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SURFACES OF CONSTANT CURVATURE IN 3-DIMENSIONAL SPACE FORMS

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1 Introduction.

The study of surfaces immersed in a 3-dimensional ambient space plays a central role in the theory of submanifolds. In addition, Riemannian manifolds with constant sectional curvature can be considered as the most simple examples. Thus, one can think of surfaces with constant Gauss curvature in the Euclidean space $\mathbb{R}^3$, hyperbolic space $\mathbb{H}^3$ or 3-sphere $\mathbb{S}^3$ as very natural objects of study.

Through these notes we will study some classical results on complete surfaces with constant Gauss curvature in 3-dimensional space forms, using a modern approach. The notes are organized as follows.

In Section 2 we establish some notation and recall some necessary preliminary concepts. In Section 3 we center our attention on the complete surfaces with positive constant Gauss curvature in $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$. We shall prove the Liebmann theorem and show that the only complete examples must be totally umbilical round spheres. In particular, there is no complete surface in $\mathbb{S}^3$ with constant Gauss curvature $K(I) \in (0,1)$.

In Section 4 we prove the existence of a conformal representation for flat surfaces in the hyperbolic 3-space. We shall associate to each surface a pair of holomorphic 1-forms and see that the flat surface can be recovered in terms of this holomorphic pair. We prove that the two hyperbolic Gauss maps of a flat surface are holomorphic with respect to the conformal structure given by the second fundamental form, and classify the complete flat surfaces.

Section 5 is devoted to the Hartman-Nirenberg theorem for flat surfaces in the

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Euclidean 3-space. We shall show that a complete flat surface in $\mathbb{R}^3$ must be a right cylinder on a planar curve which is defined for all values of its arc length parameter.

In Section 6 we shall study the Bianchi-Spivak representation of a flat surface in $\mathbb{S}^3$. We start showing the existence of global Tschebyscheff coordinates in a complete flat surface. We prove that the asymptotic curves of a flat surface have torsion ± 1 when the curvature of the curves does not vanish. Moreover, these asymptotic curves are characterized in a simple way which generalizes the torsion ± 1 concept. We recall the quaternionic model of $\mathbb{S}^3$ and show that any complete flat surface Σ in $\mathbb{S}^3$ can be recovered by multiplication of the two asymptotic curves passing across a point $p \in \Sigma$. We also prove the converse of this result in terms of curves with generalized torsion ± 1. Finally, we note that every asymptotic curve of a flat torus is closed. Thus, flat tori can be classified.

In Section 7 we prove the Hilbert theorem. Thus, we show that there is no complete surface with constant negative Gauss curvature $K(I)$ in $\mathbb{R}^3$ or $\mathbb{S}^3$. The same happens for complete surfaces in $\mathbb{H}^3$ when $K(I) < -1$. There exist complete surfaces in $\mathbb{H}^3$ with constant Gauss curvature $K(I) \in [-1, 0[$.

2 Some preliminary concepts.

Let us denote by $\mathbb{M}^3(c)$ to the simply-connected Riemannian 3-space with constant sectional curvature $c = -1, 0, 1$. That is, $\mathbb{M}^3(c)$ denotes to the hyperbolic space $\mathbb{H}^3$ if $c = -1$, to the Euclidean space $\mathbb{R}^3$ when $c = 0$ or to the sphere $\mathbb{S}^3$ if $c = 1$.

Now, consider an oriented Riemannian surface Σ with induced metric $I = \langle , \rangle$ and $f : \Sigma \longrightarrow \mathbb{M}^3(c)$ an isometric immersion. Let N be a unit normal along Σ which is compatible with the orientation, and II its associated second fundamental form. That is,

$$II(X, Y) = \langle -\overline{\nabla}_X N, Y \rangle, \qquad X, Y \in \mathfrak{X}(\Sigma),$$

where $\mathfrak{X}(\Sigma)$ denotes to the set of smooth vector fields in Σ, and $\overline{\nabla}$ the Levi-Civita connection of the ambient space.

If k_1, k_2 are the principal curvatures of the immersion, namely, the eigenvalues of the shape operator S, given by $SX = -\overline{\nabla}_X N$, then we shall denote by

$$K = k_1 k_2, \qquad H = \frac{k_1 + k_2}{2}$$

to the extrinsic curvature and to the mean curvature of the immersion, respectively. These two quantities are extrinsic and depend on how S is immersed into $\mathbb{M}^3(c)$.

On the other hand, we denote by $K(I)$ to the curvature of the metric I, namely, the Gauss curvature, which is an intrinsic quantity.

For the previous immersion $f : \Sigma \longrightarrow \mathbb{M}^3(c)$ the following relations must be satisfied

Gauss equation: $\quad K(I) = K + c,$

Mainardi-Codazzi equation: $\quad \nabla_X SY - \nabla_Y SX - S[X, Y] = 0, \quad X, Y \in \mathfrak{X}(\Sigma),$

where ∇ is the Levi-Civita connection of the metric I.

In fact, Bonnet's theorem asserts that given a simply-connected Riemannian surface $\widetilde{\Sigma}$ with induced metric I and a self-adjoint endomorphism S, the Gauss and Mainardi-Codazzi equations are sufficient for the existence of an immersion $\widetilde{f} : \widetilde{\Sigma} \longrightarrow \mathbb{M}^3(c)$, whose induced metric is I and whose shape operator is S. Moreover, that immersion is unique up to rigid motions.

It is a remarkable fact that the Mainardi-Codazzi equation does not depend on c. Hence, it agrees for the ambient spaces $\mathbb{H}^3$, $\mathbb{R}^3$ and $\mathbb{S}^3$.

Moreover, this equation appears in different contexts, which motivates the study of the equation from an abstract point of view, since every result obtained from the abstract setting will be applied in different situations. Thus, we introduce the concept of *Codazzi pair*:

Codazzi pairs.

We start by defining the framework where the extrinsic curvature and mean curvature can be defined.

Definition 1. *A fundamental pair on an oriented surface Σ is a pair of real quadratic forms (I, II) on Σ, where I is a Riemannian metric.*

Associated with a fundamental pair (I, II) we define the shape operator S of the pair as

$$II(X,Y) = I(S(X),Y) \tag{1}$$

for any vector fields X, Y on Σ.

Conversely, from (1) it becomes clear that the quadratic form II is totally determined by I and S. In other words, to give a fundamental pair on Σ is equivalent to give a Riemannian metric on Σ and a self-adjoint endomorphism S.

We define the *mean curvature*, the *extrinsic curvature* and the *principal curvatures* of (I, II) as half the trace, the determinant and the eigenvalues of the endomorphism S, respectively.

In particular, given local parameters (x, y) on Σ such that

$$I = E\,dx^2 + 2F\,dxdy + G\,dy^2, \qquad II = e\,dx^2 + 2f\,dxdy + g\,dy^2,$$

the mean curvature and the extrinsic curvature of the pair are given, respectively, by

$$H = H(I, II) = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \qquad K = K(I, II) = \frac{eg - f^2}{EG - F^2}.$$

Moreover, the principal curvatures of the pair are $H \pm \sqrt{H^2 - K}$.

We shall say that the pair (I, II) is *umbilical* at $p \in \Sigma$ if II is proportional to I at p, or equivalently:

- if both principal curvatures coincide at p, or
- if S is proportional to the identity map on the tangent plane at p, or
- if $H^2 - K = 0$ at p.

We define the *Hopf differential* of the fundamental pair (I, II) as the (2,0)-part of II for the Riemannian metric I. That is, if we consider Σ as a Riemann surface with respect to the metric I and take a local conformal parameter z, then

$$\begin{aligned} I &= 2\lambda\,|dz|^2 \\ II &= Q\,dz^2 + 2\lambda\,H\,|dz|^2 + \overline{Q}\,d\overline{z}^2. \end{aligned} \tag{2}$$

The quadratic form $Q\,dz^2$, which does not depend on the chosen parameter, is known as the Hopf differential of the pair (I, II). We note that (I, II) is umbilical at $p \in \Sigma$ if, and only if, $Q(p) = 0$.

All the above definitions can be understood as natural extensions of the corresponding ones for isometric immersions of a Riemann surface in a 3-dimensional ambient space, where I plays the role of the induced metric and II the role of its second fundamental form.

A specially interesting case for our study happens when the fundamental pair satisfies, in an abstract way, the Mainardi-Codazzi equation for surfaces in $\mathbb{M}^3(c)$,

Definition 2. *We say that a fundamental pair* (I, II)*, with shape operator* S*, is a* Codazzi *pair if*

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = 0, \qquad X, Y \in \mathfrak{X}(\Sigma), \tag{3}$$

where ∇ *stands for the Levi-Civita connection associated with the Riemannian metric* I.

