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C^k SOLVABILITY NEAR THE CHARACTERISTIC SET FOR A CLASS OF VECTOR FIELDS OF INFINITE TYPE

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Abstract

We consider vector fields of the form $T_\lambda \doteq (1-r)\partial/\partial r + i\lambda\partial/\partial\theta$ defined on $R_\delta \doteq \{(r,\theta) : |r-1| < \delta\}$, where $\lambda \in \mathbb{C}$ and $\mathrm{Re}\lambda \neq 0$. We look for C^k solutions of the equation $T_\lambda u = f$ in a full neighborhood of the characteristic set $\Sigma \doteq \{r = 1\}$.

1 Introduction

Let

$$T_\lambda \doteq (1-r)\frac{\partial}{\partial r} + i\lambda\frac{\partial}{\partial\theta} \tag{1.1}$$

be a complex vector field defined on $R_\delta \doteq \{(r,\theta) : |r-1| < \delta \text{ and } \theta \in \mathbb{R}\}$, where (r,θ) are the polar coordinates, $\lambda \in \mathbb{C}$ and $\mathrm{Re}\lambda \neq 0$.

Observe that the change of variables $r' = r, \theta' = -\theta$ transforms T_λ into $T_{-\lambda}$, therefore it suffices to study this operator in the case $\mathrm{Re}\lambda > 0$.

The set $\Sigma \doteq \{r = 1\}$ is the characteristic set of T_λ, that is, the region where T_λ fails to be elliptic.

This work deals with solvability of T_λ in following sense: for a fixed $k \in \mathbb{Z}^+$, we say that T_λ is C^k solvable at Σ if given $f \in C^\infty(R_\delta)$ satisfying

$$\int_0^{2\pi} f(1,\theta)\, d\theta = 0, \tag{1.2}$$

there is a C^k solution u of the equation

$$T_\lambda u(r,\theta) = f(r,\theta) \tag{1.3}$$

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in a neighborhood of Σ.

The vector field (1.1) is a model of a class of rotationally invariant complex vector fields of infinite type along a closed smooth curve. For further information on these concepts and ideas we refer the reader to the works [7, 11, 14, 16].

The equation (1.3) was studied by Berhanu and Meziani in the C^0 category, in [12], where the authors constructed continuous solutions near the characteristic set by analyzing two distinct cases, namely: $\lambda \in \mathbb{Q}^+$ and $\lambda \notin \mathbb{Q}^+$.

In [6], Bergamasco and Meziani proved that for every λ, there exist C^∞ functions f satisfying (1.2) such that equation (1.3) has no C^∞ solution in any neighborhood of Σ.

A natural question then arises: are there C^k solutions to this equation in a neighborhood of Σ?

In [16], Meziani proved that if $\lambda \notin \mathbb{Q}^+$ and $f \in C^\infty(R_\delta)$ satisfies (1.2) then, for every $k \in \mathbb{Z}^+$, the equation $T_\lambda u = f$ has a C^k solution defined in a neighborhood of Σ.

The main goals of the present work are to give a new proof of the result about C^k solvability of T_λ obtained by Meziani, in [16], and also to present results about C^k-solvability of T_λ when $\lambda \in \mathbb{Q}^+$. Our first result is the following.

Theorem 1.1. *Let $f \in C^\infty(R_\delta)$ and $k \in \mathbb{N}$.*

(i) If $\lambda \notin \mathbb{Q}^+$ and

$$\int_0^{2\pi} f(1,\theta)\, d\theta = 0, \tag{1.4}$$

then the equation $T_\lambda u = f$ has a C^k solution u in R_δ;

(ii) If $\lambda = p/q \in \mathbb{Q}^+$ with $\gcd(p,q) = 1$ and

$$\int_0^{2\pi} \frac{\partial^{lp} f}{\partial r^{lp}}(1,\theta) e^{i\theta l q}\, d\theta = 0, \ \forall l \in \mathbb{Z}^+, \tag{1.5}$$

then the equation $T_\lambda u = f$ has a C^k solution u in R_δ.

The strategy to prove this theorem is to use the same approach that was used by Berhanu and Meziani, in [12], introducing an essencial modification in the way that the solution is constructed.

