{R}^2, \mathbb{R})$, problem

$$\begin{aligned} \mathcal{M}(v) &= g(\|x\|, v) + h(\|x\|, v, \partial v/\partial\nu) \quad in \quad A_\rho, \\ \frac{\partial v}{\partial \nu} &= \frac{C}{\sqrt{1+C^2}} \quad on \quad \partial B_0(\rho), \quad \frac{\partial v}{\partial \nu} = \frac{D}{\sqrt{1+D^2}} \quad on \quad \partial B_0(1) \end{aligned} \tag{24}$$

has at least one radial solution.

Corollary 6 *For all $g \in C([\rho,1] \times \mathbb{R}, \mathbb{R})$ such that $g(r,\cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism for all $r \in [\rho,1]$, problem*

$$\begin{aligned} \mathcal{M}(v) &= g(\|x\|, v) \quad in \quad A_\rho, \\ \frac{\partial v}{\partial \nu} &= \frac{C}{\sqrt{1+C^2}} \quad on \quad \partial B_0(\rho), \quad \frac{\partial v}{\partial \nu} = \frac{D}{\sqrt{1+D^2}} \quad on \quad \partial B_0(1) \end{aligned} \tag{25}$$

has an unique radial solution.

This is in particular the case for $g(\|x\|, v) = |v|^{p-1}v + e(\|x\|) \quad (p > 1)$, with $e \in C[\rho,1]$.

We can also deduce from Theorem 2 some Landesman-Lazer type results.

Corollary 7 *Let $g \in C(\mathbb{R}, \mathbb{R})$. Then problem*

$$\mathcal{M}(v) + g(v) = e(\|x\|) \quad in \quad A_\rho, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial A_\rho \tag{26}$$

has a radial solution if either

$$\overline{\lim}_{s \to -\infty} g(s) < \frac{N}{1-\rho^N} \int_\rho^1 r^{N-1} e(r)\, dr < \underline{\lim}_{s \to +\infty} g(s) \tag{27}$$

or

$$\overline{\lim}_{s\to+\infty}g(s) < \frac{N}{1-\rho^N}\int_\rho^1 r^{N-1}e(r)dr < \underline{\lim}_{s\to-\infty}g(s) \tag{28}$$

For example, of $g(v) = \arctan v + \sin v$, the conditions become

$$1-\frac{\pi}{2} < \frac{N}{1-\rho^N}\int_\rho^1 r^{N-1}e(r)dr < \frac{\pi}{2}-1$$

Remark 1 Conditions (27) or (28) are sharp if either

$$\overline{\lim}_{s\to-\infty}g(s) < g(v) < \underline{\lim}_{s\to+\infty}g(s)$$

or

$$\overline{\lim}_{s\to+\infty}g(s) < g(v) < \underline{\lim}_{s\to-\infty}g(s) \quad for \quad all \quad v\in\mathbb{R}.$$

For example, if $g(v) = \pm e^v$, (26) has a radial solution if and only if

$$\int_\rho^1 r^{N-1}e(r)dr \gtrless 0.$$

5 Lower and upper solutions method for Neumann boundary conditions

Let again $\phi : (-a,a) \to \mathbb{R}$ be an increasing homeomorphism such that $\phi(0) = 0$, $\rho \in (0,1)$, $f \in C([\rho,1]\times\mathbb{R}^2,\mathbb{R})$ and let us consider the Neumann problem

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(\rho) = \phi^{-1}(C),\ u'(1) = \phi^{-1}(D) \tag{29}$$

where $C, D \in \mathbb{R}$ and $\rho > 0$.