A first important remark is the following: given a fundamental pair (I, II) and a conformal parameter z for I then, from (2), a straightforward computation gives that (I, II) is a Codazzi pair if and only if

$$Q_{\overline{z}} = \lambda H_z. \tag{4}$$

In particular, this means the Hopf differential Qdz^2 is holomorphic if and only if the mean curvature is constant.

We observe that the existence of a holomorphic quadratic form for a class of surfaces is a very useful tool, which can be used in order to obtain geometric properties about the behavior of those surfaces. As an example, we get

Theorem 1 (Hopf)**.** *Let* (I, II) *be a Codazzi pair of constant mean curvature* H *on a topological sphere* Σ*. Then* (I, II) *is totally umbilical.*

When this result is used for immersions into $\mathbb{M}^3(c)$ asserts that a topological sphere of constant mean curvature in $\mathbb{M}^3(c)$ must be necessarily a round sphere [17].

Proof of Theorem 1. *If the pair (I, II) has constant mean curvature then $Q_{\overline{z}} \equiv 0$. The only holomorphic quadratic form on a sphere vanishes identically. Thus $Qdz^2 \equiv 0$ and (I, II) is totally umbilical.*

□

We shall apply the same technique in the next Section in order to prove the Liebmann theorem.

Finally, it should be observed that U. Abresch and H. Rosenberg have extended the Hopf theorem for surfaces immersed in 3-dimensional homogeneous manifolds with a 4-dimensional isometry group. They obtain a quadratic holomorphic differential which allows them to characterize the spheres of constant mean curvature as the revolution examples [1, 2]. For the 3-dimensional homogeneous ambient space Sol_3, a recent result has been obtained by B. Daniel and P. Mira [10].

3 Surfaces with positive constant Gauss curvature.

Let us start with a surface Σ endowed with a complete Riemannian metric I of constant Gauss curvature $K(I) = k > 0$. Thus, if Σ is simply-connected then Σ is isometric to the standard sphere $\mathbb{S}^2(r)$ of radius $r = 1/\sqrt{k}$, from the Cartan-Hadamard theorem.

The following theorem asserts how a complete Riemannian surface with positive constant Gauss curvature can be immersed in $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$.

Theorem 2 (Liebmann)**.** *Let Σ be a surface and $f : \Sigma \longrightarrow \mathbb{M}^3(c)$ a complete immersion with positive constant Gauss curvature. Then $f(\Sigma)$ is a totally umbilical round sphere.*

Observe that, from the Gauss equation, a surface with constant Gauss curvature must also have constant extrinsic curvature. In $\mathbb{R}^3$ both curvatures agree, and they differ by a constant in $\mathbb{H}^3$ and $\mathbb{S}^3$. Moreover, if the Gauss curvature is positive then the extrinsic curvature of the surface is also positive in $\mathbb{H}^3$ and $\mathbb{R}^3$. However, if the Gauss curvature $K(I)$ is positive in $\mathbb{S}^3$ then the extrinsic curvature is only positive if $K(I) > 1$.

Bearing in mind these comments, the Liebmann theorem for $\mathbb{M}^3(c)$ (with $K(I) > 1$ in $\mathbb{S}^3$) will be a consequence of the following result [23].

Theorem 3. *Let (I, II) be a Codazzi pair on a surface Σ with positive constant extrinsic curvature. Then the (2,0)-part of I with respect to the Riemannian metric II is a holomorphic quadratic form.*

In particular, if Σ is a topological sphere then the pair (I, II) is totally umbilical.

Proof. *Since $K = k_1 k_2 > 0$, we can orientate Σ such that k_1 and k_2 are positive. In particular, with this orientation II is a Riemannian metric.*

Thus, if we take a local conformal parameter z for the Riemannian metric II, then we can write

$$I = P dz^2 + 2\lambda |dz|^2 + \overline{P} d\overline{z}^2, \qquad II = 2\rho |dz|^2.$$

Now, let us denote by Γ_{ij}^k to the Christoffel symbols of the Levi-Civita connection of I with respect to the parameter z. That is,

$$\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \Gamma_{11}^1 \frac{\partial}{\partial z} + \Gamma_{11}^2 \frac{\partial}{\partial \overline{z}}, \qquad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial \overline{z}} = \Gamma_{12}^1 \frac{\partial}{\partial z} + \overline{\Gamma_{12}^1} \frac{\partial}{\partial \overline{z}}.$$

If we denote by $D = \lambda^2 - |P|^2$, a direct computation gives

$$\frac{D_z}{2D} = \Gamma_{11}^1 + \overline{\Gamma_{12}^1}.$$

The Codazzi equation for the parameter z gives

$$\frac{\rho_z}{\rho} = \Gamma_{11}^1 - \overline{\Gamma_{12}^1}.$$

And $K = \rho^2/D$ is constant, one has $2\rho_z/\rho = D_z/D$. Or equivalently, $\Gamma_{12}^1 = 0$. Therefore,

$$P_{\overline{z}} = \frac{\partial}{\partial \overline{z}} I\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = 0,$$

as we wanted to show.

Since a holomorphic 2-form in a sphere Σ must vanish identically then $Pdz^2 \equiv 0$ in Σ. Hence, (I, II) must be totally umbilical.

$\square$

In order to finish the proof of the Liebmann theorem, it is necessary to study the case of an immersion $f : \Sigma \longrightarrow \mathbb{S}^3$ with constant Gauss curvature $K(I) = k \in (0, 1]$.

When $k \in (0, 1)$ then the extrinsic curvature is negative. Thus, the principal curvatures satisfy $k_1 k_2 < 0$, and $k_1 \neq k_2$ on Σ. But this fact contradicts the Poincaré index theorem, since the universal cover of Σ is a sphere.

The case $k = 1$ depends more strongly on the Gauss equation. For that, we argue as follows: Let p be a non-umbilical point on an immersed sphere Σ of constant Gauss curvature 1 in $\mathbb{S}^3$. Then, let us take doubly orthogonal parameters (u, v) in a neighborhood of p, that is,

$$I = Edu^2 + Gdv^2, \qquad II = k_2 G dv^2.$$

Here, one principal curvature must vanish since, from the Gauss equation, the extrinsic curvature also vanishes. And k_2 is different from 0.

The Codazzi equation yields $E_v = 0$ and $(k_2^2 G)_u = 0$.

Thus, taking $\widetilde{u}$ with $d\widetilde{u} = \sqrt{E(u)}du$, if necessary, we can assume $E \equiv 1$. And analogously, from the equation $(k_2^2 G)_u = 0$ we have

$$I = du^2 + \frac{1}{k_2^2}\, dv^2, \qquad II = \frac{1}{k_2}\, dv^2.$$

Therefore, the Gauss curvature takes the form

$$1 = -|k_2| \left(\frac{1}{|k_2|} \right)_{uu}. \tag{5}$$

Since Σ is compact, either Σ is totally umbilical or there exists a non-umbilical point where k_2 has a maximum or minimum. But the latter case is not possible since every solution of the real equation $h''(t) + h(t) = 0$ has no positive local minimum, and $1/|k_2|$ would have a positive local minimum, which contradicts (5).

Therefore, Σ is a totally umbilical sphere, which finishes Liebmann theorem.

$\square$

The techniques of Codazzi pairs in Theorem 3 have been used in the ambient spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ in order to obtain a Liebmann type theorem [3] (see also [8]).

There are some interesting open problems related with the existence of surfaces with positive Gauss curvature:

If Σ is a surface with positive Gauss curvature in $\mathbb{R}^3$ and boundary lying on a plane P then it is easy to see that $\partial\Sigma$ is a convex Jordan curve. Moreover, given a convex Jordan curve Γ lying on a plane P, there exists a constant $k > 0$ such that for all $k' \in (0, k]$ there is a surface Σ with Gauss curvature k' whose boundary is Γ (see for instance [15]).

An open problem is to find the best value of k in terms of the curve Γ. It is known that some restrictions must be satisfied. For instance, k and Γ must satisfy the following inequality

$$k \leq \frac{\pi}{A_\Gamma},$$

where A_Γ is the planar area enclosed by Γ [11]. Moreover, the equality in the inequality above is only possible when Γ is a circle and Σ is a hemisphere.

On the other hand, given a convex planar Jordan curve Γ it is not known if fixed a constant $k > 0$ there exist exactly two non-isometric surfaces with Gauss curvature k and boundary Γ.

A specially interesting problem is to find the closed surfaces in $\mathbb{R}^3$ with constant Gauss curvature and with a small number of singularities, that is, with a finite number of isolated singularities. As we have proved, if there is no singularity then the surface must be a round sphere. It is not difficult to prove that there is no closed surface with only one singularity, by using Alexandrov reflection principle. And revolution surfaces are the only surfaces with two singularities. However, the existence of these surfaces for a finite number of singularities greater than two is not known.