The study of existence and regularity of solutions near of the characteristic set is closely related to the study of global solvability and global hipoellipticity. In the articles [5, 6, 7, 14, 17] the authors study solvability near the characteristic set, while the articles [3, 4, 15, 16] study global problems in this approach. Other useful references are [1, 2, 8, 9, 10, 13] where the authors study vector fields and systems of vector fields on compact surfaces.

2 Preliminaries and a new formulation of the theorem

Since the operator $T_\lambda = (1-r)\partial/\partial r + i\lambda\partial/\partial\theta$ is elliptic outside of Σ, given $\delta, \varepsilon > 0$, such that $\varepsilon < \delta < 1$, it is possible to find $v_\varepsilon \in C^\infty(R_\delta)$ solution of $T_\lambda v_\varepsilon = f$, in $R_\delta \setminus R_\varepsilon$ and $v_\varepsilon \equiv 0$ in $R_{\frac{1}{2}\varepsilon}$. To obtain this solution we consider the structure of Riemann surface given by the operator T_λ, in each one of the connected components of $R_\delta \setminus R_\varepsilon$, and we use the Uniformization Theorem to construct a smooth change of variables in this component which transforms T_λ in a multiple of the Cauchy-Riemann operator $\partial/\partial\overline{z}$. The solvability, in the C^∞ sense, of $\partial/\partial\overline{z}$ provides a solution $v_\varepsilon \in C^\infty(R_\delta \setminus R_\varepsilon)$ of the equation $T_\lambda v_\varepsilon = f$ in $R_\delta \setminus R_\varepsilon$. Finally, by using a convenient cut-off function, we can assume $v_\varepsilon \in C^\infty(R_\delta)$ and $v_\varepsilon \equiv 0$ in $R_{\frac{1}{2}\varepsilon}$.

Therefore, after replacing f by $f - T_\lambda v_\varepsilon$, we may assume that the right side of the equation $T_\lambda u = f$ is supported on $R_\varepsilon = \{(r,\theta) : |r-1| < \epsilon\}$.

Thus from now on we will assume that

$$\operatorname{supp} f \subset R_\varepsilon, \quad 0 < \varepsilon < \delta < 1/2.$$

We say that a function $f \in C^\infty(R_\delta)$ is flat at $r = 1$ if its partial derivatives of all orders vanish at Σ.

In this work we will use the following result, which was obtained from lemma 4.2 of [12] by a change notation.

Lemma 2.1. *Let $f \in C^\infty(R_\delta)$.*

(i) If $\lambda \notin \mathbb{Q}^+$, there is $v \in C^\infty(R_\delta)$ with support in R_ε such that $T_\lambda v - f$ is flat on $r = 1$ if and only if (1.4) holds;

(ii) If $\lambda = p/q \in \mathbb{Q}^+$ with $\gcd(p, q) = 1$, then there is $v \in C^\infty(R_\delta)$ with support in R_ε, such that $T_\lambda v - f$ is flat on $r = 1$ if and only if (1.5) holds.

Thus, with the equivalent conditions presented in lemma 2.1, after replacing the original f by $f - T_\lambda v_\varepsilon - T_\lambda v$, we may assume that the right side of the equation (1.3) is flat in $r = 1$ and supp$f \subset R_\varepsilon$.

Therefore, in order to prove theorem 1.1, it is enough to prove the following result.

Proposition 2.2. *If $f \in C^\infty(R_\delta)$ is flat on $r = 1$ and supp$f \subset R_\varepsilon$ then, given $k \in \mathbb{Z}^+$, there exists $u \in C^k(R_\delta)$ such that $T_\lambda u = f$.*

We will prove this result providing Fourier coefficients of the solution and proving that these coefficients in fact correspond to a C^k function that satisfies the equation $T_\lambda u = f$.

3 Construction of the C^k solution

To study equation (1.3), we use the characterization by Fourier series, relatively to θ, namely:

$$f(r, \theta) = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}, \quad u(r, \theta) = \sum_{n \in \mathbb{Z}} u_n(r) e^{in\theta}, \tag{3.1}$$

where

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) e^{-in\theta} \, d\theta \tag{3.2}$$

and $u_n(r)$ is defined likewise.

Note that each $f_n(r)$ is supported in $[1-\varepsilon, 1+\varepsilon]$.