Definition 1 *A* lower solution *for (29) is a function* $\alpha \in C^1$ *such that*

$$\|\alpha'\|_\infty < a, \quad r^{N-1}\phi(\alpha') \in C^1,$$

$$\alpha'(\rho) \ge \phi^{-1}(C), \quad \alpha'(1) \le \phi^{-1}(D) \tag{30}$$

and

$$(r^{N-1}\phi(\alpha'(r)))' \geq r^{N-1}f(r,\alpha(r),\alpha'(r)) \quad for \quad all \quad r \in [\rho,1]. \tag{31}$$

An upper solution *for (29) is a function $\beta \in C^1$ such that*

$$\|\beta'\|_\infty < a, \quad r^{N-1}\phi(\beta') \in C^1,$$

$$\beta'(\rho) \leq \phi^{-1}(C), \quad \beta'(1) \geq \phi^{-1}(D) \tag{32}$$

and

$$(r^{N-1}\phi(\beta'(r)))' \leq r^{N-1}f(r,\beta(r),\beta'(r))) \quad for \quad all \quad r \in [\rho,1]. \tag{33}$$

Theorem 3 *The existence of a lower solution α and an upper solution β for (29) implies the existence of a solution to (29).*

Proof. We follow the ideas of [4]. One first proves the result when $\alpha(r) \leq \beta(r)$ for all $r \in [\rho,1]$ by the standard approach : introduction of a modified problem outside $[\alpha,\beta]$, existence of the modified problem using Corollary 5, and finally proof that any such solution lies in $[\alpha,\beta]$, and solves the original problem. If α and β are unordered, we adapt an argument introduced by Amann-Ambrosetti-Mancini [1] in semilinear Dirichlet problems. If $QN_f(u) - \frac{N}{1-\rho^N}\left(D - \rho^{N-1}C\right) = 0$ for some $u \in \mathcal{C}$, we are done. If $QN_f(u) - \frac{N}{1-\rho^N}\left(D - \rho^{N-1}C\right) < 0$ (resp. > 0) on $\mathcal{C}$, one gets the existence of an upper (resp. lower) solution greater (resp. smaller) than α (resp. β), and then apply the first part of the proof.

□

Remark 2 No Nagumo-type growth condition with respect to u' is required in Theorem 3.

Corollary 8 *Problem*

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(\rho) = 0 = u'(1)$$

has at least one solution if there exists $A, B \in \mathbb{R}$ *such that*

$$f(r,A,0)\cdot f(r,B,0) \le 0 \quad for \quad all \quad r \in [\rho,1].$$

Proof. Take constant lower and upper solutions $\alpha = A$ and $\beta = B$.

□

We can apply the method of lower and upper solutions to get an existence result of the type introduced by Kazdan-Warner [8] for the Dirichlet problem of an equation involving a second order elliptic operator.

Corollary 9 *If* $f \in C([\rho,1]\times\mathbb{R},\mathbb{R})$ *is such that* $f(r,\cdot)$ *is either nondecreasing or nonincreasing for all* $r \in [\rho,1]$*, then problem*

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u), \quad u'(\rho) = \phi^{-1}(C),\ u'(1) = \phi^{-1}(D) \tag{34}$$

is solvable if and only if there exists $c \in \mathbb{R}$ *such that*

$$\int_\rho^1 r^{N-1}f(r,c)\,dr = D - C\rho^{N-1} \tag{35}$$

Corollary 10 *If* $f \in C([\rho,1]\times\mathbb{R},\mathbb{R})$ *is such that* $f(r,\cdot)$ *is either nondecreasing or nonincreasing for all* $r \in [\rho,1]$*, then problem*

$$\begin{aligned} \mathcal{M}(v) &= f(||x||,v) \quad in \quad A_\rho, \\ \frac{\partial v}{\partial \nu} &= \frac{C}{\sqrt{1+C^2}} \quad on \quad \partial B_0(\rho), \quad \frac{\partial v}{\partial \nu} = \frac{D}{\sqrt{1+D^2}} \quad on \quad \partial B_0(1) \end{aligned} \tag{36}$$

has a radial solution if and only if there exists $c \in \mathbb{R}$ *such that (35) holds.*

Uniqueness holds if $f(r,\cdot)$ is strictly monotone.

Example 1 *For all* $p > 1$,

$$\mathcal{M}(v) \pm |v|^{p-1}v^{+} = e(\|x\|) \quad in \quad A_\rho, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial \mathcal{A}_\rho$$

has a radial solution if and only if $\int_\rho^1 r^{N-1}e(r)\,dr \geq 0 \quad (resp. \ \leq 0)$.

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