4 Flat surfaces in the hyperbolic 3-space.

In the Lorentz-Minkowski model, $\mathbb{L}^4$, the hyperbolic 3-space is described by one connected component of a two-sheeted hyperboloid. More precisely,

$$\mathbb{H}^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1, \, x_0 > 0\right\} \qquad (6)$$

endowed with the induced metric of $\mathbb{L}^4$ given by the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$.

Let

$$\mathbb{N}^3_+ = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \, x_0 > 0\right\}$$

be the positive null cone in $\mathbb{L}^4$. And consider the following relation in $\mathbb{N}^3_+$

$$x, y \in \mathbb{N}^3_+ \text{ are identified if and only if there exists } \lambda > 0 : \ x = \lambda y.$$

With this identification $\mathbb{N}^3_+/\mathbb{R}^+$ is a topological sphere. In fact $\mathbb{N}^3_+/\mathbb{R}^+$ inherits a natural conformal structure and can be regarded as the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$.

We shall regard $\mathbb{L}^4$ as the space of 2×2 Hermitian matrices, Herm(2), in the standard way, by identifying $(x_0, x_1, x_2, x_3) \in \mathbb{L}^4$ with the matrix

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}. \qquad (7)$$

Under this identification the inner product of $\mathbb{L}^4$ can be recovered from the equality $\langle m, m\rangle = -\det(m)$ for all $m \in$ Herm(2). Then $\mathbb{H}^3$ is the set of $m \in$ Herm(2) with $\det(m) = 1$. The action of $\mathbb{SL}(2, \mathbb{C})$ on these Hermitian matrices defined by

$$g \cdot m = gmg^*,$$

where $g \in \mathbb{SL}(2,\mathbb{C})$, $g^* = {}^t\overline{g}$, preserves the inner product, leaves $\mathbb{H}^3$ invariant and is transitive. The kernel of this action is $\{\pm I_2\} \subseteq \mathbb{SL}(2, \mathbb{C})$ and $\mathbb{PSL}(2, \mathbb{C}) = \mathbb{SL}(2, \mathbb{C})/\{\pm I_2\}$ can be regarded as the identity component of the special Lorentzian group $\mathbb{SO}(1, 3)$.

The space $\mathbb{N}^3_+$ is seen as the space of positive semi-definite 2×2 Hermitian matrices of determinant 0 and its elements can be written as $w\, {}^t\overline{w}$, where ${}^t w = (w_1, w_2)$ is

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Now, let Σ be a surfa
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$G^+ = [\psi + \eta] : \Sigma \longrightarrow \mathbb{S}$

Analogously, the neg
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$\gamma'(0) = \eta(p)$. Then G^+
and the ideal boundary $\mathbb{S}$

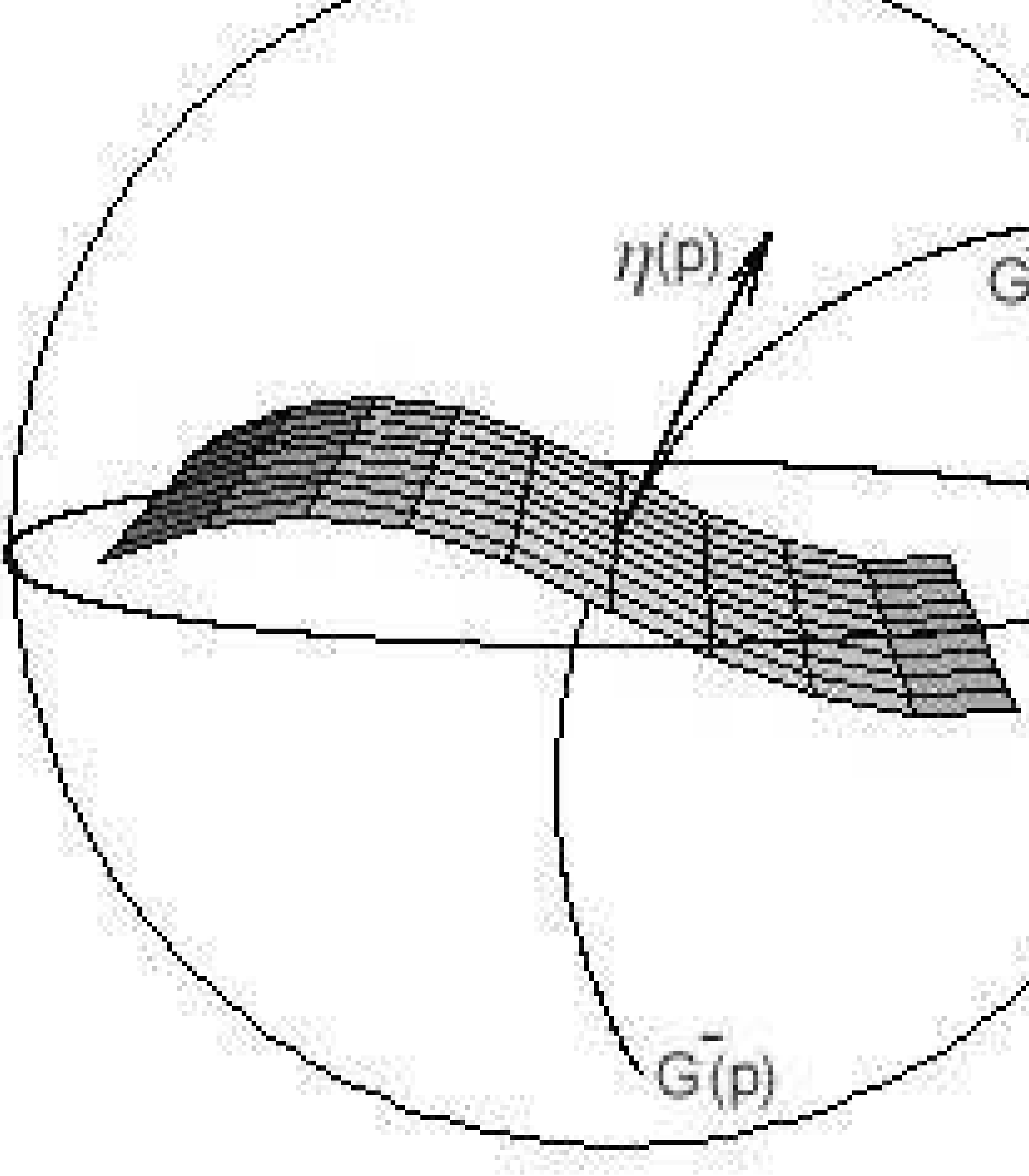

Figure 1: $G^+(p)$ and $G^-(p)$ in the Poincaré model of $\mathbb{H}^3$.

Let us start with a flat surface Σ in $\mathbb{H}^3$, that is, with intrinsic curvature $K(I) = 0$. From the Gauss equation the product of its principal curvatures $k_1k_2 = 1$. Thus, we can choose a unit normal in Σ such that k_1 and k_2 are positive, or equivalently, the second fundamental form is a Riemannian metric. We shall always fix that normal throughout this section.

Proposition 1. *Let Σ be a surface and $\psi : \Sigma \longrightarrow \mathbb{H}^3$ a flat immersion with unit normal η. If we consider Σ as a Riemann surface with the conformal structure induced by II, then $\psi + \eta, \psi - \eta : \Sigma \longrightarrow \mathbb{N}^3_+$ are conformal maps . In particular, the hyperbolic Gauss maps $G^+, G^- : \Sigma \longrightarrow \mathbb{S}^2_\infty$ are conformal for II.*

Proof. *Let $p \in \Sigma$ and $\{e_1, e_2\}$ an orthonormal basis at p such that $-d\eta(e_1) = k_1 e_1$ and $-d\eta(e_2) = k_2 e_2$. Then at p,*

$$\begin{array}{ll} II(e_1,e_1) = k_1 & \langle d(\psi+\eta)(e_1), d(\psi+\eta)(e_1)\rangle = (1-k_1)^2 \\ II(e_1,e_2) = 0 & \langle d(\psi+\eta)(e_1), d(\psi+\eta)(e_2)\rangle = 0 \\ II(e_2,e_2) = k_2 & \langle d(\psi+\eta)(e_2), d(\psi+\eta)(e_2)\rangle = (1-k_2)^2 \end{array}$$

Therefore, $\psi + \eta$ is conformal at p if and only if

$$\frac{(1-k_1)^2}{k_1} = \frac{(1-k_2)^2}{k_2}.$$

Or equivalently, $(k_1k_2 - 1)(k_1 - k_2) = 0$; which is satisfied because $k_1k_2 = 1$ in Σ.