Equation (1.3) implies that each $u_n(r)$ satisfies the equation

$$(1-r)u_n'(r) - n\lambda u_n(r) = f_n(r), \quad \forall n \in \mathbb{Z}, \tag{3.3}$$

Since we are assuming that f is and flat on $r=1$, we choose $u_n(1)=0$, for every $n \in \mathbb{Z}$.

If $n < -k/\mathrm{Re}\lambda$ we take

$$\begin{aligned} u_n(r) &= \frac{1}{(1-r)^{n\lambda}} \int_0^r (1-t)^{n\lambda-1} f_n(t)\,dt, \text{ if } r<1, & (3.4)\\ u_n(r) &= \frac{1}{(r-1)^{n\lambda}} \int_r^\infty (t-1)^{n\lambda-1} f_n(t)\,dt, \text{ if } r>1. & (3.5)\end{aligned}$$

If $n \geqslant -k/\mathrm{Re}\lambda$ and $n \neq 0$ we take

$$\begin{aligned} u_n(r) &= \frac{-1}{(1-r)^{n\lambda}} \int_r^1 (1-t)^{n\lambda-1} f_n(t)\,dt, \text{ if } r<1, & (3.6)\\ u_n(r) &= \frac{-1}{(r-1)^{n\lambda}} \int_1^r (t-1)^{n\lambda-1} f_n(t)\,dt, \text{ if } r>1. & (3.7)\end{aligned}$$

Finally, for $n=0$ we take

$$\begin{aligned} u_0(r) &= -\int_r^1 \frac{f_0(t)}{1-t}\,dt, \text{ if } r<1, & (3.8)\\ u_0(r) &= -\int_1^r \frac{f_0(t)}{t-1}\,dt, \text{ if } r>1. & (3.9)\end{aligned}$$

Remark 3.1. The formulas (3.4), (3.5), (3.6) and (3.7) above are exactly the ones used by Berhanu and Mezziani in [11], page 139, however, there is an essential difference in the form that this expressions are chosen. For us, this choice takes into account the degree of regularity that is expected of the solution (we choose different formulas to $0 > n \geqslant -k/\mathrm{Re}\lambda$).

In order to show that the function u, whose Fourier coefficients are defined above, is in fact a C^k function, we present four lemmas which will help us to develop this proof.

Lemma 3.2. *If $f \in C^\infty(R_\delta)$ is flat on $r=1$, supp$f \subset R_\varepsilon$ and $m,q \in \mathbb{Z}^+$, then for all $r \in (1-\delta, 1+\delta)$ and $n \neq 0$ we have*

$$\left|f_{n,p}^{(m)}(r)\right| \leqslant \frac{Q_{m+p,q}}{|n|^q}, \tag{3.10}$$

where the constant $Q_{m+p,q}$ depends only on $m+p$ and q, and $f_{n,p}^{(m)}(r)$ denotes the function defined by

$$f_{n,p}^{(m)}(r) = \begin{cases} \dfrac{f_n^{(m)}(r)}{(r-1)^p}, & \textit{if } r \neq 1 \\ 0, & \textit{if } r = 1 \end{cases} \tag{3.11}$$

for all $n \in \mathbb{Z}$ and $p \in \mathbb{N}$.

Proof: If we differentiate $m+p$ times the expression (3.2) and integrate by parts q times we obtain

$$f_n^{(m+p)}(r) = \frac{1}{(in)^q}\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^q}{\partial\theta^q}\left[\frac{\partial^{m+p}}{\partial r^{m+p}} f(r,\theta)\right] e^{-in\theta}\, d\theta.$$