Analogously, it can be proved that $\psi - \eta$ is conformal.

$\square$

Now, let us start with a simply connected surface Σ and a flat immersion $\psi : \Sigma \longrightarrow \mathbb{H}^3$ with unit normal η. Then, from Proposition 1, $G^+ : \Sigma \longrightarrow \mathbb{S}^2_\infty \equiv \mathbb{CP}^1$ is conformal and, so, there exist holomorphic functions A, B such that $G^+ = [\psi + \eta] = [(A, B)]$. Then,

$$\psi + \eta = R_1 \begin{pmatrix} A \\ B \end{pmatrix} (\overline{A}, \overline{B}) = R_1 \begin{pmatrix} A\overline{A} & A\overline{B} \\ \overline{A}B & B\overline{B} \end{pmatrix},$$

for some positive regular function R_1.

Analogously, since $G^- : \Sigma \longrightarrow \mathbb{S}^2_\infty \equiv \mathbb{CP}^1$ is conformal, there exist holomorphic functions C, D such that $G^- = [\psi - \eta] = [(C, D)]$. Then,

$$\psi - \eta = R_2 \begin{pmatrix} C \\ D \end{pmatrix} (\overline{C}, \overline{D}) = R_2 \begin{pmatrix} C\overline{C} & C\overline{D} \\ \overline{C}D & D\overline{D} \end{pmatrix},$$

for some positive regular function R_2.

By taking

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

we have

$$\psi+\eta=g\begin{pmatrix} R_1 & 0\\ 0 & 0\end{pmatrix}g^*,\qquad \psi-\eta=g\begin{pmatrix} 0 & 0\\ 0 & R_2\end{pmatrix}g^*. \tag{8}$$

So,

$$\psi=\frac{1}{2}\,g\begin{pmatrix} R_1 & 0\\ 0 & R_2\end{pmatrix}g^*,\qquad \eta=\frac{1}{2}\,g\begin{pmatrix} R_1 & 0\\ 0 & -R_2\end{pmatrix}g^*. \tag{9}$$

Since, $\langle\psi,\psi\rangle=-1$ we have

$$1=\frac{1}{4}\det\left(g\begin{pmatrix} R_1 & 0\\ 0 & R_2\end{pmatrix}g^*\right)=\frac{1}{4}\,R_1\,R_2\,|\det(g)|^2. \tag{10}$$

As $|\det(g)|\neq 0$ and $\det(g)$ is holomorphic, there exists a well defined holomorphic function h_1 in Σ such that $h_1=\sqrt{\det(g)}$. Thus, by substituting the previous A,B,C,D by $A/h_1,B/h_1,C/h_1,D/h_1$, respectively, one has $\det g=1$. Moreover, $R_1R_2=4$ from (10), that is, $R_2=4/R_1$.

With this

$$g^{-1}g_z=\begin{pmatrix} D & -C\\ -B & A\end{pmatrix}\begin{pmatrix} A' & C'\\ B' & D'\end{pmatrix}=:\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & -a_{11}\end{pmatrix}, \tag{11}$$

and so

$$(\psi+\eta)_z=g\left[\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & -a_{11}\end{pmatrix}\begin{pmatrix} R_1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix} (R_1)_z & 0\\ 0 & 0\end{pmatrix}\right]g^*. \tag{12}$$

By using $\langle(\psi+\eta)_z,\psi-\eta\rangle=0$ one has from (8) and (12)

$$a_{11}+\frac{(R_1)_z}{R_1}=0. \tag{13}$$

since $R_2=4/R_1$.

Thus, $h_2=\log(R_1)$ is a harmonic function. That is, there exists a non vanishing holomorphic function h_3 in Σ such that $R_1=e^{h_2}=2|h_3|^2$.

Therefore, replacing again A,B,C,D by $Ah_3,Bh_3,C/h_3,D/h_3$, one has from (9)

$$\psi=gg^*,\qquad \eta=g\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}g^*. \tag{14}$$

Moreover, from (13), the new holomorphic function a_{11} given by (11) vanishes identically. That is, we can write

$$g^{-1}g_z = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

With all of this, we have found a holomorphic map $g : \Sigma \longrightarrow \mathbb{SL}(2,\mathbb{C})$ given by

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{15}$$

such that the immersion ψ and its unit normal η can be recovered by (14). Moreover, the hyperbolic Gauss maps are given by

$$G^+ = \frac{A}{B} \in \mathbb{S}^2_\infty, \qquad G^- = \frac{C}{D} \in \mathbb{S}^2_\infty \tag{16}$$

and g satisfies

$$g^{-1}dg = \begin{pmatrix} 0 & \omega \\ \theta & 0 \end{pmatrix},$$

where ω, θ are holomorphic 1-forms in Σ.

Let us suppose there exists another holomorphic map $g_0 : \Sigma \longrightarrow \mathbb{SL}(2,\mathbb{C})$ such that

$$\psi = g_0 g_0^*, \qquad \eta = g_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g_0^*.$$

Then, from (14), $gg^* = g_0 g_0^*$ and so $g^{-1}g_0(g^{-1}g_0)^* = I_2$, where I_2 denotes the identity matrix. Thus,

$$g^{-1}g_0 = \begin{pmatrix} p & \overline{q} \\ -q & \overline{p} \end{pmatrix}, \tag{17}$$

for certain complex functions p, q such that $|p|^2 + |q|^2 = 1$.

Since g, g_0 are holomorphic then p, q must be constant. In addition, as

$$g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* = g_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g_0^*$$

one has that q must vanish identically. Hence, from (17),

$$g_0 = g \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix},$$

for a real constant φ.

In particular, g_0 also satisfies (16) and

$$g_0^{-1}dg_0 = \begin{pmatrix} 0 & e^{-2i\varphi}\omega \\ e^{2i\varphi}\theta & 0 \end{pmatrix}. \tag{18}$$

Thus, we obtain the following representation of flat immersions in terms of holomorphic data [12].

Theorem 4 (Conformal representation)**.**
(i) Let Σ be a simply connected surface and $\psi : \Sigma \longrightarrow \mathbb{H}^3$ a flat immersion with unit normal η. Consider Σ as a Riemann surface with the conformal structure induced by its second fundamental form. Then, there exists a holomorphic map $g : \Sigma \longrightarrow \mathbb{SL}(2,\mathbb{C})$ and two holomorphic 1-forms ω, θ with $|\omega| > |\theta|$ such that

$$\psi = gg^*, \qquad \eta = g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^*, \tag{19}$$

and

$$g^{-1}dg = \begin{pmatrix} 0 & \omega \\ \theta & 0 \end{pmatrix}. \tag{20}$$

Besides, its hyperbolic Gauss maps are given as in (15) and (16).

If $g_0 : \Sigma \longrightarrow \mathbb{SL}(2,\mathbb{C})$ is another holomorphic curve in the conditions above then

$$g_0 = g \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix},$$

for a real constant φ.

Moreover, its induced metric and its second fundamental form are given by

$$I = \omega\theta + (|\omega|^2 + |\theta|^2) + \overline{\omega}\overline{\theta}, \qquad II = |\omega|^2 - |\theta|^2. \tag{21}$$

(ii) Conversely, let Σ be a Riemann surface and $g : \Sigma \longrightarrow \mathbb{SL}(2,\mathbb{C})$ a holomorphic map such that

$$g^{-1}dg = \begin{pmatrix} 0 & \omega \\ \theta & 0 \end{pmatrix},$$

for certain 1-forms ω, θ with $|\omega| > |\theta|$. Then, $\psi = gg^*$ is a flat immersion in $\mathbb{H}^3$ with unit normal vector field η given by (19). Moreover, its induced metric and its second fundamental form are given by (21).

Proof. *In order to prove (i), we need to compute the induced metric and second fundamental form of the immersion.*

Thus, if we take a local conformal parameter z and we write $\omega = \omega_1 dz, \theta = \theta_1 dz$, then from (19) and (20)

$$\psi_z = g \begin{pmatrix} 0 & \omega_1 \\ \theta_1 & 0 \end{pmatrix} g^*, \qquad \eta_z = g \begin{pmatrix} 0 & \omega_1 \\ \theta_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^*.$$

Hence, one obtains

$$\langle \psi_z, \psi_z \rangle = \omega_1 \theta_1, \qquad \langle \psi_z, \psi_{\overline{z}} \rangle = \frac{1}{2}(|\omega_1|^2 + |\theta_1|^2),$$
$$\langle \psi_z, -\eta_z \rangle = 0, \qquad \langle \psi_z, -\eta_{\overline{z}} \rangle = \frac{1}{2}(|\omega_1|^2 - |\theta_1|^2).$$

Thus (21) is proved, and since II is positive definite $|\omega| > |\theta|$.