Furthermore $f \in C^\infty(R_\delta)$ and supp$f \subset R_\varepsilon$, hence

$$\left|\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^q}{\partial\theta^q}\left[\frac{\partial^{m+p}}{\partial r^{m+p}} f(r,\theta)\right] e^{-in\theta}\, d\theta\right| \leqslant Q_{m+p,q},$$

therefore

$$\left|f_n^{(m+p)}(r)\right| \leqslant \frac{Q_{m+p,q}}{|n|^q}. \tag{3.12}$$

Since f_n is flat on $r=1$, the Taylor's formula for $f_n^{(m)}$ gives

$$\begin{aligned} f_n^{(m)}(r) &= \sum_{s=1}^{p} \frac{f_n^{(m+s-1)}(1)}{(s-1)!}(r-1)^{s-1} \\ &\quad + \left[\int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f_n^{(m+p)}\big(1+t(r-1)\big)\, dt\right](r-1)^p \\ &= (r-1)^p \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f_n^{(m+p)}\big(1+t(r-1)\big)\, dt, \end{aligned}$$

therefore

$$f_{n,p}^{(m)}(r) = \frac{f_n^{(m)}(r)}{(r-1)^p} = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f_n^{(m+p)}\big(1+t(r-1)\big)\, dt.$$

Thus, from (3.12) follows that

$$\begin{aligned} \left|f_{n,p}^{(m)}(r)\right| &\leqslant \int_0^1 \left|\frac{(1-t)^{p-1}}{(p-1)!}\right| \left|f_n^{(m+p)}\big(1+t(r-1)\big)\right| dt \\ &\leqslant \frac{Q_{m+p,q}}{|n|^q} \int_0^1 \left|\frac{(1-t)^{p-1}}{(p-1)!}\right| dt \leqslant \frac{Q_{m+p,q}}{|n|^q} \end{aligned}$$

This completes the proof.

□

Lemma 3.3. *Assume that $m, n, k \in \mathbb{Z}$, $n \neq 0$ and $0 < m \leqslant k$. Then*

$$\prod_{j=0}^{m-1} |n\lambda + j| \leqslant c_m |n|^m$$

where c_m is a positive constant that does not depend on n.

Proof: Let $j \in \mathbb{Z}$ such that $0 \leqslant j < m$,

i. if $n < -k/\mathrm{Re}\lambda$, then $j(j + 2n\mathrm{Re}\lambda) \leqslant 0$ hence $|n\lambda + j| \leqslant |n\lambda|$.

ii. if $0 > n \geq -k/\mathrm{Re}\lambda$ and $d_m = \max\{|\lambda + j/n|; 0 \leqslant j \leqslant m-1\}$ then $|n\lambda + j| = |n||\lambda + j/n| \leqslant d_m|n|$.

iii. if $n > 0$ then $|n\lambda + j| \leqslant |n\lambda + m| \leqslant |n||\lambda + m|$.

Therefore, we set $c_m \doteq max\{|\lambda|^m, d_m^m, |\lambda + m|^m\}$ and conclude the proof.

□

Lemma 3.4. *If $m \in \mathbb{Z}^+$ is such that $0 \leqslant m \leqslant k$ and $r \in (1-\delta, 1+\delta) \setminus \{1\}$ then,*

$$\left|\frac{d^m u_n}{dr^m}(r)\right| \leqslant c_m |n|^m \left[\int_{1-\varepsilon}^{1+\varepsilon} |f_{n,k+1}(t)|\, dt + \sum_{j=0}^{m-1} \left| f_{n,k+1-j}^{(j)}(r) \right| \right]$$

Proof: First, if $n \neq 0$ and $r \in (1-\delta, 1+\delta) \setminus \{1\}$, by induction on the order of differentiation in (3.3) we can write

$$\frac{d^m u_n}{dr^m}(r) = \prod_{j=0}^{m-1} [n\lambda + j] \frac{u_n(r)}{(1-r)^m} + g_m(r) \tag{3.13}$$

where

$$g_m(r) = \sum_{j=0}^{m-2}\left(\prod_{k=j}^{m-2}\big(n\lambda+k+1\big)\frac{f_n^{(j)}(r)}{(1-r)^{m-j}}\right)+\frac{f_n^{(m-1)}(r)}{(1-r)}$$

and when $n=0$ observe that $u_n(r)$ satisfies (3.8) and (3.9), thus

$$\left|\frac{d^m u_0}{dr^m}(r)\right| = \left|\frac{d^{m-1}}{dr^{m-1}}\left(\frac{f_0(r)}{1-r}\right)\right|$$

We shall analyze each one of the four possibilities, given by the formulas (3.4), (3.5), (3.6) and (3.7), to insert u_n in the expression (3.13). For this we start by analyzing the expression $\prod_j[n\lambda+j]u_n(r)(1-r)^{-m}$.

Case 1: $n<-k/\mathrm{Re}\lambda$, $r<1$ and $m>0$;

By formula (3.4) and lemma 3.3 we obtain

$$\begin{aligned}
\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| &\leqslant c_m|n|^m\left|\frac{1}{(1-r)^{n\lambda+m}}\int_0^r(1-t)^{n\lambda-1}f_n(t)\,dt\right|\\
&= c_m|n|^m\left|\int_0^r\left(\frac{1-t}{1-r}\right)^{n\lambda+m}\frac{f_n(t)}{(1-t)^{m+1}}\,dt\right|\\
&\leqslant c_m|n|^m\int_0^r\left|\left(\frac{1-t}{1-r}\right)^{n\lambda+m}f_{n,m+1}(t)\right|\,dt\\
&= c_m|n|^m\int_0^r\left(\frac{1-t}{1-r}\right)^{m+n\mathrm{Re}\lambda}\big|f_{n,m+1}(t)\big|\,dt.
\end{aligned}$$

Since $0\leqslant t\leqslant r<1$ and $m\leqslant k<-n\mathrm{Re}\lambda$ hence $1-t\geqslant 1-r>0$ and $m+n\mathrm{Re}\lambda<0$, therefore

$$\left(\frac{1-t}{1-r}\right)^{m+n\mathrm{Re}\lambda}\leqslant 1.$$

Moreover, $\mathrm{supp} f_{n,m+1}\subset[1-\varepsilon,1+\varepsilon]$ and for all $t\in[1-\varepsilon,1+\varepsilon]$ we have

$$|f_{n,m+1}(t)|\leqslant|f_{n,k+1}(t)|,$$

Thus
$$\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| \leqslant c_m|n|^m\int_{1-\varepsilon}^{1+\varepsilon}|f_{n,k+1}(t)|\;dt. \tag{3.14}$$

Case 2: If $n<-k/\mathrm{Re}\lambda$, $r>1$ and $m>0$;

By formula (3.5) and lemma 3.3,

$$\begin{aligned}\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| &\leqslant c_m|n|^m\int_r^\infty\left|\left(\frac{t-1}{r-1}\right)^{m+n\lambda}\frac{f_n(t)}{(t-1)^{m+1}}\right|dt\\ &= c_m|n|^m\int_r^\infty\left(\frac{t-1}{r-1}\right)^{m+n\mathrm{Re}\lambda}\big|f_{n,m+1}(t)\big|\,dt.\end{aligned}$$

Since $1<r\leqslant t$ and $m\leqslant k<-n\mathrm{Re}\lambda$ hence $t-1\geqslant r-1>0$ and $m+n\mathrm{Re}\lambda<0$, thus
$$\left(\frac{t-1}{r-1}\right)^{m+n\mathrm{Re}\lambda}\leqslant 1,$$
Moreover, $\mathrm{supp}f_{n,m+1}\subset[1-\varepsilon,1+\varepsilon]$ and for all $t\in[1-\varepsilon,1+\varepsilon]$ we have

$$|f_{n,m+1}(t)|\leqslant|f_{n,k+1}(t)|,$$

thus
$$\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| \leqslant c_m|n|^m\int_{1-\varepsilon}^{1+\varepsilon}|f_{n,k+1}(t)|\;dt. \tag{3.15}$$

Case 3: If $n\neq 0$, $n\geqslant -k/\mathrm{Re}\lambda$, $r<1$ and $m>0$;

By formula (3.6) and lemma 3.3,

$$\begin{aligned}\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| &\leqslant c_m|n|^m\left|\int_r^1\left(\frac{1-t}{1-r}\right)^{k+n\mathrm{Re}\lambda}\frac{f_n(t)}{(t-1)^{k+1}}\right|dt\\ &= c_m|n|^m\int_r^1\left(\frac{1-t}{1-r}\right)^{na+k}\big|f_{n,k+1}(t)\big|\,dt.\end{aligned}$$

Since $0 \leqslant r \leqslant t < 1$, then $1-t \leqslant 1-r$, moreover $k + n\mathrm{Re}\lambda \geqslant 0$, hence

$$\left(\frac{1-t}{1-r}\right)^{k+n\mathrm{Re}\lambda} \leqslant 1.$$

Thus

$$\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| \leqslant c_m|n|^m \int_{1-\varepsilon}^{1+\varepsilon} \left|f_{n,k+1}(t)\right| dt. \tag{3.16}$$

Case 4: If $n \neq 0$, $n \geqslant -k/\mathrm{Re}\lambda$, $r > 1$ and $m > 0$.