The converse is a straightforward computation.

□

The pair (ω, θ) given by the theorem above is usually called the *Weierstrass data* associated with the flat immersion. As it was proved in (18), they are unique up to multiplication by a unit complex number p in the following way $(p\,\omega, \overline{p}\,\theta)$.

In particular, $\omega\theta$, $|\omega|$ and $|\theta|$ are uniquely defined.

Remark 1. *Since $|\omega| > 0$ one obtains that G^- is a local diffeomorphism because $(G^-)'(z) \neq 0$ at every point. This was first observed by L. Bianchi [5]. In fact, he proved that the map $G^- \to G^+$ is holomorphic and gave a representation of flat immersions using Euclidean congruence of spheres.*

Using the conformal representation above, we give a new proof of the classification of the complete flat surfaces in $\mathbb{H}^3$, obtained by Volkov and Vladimirova [31].

Theorem 5. *Let Σ be a surface and $\psi : \Sigma \longrightarrow \mathbb{H}^3$ a complete flat immersion. Then $\psi(\Sigma)$ is either a horosphere or the set of points at a fixed distance from a geodesic.*

Proof. *Passing to the universal cover of Σ, if necessary, we can assume that Σ is simply connected. Hence, with the conformal structure determined by the second* *fundamental form,* Σ *must be biholomorphic either to the unit disk* $\mathbb{D}$ *or to the complex plane* $\mathbb{C}$*.*

If (ω, θ) *is the Weierstrass data associated with the immersion, then from (21) and since* $|\omega| > |\theta|$

$$I \leq 4|\omega|^2.$$

Thus, $4|\omega|^2$ *is a conformal complete flat metric in* Σ*. Thus,* Σ *with this metric must be isometric to the Euclidean plane. In particular,* Σ *is conformal to* $\mathbb{C}$*.*

On the other hand, the modulus of the holomorphic function θ/ω *is less than one in* $\Sigma \equiv \mathbb{C}$*. Hence,* θ/ω *is a complex constant* c_0*, with* $|c_0| < 1$*.*

By using $|\omega| \neq 0$*, we can take a local conformal parameter* z *such that* $\omega = dz$*. Thus, from (20), we must find a holomorphic immersion*

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathbb{SL}(2, \mathbb{C})$$

satisfying

$$g^{-1} g_z = \begin{pmatrix} 0 & 1 \\ c_0 & 0 \end{pmatrix}.$$

Or equivalently,

$$\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} = \begin{pmatrix} c_0\, C & A \\ c_0\, D & B \end{pmatrix}.$$

Thereby, A, B, C, D *are solutions of the differential equation* $X'' = c_0\, X$*.*

If $c_0 = 0$ *the solutions of the previous ODE are* $X(z) = a + bz$ *for suitable constants* $a, b \in \mathbb{C}$*. In such a case*

$$g = m_0 \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where m_0 *is a constant matrix in* $\mathbb{SL}(2, \mathbb{C})$*.*

Therefore, the flat immersion is, up to the isometry $m_0 \psi m_0^*$*, given by*

$$\psi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{z} & 1 \end{pmatrix} = \begin{pmatrix} 1 + |z|^2 & z \\ \overline{z} & 1 \end{pmatrix}.$$

Since the last coordinate of the matrix is constant one has from (7), $\psi_0 + \psi_3 = 1$*. That is,* $\psi(\Sigma)$ *lies on a horosphere, and* $\psi(\Sigma)$ *is the whole horosphere by completeness. (See Figure 2.)*

Figure 2: A horosphere in the half-space model of $\mathbb{H}^3$.

If $c_0 \neq 0$ *the solutions of the previous ODE are* $X(z) = ae^{\sqrt{c_0}z} + be^{-\sqrt{c_0}z}$ *for suitable constants* $a, b \in \mathbb{C}$. *In such a case*

$$g = m_0 \begin{pmatrix} \sqrt[4]{\frac{c_0}{4}}\, e^{-\sqrt{c_0}z} & \frac{-1}{\sqrt[4]{4c_0}}\, e^{-\sqrt{c_0}z} \\ \sqrt[4]{\frac{c_0}{4}}\, e^{\sqrt{c_0}z} & \frac{1}{\sqrt[4]{4c_0}}\, e^{\sqrt{c_0}z} \end{pmatrix},$$

where m_0 *is a constant matrix in* $\mathbb{SL}(2, \mathbb{C})$.

Then, up to an isometry, the flat immersion is given by

$$\psi(z) = \frac{1}{2|c_1|} \begin{pmatrix} (|c_1|^2 + 1)e^{-c_1 z - \overline{c_1 z}} & (|c_1|^2 - 1)e^{-c_1 z + \overline{c_1 z}} \\ (|c_1|^2 - 1)e^{c_1 z - \overline{c_1 z}} & (|c_1|^2 + 1)e^{c_1 z + \overline{c_1 z}} \end{pmatrix}$$

where $c_1 = \sqrt{c_0}$.

The surface, given by the points whose distance to the geodesic

$$\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}$$

is a constant $R > 0$, *can be parametrized by*

$$F(s,t) = \begin{pmatrix} \cosh R\, e^s & \sinh R\, e^{it} \\ \sinh R\, e^{-it} & \cosh R\, e^{-s} \end{pmatrix}.$$

Thus, $\psi(z)$ *lies on the surface* $F(s,t)$ *for* $R = \operatorname{arcsinh}\left(\frac{1-|c_1|^2}{2|c_1|}\right)$ *as we wanted to show. (See Figure 3.)*

Figure 3: A surface equidistant from a geodesic in the half-space model of $\mathbb{H}^3$.

□

The conformal representation is a useful tool and can be used for investigating the geometric behavior of flat immersions. For instance, it was used for proving that a properly embedded annular end must be asymptotic to a revolution surface (see [12] and [6]).

Since the complete flat immersions in $\mathbb{H}^3$ are well known, it is interesting to pose the study of flat surfaces with singularities. These "surfaces" are being intensively studied by different authors (see [6], [13], [24], [25], [26], [28]). To this respect, and bearing in mind that a flat surface is locally convex in $\mathbb{H}^3$, it seems of special interest to research the complete surfaces Σ in $\mathbb{H}^3$ such that they are the boundary of a convex body with a small number of singularities. This problem is still open (see [13], [6]).

5 Flat surfaces in the Euclidean space.

We start with a flat immersion $\psi : \Sigma \longrightarrow \mathbb{R}^3$. From the Gauss equation the product of the principal curvatures of the immersion vanishes identically. We consider two subsets of Σ given by the interior of the umbilical points, Σ_U, and the set of non umbilical points Σ_N.

It is clear that the closure of $\Sigma_U \cup \Sigma_N$ is the whole surface. Thus, we study both subsets independently in order to understand $\psi(\Sigma)$.

The set Σ_U is made of umbilical points with vanishing principal curvatures. Hence, every connected component of $\psi(\Sigma_U)$ lies on a plane.

Thus, we will focus our attention on the study of Σ_N.

Lemma 1. *Let $\Upsilon \subseteq \Sigma_N$ be a line of curvature corresponding to the vanishing principal curvature, then $\psi(\Upsilon)$ is a straight segment in $\mathbb{R}^3$ with no umbilical point.*

Moreover, if ψ is a complete immersion and Υ is a line of curvature associated to the principal curvature 0 with a non umbilical point $p_0 \in \Upsilon$, then $\psi(\Upsilon)$ is a straight line of $\mathbb{R}^3$ with no umbilical point.

Proof. *Since $\Upsilon \subseteq \Sigma_N$ there exist local coordinates (u, v) such that*

$$I = Edu^2 + Gdv^2, \qquad II = k_2 G dv^2.$$

Here, the coordinate curves are the lines of curvature of the immersion.

The Codazzi equation gives $E_v = 0$ and $(k_2^2 G)_u = 0$. Thus, up to a change of parameters as we did in Section 3, we can assume

$$I = du^2 + \frac{1}{k_2^2} dv^2, \qquad II = \frac{1}{k_2} dv^2.$$

The vector ψ_{uu} has no normal part since $k_1 \equiv 0$. Thus, $\psi_{uu} = \Gamma_{11}^1 \psi_u + \Gamma_{11}^2 \psi_v$, and from the expression of I one has $\Gamma_{11}^1 = 0 = \Gamma_{11}^2$.

Therefore the curves $\alpha(u) = \psi(u, v_0)$ which are parametrized by the length arc satisfy $\alpha''(u) = 0$. That is, $\alpha(u)$ is a segment of line in $\mathbb{R}^3$.

The Gauss equation yields

$$\left(\frac{1}{k_2}\right)_{uu} = 0. \tag{22}$$

That is, for the segment $\alpha(u)$ one has that the second principal curvature k_2 satisfies the equation (22) for the arc length parameter of the curve u. Thus, if there exists a first umbilical point when $u \to u_0$ then k_2 must tend to 0, which contradicts (22).

Therefore, $\alpha(u)$ has no umbilical point. In particular, if ψ is a complete immersion $\alpha(u)$ must be a straight line.

□

It is important to observe the following fact from the lemma above. If $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is a complete flat immersion and p_1, p_2 are non umbilical points with respective straight lines r_1, r_2 associated with the vanishing principal curvature then $r_1 = r_2$ or they do not intersect. Otherwise, there would exist a unique point $p \in r_1 \cap r_2$ satisfying that two different directions (associated with r_1 and r_2) are principal directions with vanishing principal curvatures. Then p would be umbilical, which contradicts that r_1 has no umbilical point.

With all of this, we can classify the complete flat surfaces in $\mathbb{R}^3$ [16].

Theorem 6 (Hartman-Nirenberg)**.** *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete flat immersion. Then $\psi(\Sigma)$ is a right cylinder on a planar curve which is defined for all values of its arc length parameter.*

Proof. *Passing to the universal cover of Σ, if necessary, we can assume that Σ is simply connected. So, Σ is a simply connected flat surface and thereby Σ is isometric to the Euclidean plane $\mathbb{R}^2$. Hence, we shall work with $\mathbb{R}^2$ instead of Σ.*

If ψ is totally umbilical then the image is a plane. Thus, we shall assume there is a non umbilical point.

Let $p_0 \in \mathbb{R}^2$ be a non umbilical point. By using Lemma 1 there exists a line of curvature $\Upsilon_0 \subseteq \mathbb{R}^2$ such that $\psi(\Upsilon_0)$ is a straight line. Since $\psi(\Upsilon_0)$ is a geodesic of $\mathbb{R}^3$ then Υ_0 is a geodesic of the surface and Υ_0 is a straight line in $\mathbb{R}^2$.

Up to an isometry in $\mathbb{R}^2$ we can assume that Υ_0 is a vertical line. Then, let us show that the image of every vertical line $r \subseteq \mathbb{R}^2$ is a straight line in $\mathbb{R}^3$.

Let $r \subseteq \mathbb{R}^2$ be a vertical line different from Υ_0 containing a non umbilical point p_1. Let $\Upsilon_1 \subseteq \mathbb{R}^2$ be its associated line of curvature such that $\psi(\Upsilon_1)$ is a straight line. Reasoning as above Υ_1 must be a straight line in $\mathbb{R}^2$. In addition, Υ_1 must be vertical because otherwise would intersect Υ_0 which contradicts the previous remark. Hence,

$r = \Upsilon_1$ and its image is a straight line.

Now, let $r \subseteq \mathbb{R}^2$ be a vertical line contained in the interior of the set of umbilical points $\Sigma_U \subseteq \mathbb{R}^2$. Since r is a geodesic in $\mathbb{R}^2$ then $\psi(r)$ is a geodesic in $\psi(\Sigma_U)$. And since $\psi(\Sigma_U)$ is a piece of a plane then $\psi(r)$ is a straight line.

If $r \subseteq \mathbb{R}^2$ is a vertical line which is not in the previous conditions then r is the limit of vertical lines r_n in the conditions above. And since $\psi(r_n)$ is a straight line for any n then $\psi(r)$ must also be a straight line.

With all of this, if we consider usual coordinates (x, y) in $\mathbb{R}^2$ the immersion ψ is parametrized as

$$\psi(x, y) = \alpha(x) + y\beta(x).$$

Here, $\psi(x_0, y)$ is a straight line with unit vector $\beta(x_0)$.

Thus,

$$\begin{aligned} 1 &= \langle \psi_x, \psi_x \rangle = \langle \alpha'(x), \alpha'(x) \rangle\rangle + 2y\langle \alpha'(x), \beta'(x)\rangle + y^2\langle \beta'(x), \beta'(x)\rangle, \\ 0 &= \langle \psi_x, \psi_y \rangle = \langle \alpha'(x), \beta(x)\rangle + y\langle \beta'(x), \beta(x)\rangle, \\ 1 &= \langle \psi_y, \psi_y \rangle = \langle \beta(x), \beta(x)\rangle. \end{aligned}$$

In particular $\langle \beta'(x), \beta'(x)\rangle = 0$ and $\beta(x)$ is a constant unit vector β_0. That is, the image of vertical lines in $\mathbb{R}^2$ are parallel straight lines in $\mathbb{R}^3$.

Moreover $\langle \alpha'(x), \beta_0\rangle = 0$ and by integrating $\langle \alpha(x) - \alpha(x_0), \beta_0\rangle = 0$. This means $\alpha(x)$ is contained in a plane perpendicular to the vector β_0.

$\square$

6 Flat surfaces in the 3-sphere.

This section is devoted to expose the basics of the theory of flat surfaces in the unit 3-sphere $\mathbb{S}^3$. We shall begin by describing the fundamental equations of flat surfaces in terms of asymptotic parameters. Then we shall describe $\mathbb{S}^3$ as well as the usual Hopf fibration in terms of quaternions. By means of this model for $\mathbb{S}^3$, we shall explain the classical Bianchi method via which flat surfaces in $\mathbb{S}^3$ are constructed by multiplying two intersecting asymptotic curves. We shall also describe a refinement of this method due to Kitagawa [18], which has been the fundamental tool for studying flat surfaces in $\mathbb{S}^3$ from a global viewpoint. Afterwards, we shall expose the most significative global results regarding complete flat surfaces and flat tori in $\mathbb{S}^3$. Finally, we shall discuss some of the most important open problems of the theory. The basic references for most of what follows are [14, 18, 30, 33].

6.1 Asymptotic parameters

Generally, the fundamental equations of a flat surface in $\mathbb{S}^3$ are better understood by means of parameters whose coordinate curves are asymptotic curves on the surface. First, let us observe that, as the surface is flat, its intrinsic Gauss curvature vanishes identically. Consequently, by the Gauss equation, the extrinsic curvature of the surface is $K = -1$. In this situation, as K is a negative constant, there exist *Tschebyscheff coordinates* around every point. This simply means that we can choose local coordinates (u, v) such that: (a) the u-curves and the v-curves are asymptotic curves of the surface, and (b) these curves are parametrized by arclength.

Theorem 7. *Let* (I, II) *be a Codazzi pair in a surface* Σ *with negative constant extrinsic curvature* K*. Then there exist local coordinates* (u, v) *and a smooth function* $\omega(u, v) \in (0, \pi)$ *such that*

$$\begin{aligned} I &= du^2 + 2\cos\omega\, dudv + dv^2, \\ II &= 2\sqrt{-K}\sin\omega\, dudv. \end{aligned} \tag{23}$$

Moreover,

1. *If* Σ *is simply connected there exist global functions* $u, v : \Sigma \longrightarrow \mathbb{R}$ *such that* (u, v) *are local coordinates in the conditions above in a neighborhood of every point.*

2. *If* Σ *is simply connected and* I *is complete then the previous map* $(u, v) : \Sigma \longrightarrow \mathbb{R}^2$ *is a global diffeomorphism.*

Proof. *Let us consider local coordinates (global coordinates if* Σ *is simply connected) such that*

$$\begin{aligned} I &= E\, du^2 + 2\, F\, dudv + G\, dv^2, \\ II &= 2\, f\, dudv, \qquad f > 0. \end{aligned}$$

Let us denote by Γ_{ij}^k to the Christoffel symbols of the Levi-Civita connection of the induced metric with respect to the parameters (u, v). That is,

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \Gamma_{11}^1 \frac{\partial}{\partial u} + \Gamma_{11}^2 \frac{\partial}{\partial v}, \quad \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = \Gamma_{12}^1 \frac{\partial}{\partial u} + \Gamma_{12}^2 \frac{\partial}{\partial v},$$

$$\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = \Gamma_{22}^1 \frac{\partial}{\partial u} + \Gamma_{22}^2 \frac{\partial}{\partial v}.$$

Denote $D = EG - F^2$, then a straightforward computation gives

$$\frac{D_u}{2D} = \Gamma_{11}^1 + \Gamma_{12}^2, \qquad \frac{D_v}{2D} = \Gamma_{22}^2 + \Gamma_{12}^1.$$

Besides, from the Codazzi equation for (I, II) one has

$$\frac{f_u}{f} = \Gamma_{11}^1 - \Gamma_{12}^2, \qquad \frac{f_v}{f} = \Gamma_{22}^2 - \Gamma_{12}^1.$$

Now, we use that $K = -f^2/D$ is a negative constant. Thus, we have

$$0 = -\frac{f_u}{f} + \frac{D_u}{2D} = 2\Gamma_{12}^2, \qquad 0 = -\frac{f_v}{f} + \frac{D_v}{2D} = 2\Gamma_{12}^1.$$

Hence, $\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = 0$ and so $E_v = 0 = G_u$.