By formula (3.7) and lemma 3.3,

$$\begin{aligned}\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| &\leqslant c_m|n|^m \left|\int_1^r \left(\frac{t-1}{r-1}\right)^{k+n\lambda} \frac{f_n(t)}{(t-1)^{k+1}}\,dt\right| \\ &= c_m|n|^m \int_1^r \left(\frac{t-1}{r-1}\right)^{n\mathrm{Re}\lambda+k} \left|f_{n,k+1}(t)\right| dt.\end{aligned}$$

Since $1 < t \leqslant r$ then $t-1 \leqslant r-1$, moreover $k + n\mathrm{Re}\lambda \geqslant 0$, hence

$$\left(\frac{t-1}{r-1}\right)^{k+n\mathrm{Re}\lambda} \leqslant 1.$$

Thus

$$\left|\prod_{j=0}^{m-1}[n\lambda+j]\,\frac{u_n(r)}{(1-r)^m}\right| \leqslant c_m|n|^m \int_{1-\varepsilon}^{1+\varepsilon} \left|f_{n,k+1}(t)\right| dt. \tag{3.17}$$

Note that, in the above four cases, we have the same estimate for the term containing the product sign.

Now, to estimate the term $g_m(r)$ that appears in the expression (3.13), when $m > 0$, we have

$$\begin{aligned}
\left|g_m(r)\right| &= \left|\sum_{j=0}^{m-2}\left(\prod_{l=j}^{m-2}\left(n\lambda+l+1\right)\frac{f_n^{(j)}(r)}{(1-r)^{m-j}}\right)+\frac{f_n^{(m-1)}(r)}{(1-r)}\right| \\
&\leqslant c_m|n|^m\sum_{j=0}^{m-2}\left|\frac{f_n^{(j)}(r)}{(1-r)^{m-j}}\right|+\left|\frac{f_n^{(m-1)}(r)}{(1-r)}\right| \\
&\leqslant c_m|n|^m\sum_{j=0}^{m-2}\left|\frac{f_n^{(j)}(r)}{(1-r)^{k+1-j}}\right|+\left|\frac{f_n^{(m-1)}(r)}{(1-r)}\right| \\
&\leqslant c_m|n|^m\sum_{j=0}^{m-1}\left|f_{n,k+1-j}^{(j)}(r)\right|.
\end{aligned}$$

Finally, when $m = 0$, note that in all cases above the inequalities hold if we set $g_0 \equiv 0$ and $c_0 = 1$, which concludes the proof.

□

Lemma 3.5. *If $m, n \in \mathbb{Z}$ and $0 \leqslant m \leqslant k$, then*

$$\lim_{r\to 1} u_n^{(m)}(r) = 0. \tag{3.18}$$

Proof: First, if $n < -k/\mathrm{Re}\lambda$, we choose a number α $(0 < \alpha < 1)$ such that $(n\mathrm{Re}\lambda + k + \alpha) < 0$. When $r < 1$, it follows from the formula (3.4) that

$$\begin{aligned}
\lim_{r\to 1^-}\left|\frac{u_n(r)}{(1-r)^k}\right| &= \lim_{r\to 1^-}\left|(1-r)^\alpha\int_0^r\left(\frac{1-t}{1-r}\right)^{n\lambda+k+\alpha}\frac{f_n(t)}{(1-t)^{1+k+\alpha}}\,dt\right| \\
&\leqslant \lim_{r\to 1^-}(1-r)^\alpha\int_0^r\left(\frac{1-t}{1-r}\right)^{n\mathrm{Re}\lambda+k+\alpha}\left|\frac{f_n(t)}{(1-t)^{1+k+\alpha}}\right|dt \\
&= \lim_{r\to 1^-}(1-r)^\alpha\int_0^r\left|f_{n,k+2}(t)\right|dt = \lim_{r\to 1^-}(1-r)^\alpha\, r\, R_n = 0,
\end{aligned}$$

where the constant R_n is given by the mean value theorem for integrals.

When $r > 1$ we use the formula (3.5) and recall that the support of f_n is contained in $[1-\varepsilon, 1+\varepsilon]$ hence

$$\begin{aligned}
\lim_{r\to 1^+}\left|\frac{u_n(r)}{(1-r)^k}\right| &= \lim_{r\to 1^+}\left|(r-1)^\alpha\int_r^{r+1}\Big(\frac{t-1}{r-1}\Big)^{n\lambda+k+\alpha}\frac{f_n(t)}{(1-t)^{1+k+\alpha}}\,dt\right|\\
&\leqslant \lim_{r\to 1^+}(r-1)^\alpha\int_r^{r+1}\Big(\frac{t-1}{r-1}\Big)^{n\mathrm{Re}\lambda+k+\alpha}\left|\frac{f_n(t)}{(1-t)^{1+k+\alpha}}\right|dt\\
&\leqslant \lim_{r\to 1^+}(r-1)^\alpha\int_r^{r+1}\Big|f_{n,k+2}(t)\Big|\,dt = \lim_{r\to 1^+}(1-r)^\alpha\cdot R_n = 0.
\end{aligned}$$

Now, if $n > -k/\mathrm{Re}\lambda,\ n\neq 0$ and $r<1$, using the formula (3.6) we have

$$\begin{aligned}
\lim_{r\to 1^-}\left|\frac{u_n(r)}{(1-r)^k}\right| &= \lim_{r\to 1^-}\left|\frac{1}{(1-r)^{n\lambda+k}}\int_r^1(1-t)^{n\lambda-1}f_n(t)\,dt\right|\\
&\leq \lim_{r\to 1^-}\int_r^1\left|\Big(\frac{1-t}{1-r}\Big)^{n\lambda+k}\frac{f_n(t)}{(1-t)^{1+k}}\right|dt\\
&\leq \lim_{r\to 1^-}\int_r^1\Big|f_{n,1+k}(t)\Big|\,dt \ \leq\ \lim_{r\to 1^-}R_n(1-r)=0
\end{aligned}$$

Finally, if $n > -k/\mathrm{Re}\lambda,\ n\neq 0$ and $r>1$, using the formula (3.7) we have

$$\begin{aligned}
\lim_{r\to 1^+}\left|\frac{u_n(r)}{(1-r)^k}\right| &\leq \lim_{r\to 1^+}\int_1^r\left|\Big(\frac{t-1}{r-1}\Big)^{n\lambda+k}\frac{f_n(t)}{(t-1)^{1+k}}\right|dt\\
&\leq \lim_{r\to 1^+}\int_1^r\Big|f_{n,1+k}(t)\Big|\,dt \leq \lim_{r\to 1^+}R_n(r-1)=0
\end{aligned}$$

Therefore we can conclude that $\lim_{r\to 1}u_n(r)(1-r)^{-k}=0$, for all $n\neq 0$. Moreover, if $0\leqslant m\leqslant k$ then

$$\lim_{r\to 1}\frac{u_n(r)}{(1-r)^m}=\lim_{r\to 1}\frac{u_n(r)}{(1-r)^k}\,(1-r)^{k-m}=0.$$

Finally, from (3.13) we have $\lim_{r\to 1}u_n^{(m)}(r)=0$.

□

Now we are able to prove that the solution u given by the expressions (3.4) – (3.9) is, in fact, a C^k solution to the equation $T-\lambda u=f$.

Proof of the proposition 2.2: Since, for every $0\leqslant m\leqslant k$, $u_n^{(m)}(r)$ is continuous in $(1-\delta,1+\delta)\setminus\{1\}$ and $\lim_{r\to 1}u_n^{(m)}(r)=0$, hence if we set $u_n^{(m)}(1)=0$, then $u_n^{(m)}(r)$ will be continuous in $(1-\delta,1+\delta)$, for all $n\in\mathbb{Z}$.