By replacing u, v by the new local coordinates

$$\widetilde{u} = \int \sqrt{E(u)} du, \qquad \widetilde{v} = \int \sqrt{G(v)} dv$$

one has

$$I = du^2 + 2\, F\, dudv + dv^2, \qquad II = 2\, f\, dudv.$$

Since the extrinsic curvature is a negative constant $K = -f^2/(1 - F^2)$, we have $F^2 + (f/\sqrt{-K})^2 = 1$. And so there exists a smooth function ω such that

$$I = du^2 + 2\, \cos\omega\, dudv + dv^2, \qquad II = 2\, \sin\omega\, dudv,$$

with $0 < \omega < \pi$, since $1 - F^2 > 0$.

When Σ is simply connected then we have proved that $(u, v) : \Sigma \longrightarrow \mathbb{R}^2$ is a local diffeomorphism. Thus, if we consider the new Riemannian metric $III = du^2 + 2\cos\omega\, dudv + dv^2$, then $I + III = 2(du^2 + dv^2)$. Hence $du^2 + dv^2$ is a *complete flat metric when I is complete. So, $(u, v) : (\Sigma, du^2 + dv^2) \longrightarrow \mathbb{R}^2$ is a local isometry and must be a global diffeomorphism.*

$\square$

As a consequence we have

Theorem 8. *Let Σ be a surface and $\psi : \Sigma \longrightarrow \mathbb{M}^3(c)$ an immersion with negative constant extrinsic curvature K in a space form. Then the asymptotic curves of ψ have constant torsion τ, with $\tau^2 = -K$ at points where the curvature of the curve does not vanish. Moreover, two asymptotic curves through a point have torsions of opposite signs if they have non vanishing curvature at that point.*

Proof. *Using Theorem 7 there exist local coordinates (u, v) such that its induced metric and second fundamental form can be written as in (23). Thus, with this parametrization, the coordinate curves are the asymptotic curves of the immersion.*

Let us take the asymptotic curve $\alpha(u) = \psi(u, v_0)$. Then α is parametrized by the arc length and $\langle \nabla_{\alpha'(u)} \alpha'(u), N(u, v_0)\rangle = 0$, where N is the normal vector of the immersion and ∇ is the Levi-Civita connection in $\mathbb{M}^3(c)$.

Hence, if we assume the curvature of $\alpha(u)$ does not vanishes at $u = u_0$ then $N(u_0, v_0)$ is the binormal vector at $u = u_0$. And the normal vector of $\alpha(u)$ is $N(u, v_0) \wedge \alpha'(u)$ at the points where the curvature of $\alpha(u)$ does not vanish.

If we write $N(u, v_0) \wedge \alpha'(u) = a(u)\frac{\partial}{\partial u} + b(u)\frac{\partial}{\partial v}$, for suitable functions a, b, then the torsion $\tau(u)$ of $\alpha(u)$ can be computed as

$$\begin{aligned}
\tau(u) &= -\langle \nabla_{\alpha'(u)} N(u, v_0), N(u, v_0) \wedge \alpha'(u)\rangle \\
&= -\langle \nabla_{\alpha'(u)} N(u, v_0), a(u)\frac{\partial}{\partial u} + b(u)\frac{\partial}{\partial v}\rangle \\
&= b(u)\langle N(u, v_0), \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v}\rangle = b(u)\sqrt{-K}\sin\omega(u, v_0).
\end{aligned}$$

Here, $b(u)$ can be calculated by considering the inner product with $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ in the *equality* $N(u,v_0)\wedge\alpha'(u)=a(u)\frac{\partial}{\partial u}+b(u)\frac{\partial}{\partial v}$

$$\begin{aligned} 0=\langle N(u,v_0)\wedge\alpha'(u),\frac{\partial}{\partial u}\rangle &= a(u)+b(u)\cos\omega\\ \sin\omega=\langle N(u,v_0)\wedge\alpha'(u),\frac{\partial}{\partial v}\rangle &= a(u)\cos\omega+b(u). \end{aligned}$$

That is, $b(u)=1/\sin\omega$ *and* $\tau(u)=\sqrt{-K}$ *as we wanted to show. Analogously, it can be proved that that the torsion of* $\beta(v)=\alpha(u_0,v)$ *is* $-\sqrt{-K}$*, where* $N(u_0,v)$ *is the binormal vector of the curves.*

□

For a flat immersion in $\mathbb{S}^3$ the function ω, called the *angle function*, has two basic properties: Firstly, as I is regular then $0<\omega<\pi$, that is ω is bounded. Secondly, the Gauss equation of the surface translates into $\omega_{uv}=0$. In other words, the angle ω verifies the homogeneous wave equation, and thus it can be locally decomposed as $\omega(u,v)=\omega_1(u)+\omega_2(v)$, where ω_1 and ω_2 are smooth real functions. Let us point out here that as the flat surfaces in $\mathbb{S}^3$ are described by the homogeneous wave equation, which is hyperbolic, it turns out that flat surfaces in $\mathbb{S}^3$ will not be real analytic in general.

As these Tschebyscheff coordinates (T-coordinates from now on) are essential for the local study of flat surfaces, it is important to understand when are they *globally available* on a surface, in order to develop a global theory. We have proved that *any simply-connected complete flat surface in* $\mathbb{S}^3$ *has globally defined* T*-coordinates*. If we drop the trivial topology assumption, this is no longer true. It turns out that simply connected flat surfaces in $\mathbb{S}^3$ do not possess in general globally defined T-coordinates, but instead, they admit a globally defined *Tschebyscheff immersion*. In other words, we can take two maps u,v from the surface into $\mathbb{R}$ that verify all the properties of T-coordinates, except for the fact that the map (u,v) into $\mathbb{R}^2$ may not be injective. The existence of this T-immersion is enough in many cases to deal globally with non-complete flat surfaces in $\mathbb{S}^3$. It was proved in [14] that T*-coordinates are globally available on any (not necessarily complete) simply connected real-analytic flat surface in* $\mathbb{S}^3$.

6.2 The quaternionic model for $\mathbb{S}^3$

The best way to describe explicitly flat surfaces in $\mathbb{S}^3$ is to regard the 3-sphere as the set of unit quaternions. Let us explain this model for $\mathbb{S}^3$ briefly.

We begin by identifying $\mathbb{R}^4$ with the quaternions in the standard way, that is, (x_1, x_2, x_3, x_4) is viewed as the quaternion $x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$. Recall that in the product of quaternions

$$\begin{array}{lll} \mathbf{i}\,\mathbf{j} = \mathbf{k} & \mathbf{j}\,\mathbf{k} = \mathbf{i} & \mathbf{k}\,\mathbf{i} = \mathbf{j} \\ \mathbf{j}\,\mathbf{i} = -\mathbf{k} & \mathbf{k}\,\mathbf{j} = -\mathbf{i} & \mathbf{i}\,\mathbf{k} = -\mathbf{j} \\ \mathbf{i}\,\mathbf{i} = -1 & \mathbf{j}\,\mathbf{j} = -1 & \mathbf{k}\,\mathbf{k} = -1. \end{array}$$

In that way the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ is regarded as the space of unit quaternions, i.e. quaternions x with unit norm, $\|x\| = 1$. We also point out that $\mathbb{S}^2 \equiv \mathbb{S}^3 \cap \{x_1 = 0\}$ can be seen as the space of purely imaginary unit quaternions.

The advantage of this model is that, by using the usual product of quaternions, the space $\mathbb{S}^3$ receives in a natural way a Lie group structure. Indeed, if $x, y \in \mathbb{S}^3$, then $x\,y \in \mathbb{S}^3$. Moreover, the left and right translations $x \mapsto x\,a$ and $x \mapsto a\,x$ turn out to be isometries for the standard Riemannian metric of $\mathbb{S}^3$. In other words, the metric of $\mathbb{S}^3$ is bi-invariant with respect to this Lie group structure. Thus we have for any $x, y, a \in \mathbb{S}^3$ that $\langle x, y\rangle = \langle a\,x, a\,y\rangle = \langle x\,a, y\,a\rangle$.