Let $m_1, m_2 \in \mathbb{N}$, such that $m = m_1 + m_2$; then

$$\left|\frac{\partial^m}{\partial\theta^{m_1}\partial r^{m_2}}\left(u_n(r)e^{in\theta}\right)\right| = \left|n^{m_1}u_n^{(m_2)}(r)\right|.$$

Therefore all derivatives of $u_n(r)e^{in\theta}$ are continuous, thus in order to prove that $u \in C^k(R_\delta)$, it is sufficient to prove that, for each $0 \leqslant m \leqslant k$, the following series are uniformly convergent

$$\sum_{n\neq 0}\frac{\partial^m}{\partial r^{m_1}\partial\theta^{m_2}}(u_n(r)e^{in\theta}). \tag{3.19}$$

When $n \neq 0$ and $0 \leqslant m \leqslant k$, it follows from lemmas 3.2 and 3.4 that

$$\begin{aligned}\left|n^{m_1}u_n^{(m_2)}(r)\right| &\leqslant c_m|n|^{m_1+m_2}\left[\int_{1-\varepsilon}^{1+\varepsilon}|f_{n,k+1}(t)|\,dt + \sum_{j=0}^{m-1}\left|f_{n,k+1-j}^{(j)}(r)\right|\right]\\ &\leqslant c_k\left[\frac{(k+1)Q_{k+1,k+2}}{|n|^2}\right] = \frac{Q_k}{|n|^2}.\end{aligned}$$

where Q_k is a constant that depends only on k.

Since $\sum_{n\in\mathbb{Z}^*}Q_k/n^2$ converges, it follows from the Weierstrass M-test that (3.19) is uniformly convergent.

□

Remark 3.6. It is easy to see that the expression (1.4) is a necessary condition to the equation $T_\lambda u = f$ have solution, for all $\lambda \in \mathbb{C}$. Furthermore, differentiating both sides of the equation $T_\lambda u = f$ with respect to r and using the θ-Fourier coefficients we obtain the following equation

$$-(j+\lambda n)u_n^{(j)}(1) = f_n^{(j)}(1), \quad \forall n \in \mathbb{Z} \text{ and } 1 \leqslant j \leqslant k.$$

Therefore if $T_\lambda u = f$ has a C^k solution and $\lambda = p/q > 0$ then $f_{-lq}^{(lp)}(1) = 0$, for all integer l such that $0 \leqslant lp \leqslant k$. If $\lambda \notin \mathbb{Q}$ then we have only one condition, namely: $f_0(1) = 0$.

Remark 3.7. In order to construct a C^k solution to the equation $T_\lambda u = f$, when $\lambda \in \mathbb{Q}^+$, we do not need to ask that f be flat on $r = 1$.
In fact, we use only a finite number of compatibility conditions to obtain a C^k solution and the following condition is sufficient to develop our proof:

If $\lambda = p/q \in \mathbb{Q}^+$ with $mdc(p,q) = 1$ and $f_{-lq}^{(lp)}(1) = 0$, for all nonnegative integers l such that $lp \leqslant k$, then the equation $T_\lambda u = f$ has a C^k solution in R_δ.

Theorem 3.8. *Let $f \in C^\infty(R_\delta)$ and $k \in \mathbb{Z}^+$. Then the equation $T_\lambda u = f$ has a C^k solution u in R_δ if and only if one of the following conditions is satisfied:*

(i) If $\lambda \in \mathbb{C} \setminus \mathbb{Q}$ and $f_0(1) = 0$;

(ii) If $\lambda = p/q \in \mathbb{Q}$ with $mdc(p,q) = 1$ and $f_{-lq}^{(lp)}(1) = 0$, for all nonnegative integers l such that $0 \leqslant lp \leqslant k$.

Remark 3.9. If $f \in C^\infty(R_\delta)$ and $f_n \equiv 0$, for all $n < -k/\mathrm{Re}\lambda$ (when $\mathrm{Re}\lambda > 0$) then we do not need to consider cases 1 and 2 in the proof of the proposition 2.2 and, in cases 3 and 4, we can take $m > k$ and repeat all arguments, replacing k by m, obtaining $u \in C^\infty(R_\delta)$.

Remark 3.10. If $f_n \equiv 0$, for all $n > -k/\mathrm{Re}\lambda$ (when $\mathrm{Re}\lambda > 0$), then it is easy to see in the formulas (3.6) and (3.7) that $\mathrm{supp}\, u \subset R_\varepsilon$.

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