Apart from multiplication, there is another operation with quaternions that will be useful to us: the *conjugation* $x := x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \mapsto \bar{x} = x_1 - \mathbf{i}x_2 - \mathbf{j}x_3 - \mathbf{k}x_4$. It turns out that conjugation is geometrically an orientation reversing isometry in $\mathbb{S}^3$. Moreover we have $\bar{x} = x^{-1}$ whenever $x \in \mathbb{S}^3$, and in addition $\bar{x} = -x$ if $x \in \mathbb{S}^2 \subset \mathbb{S}^3$. Let us remark that $\langle x, x\rangle = x\overline{x}$ and $\overline{x\,y} = \bar{y}\,\bar{x}$.

We end up the description of the geometry of $\mathbb{S}^3$ via quaternions with the introduction of the usual *Hopf fibration*. Let us define $\mathrm{Ad}(x)y := x\,y\,\bar{x}$, where $x, y \in \mathbb{S}^3$. Then the Hopf fibration $h : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is given by $h(x) = \mathrm{Ad}(x)\mathbf{i} = x\,\mathbf{i}\,\bar{x}$. It follows immediately that the fibers $h^{-1}(p)$ are great circles of $\mathbb{S}^3$. In fact, if

$p = \mathbf{i}p_2 + \mathbf{j}p_3 + \mathbf{k}p_4$ is a point in $\mathbb{S}^2$ then $h^{-1}(p)$ is the great circle given by

$$h^{-1}(p) = \left\{ x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \in \mathbb{S}^3 : \begin{array}{c} -p_4x_1 + p_3x_2 - (1+p_3)x_3 = 0, \\ p_3x_1 + p_4x_2 - (1+p_2)x_4 = 0 \end{array} \right\},$$

if $p_2 \neq -1$

$$h^{-1}(-\mathbf{i}) = \{\mathbf{j}x_3 + \mathbf{k}x_4 \in \mathbb{S}^3 : \; x_3, x_4 \in \mathbb{R}\}.$$

The Hopf fibration will be crucial for describing flat surfaces in $\mathbb{S}^3$. Let us also remark that one can define *skew Hopf fibrations* by means of $h_\xi(x) := \mathrm{Ad}(x)\xi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$, where $\xi \in \mathbb{S}^2$ is fixed but arbitrary. The fibers will still be great circles of $\mathbb{S}^3$.

6.3 First examples of flat surfaces in $\mathbb{S}^3$

The simplest way to obtain flat surfaces in $\mathbb{S}^3$ is by means of the Hopf fibration. This is due to the following remark by H.B. Lawson (see [30, 27]):

Proposition 2. *If c is a regular curve in $\mathbb{S}^2$, then $h^{-1}(c)$ is a flat surface in $\mathbb{S}^3$.*

Proof. *First, we observe that from the expression above of $h^{-1}(p)$ for $p \in \mathbb{S}^2$, it is a direct computation that $h^{-1}(c) \subseteq \mathbb{S}^3$ is the product of two circles contained in perpendicular planes when c is a circle. Hence, up to an isometry, $h^{-1}(c)$ is $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $0 < r < 1$. That is, $h^{-1}(c)$ is a flat surface. These tori are usually called Clifford tori.*

On the other hand, let c be a regular curve in $\mathbb{S}^2$. For any $q \in h^{-1}(c)$ we can consider a circle $\widetilde{c} \subseteq \mathbb{S}^2$ with a contact of order two at $h(q)$ with the curve c. Then, $h^{-1}(c)$ and $h^{-1}(\widetilde{c})$ have a contact of order two at q. Thus, since the Clifford torus $h^{-1}(\widetilde{c})$ is flat and the Gauss curvature only depends on derivatives up to order two from the Gauss equation, we have that q is a flat point in $h^{-1}(c)$.

$\square$

As the fibers of the Hopf fibration are geodesics of $\mathbb{S}^3$, it turns out that $h^{-1}(c)$ has, in general, the topology of a cylinder. This is the reason why these surfaces $h^{-1}(c)$ are called *Hopf cylinders*.

Moreover, if the chosen regular curve c in $\mathbb{S}^2$ is closed, the resulting Hopf cylinder $h^{-1}(c)$ is compact and has the topology of a torus. Thus, it is named a *Hopf torus*. Moreover, if c is embedded, the resulting Hopf cylinder (or torus) is embedded. This provides a large family of flat tori and complete flat cylinders in $\mathbb{S}^3$, some of which are actually embedded. Moreover, they can be explicitly calculated once we know the curve c in $\mathbb{S}^2$.

6.4 The Bianchi-Spivak construction of flat surfaces in $\mathbb{S}^3$

Let us start with a flat immersion $\psi : \Sigma \longrightarrow \mathbb{S}^3$ and unit normal N. From Theorem 7, there exist coordinates (u_1, u_2) such that

$$\begin{aligned} I &= du_1^2 + 2\cos(\omega_1(u_1) + \omega_2(u_2))du_1 du_2 + du_2^2, \\ II &= 2\sin(\omega_1(u_1) + \omega_2(u_2))du_1 du_2, \end{aligned} \tag{24}$$

where we have used that the angle function $\omega(u_1, u_2) = \omega_1(u_1) + \omega_2(u_2)$ from the Gauss equation $\omega_{u_1 u_2} = 0$. We shall allow (u_1, u_2) vary in a rectangle containing the origin.

Let us denote $e_i = \frac{\partial}{\partial u_i}$ and $n_i = N \wedge e_i$ then

$$\begin{cases} \nabla_{e_i} e_i = k_i\, n_i \\ \nabla_{e_i} n_i = -k_i\, e_i + \tau_i\, N \\ \nabla_{e_i} N = -\tau_i\, n_i \end{cases}$$

where we proved in Theorem 8 that $\tau_1 = 1$ and $\tau_2 = -1$.

It is important to observe at this point that τ_i is the torsion of the corresponding asymptotic curve when the curvature of the curve does not vanish, but in any case the functions k_i and τ_i are well defined though the curvature of the curve vanishes at some points.

Now, we compute $k_i(u_i)$. Thus, if we write $\nabla_{e_1} e_1 = a\, e_1 + b\, e_2$ for suitable functions a, b then

$$k_1 = \langle \nabla_{e_1} e_1, n_1 \rangle = b \langle e_2, N \wedge e_1 \rangle = b \sin \omega.$$

And b can be calculated by considering the inner product with e_i in the equality $\nabla_{e_1} e_1 = a\, e_1 + b\, e_2$

$$a + b\cos\omega = \langle \nabla_{e_1} e_1, e_1 \rangle = 0,$$

$$a\cos\omega + b = \langle \nabla_{e_1} e_1, e_2 \rangle = e_1 \langle e_1, e_2 \rangle = -\omega_1' \sin\omega.$$

Hence $k_1(u_1) = b\sin\omega = -\omega_1'(u_1)$. Analogously, $k_2(u_2) = \omega_2'(u_2)$. Therefore, the asymptotic curves $\alpha(u_1) = \psi(u_1, u_2)$ have curvature and torsion which do not depend on u_2, and the same is true for the asymptotic curves $\beta(u_2) = \psi(u_1, u_2)$. With all of this we have proved:

Proposition 3. *All the asymptotic curves $u_1 \to \psi(u_1, u_2)$ are congruent each other. Analogously, the asymptotic curves $u_2 \to \psi(u_1, u_2)$ are also congruent each other.*

And as an easy consequence from the above results we have a representation result mainly due to Bianchi [4] and Spivak [30].

Theorem 9. *The flat immersion $\psi(u_1, u_2)$ can be recovered in terms of the asymptotic curves $\psi(u_1, 0)$ and $\psi(0, u_2)$ as*

$$\psi(u_1, u_2) = \psi(u_1, 0) \cdot \psi(0, 0)^{-1} \cdot \psi(0, u_2). \tag{25}$$

Proof. *Let us consider $\varphi(u_1, u_2) = \psi(u_1, 0) \cdot \psi(0, 0)^{-1} \cdot \psi(0, u_2)$.*

A direct computation gives that the induced metric and second fundamental form of $\varphi(u_1, u_2)$ agree with (24). In addition, the initial conditions for φ and ψ are the same at the origin, hence from Bonnet theorem $\varphi(u_1, u_2) = \psi(u_1, u_2)$ agree as we wanted to show.

□

In order to obtain a converse for this theorem we need to observe some facts. The asymptotic curves of a flat surface have torsion ± 1 at the points with curvature of the curves different from 0. Thus, one can think that a flat surface can be described as the product of two curves: one with torsion 1 and the other with torsion -1 using the description of Theorem 9. This is essentially true, but the asymptotic curves could have points with curvature 0.

In order to simplify computations, we can assume $\psi(0,0) = 1$ up to an isometry in $\mathbb{S}^3$. Let us take $\alpha(u_1) = \psi(u_1, 0)$ and $\beta(u_2) = \psi(0, u_2)$ then from (25) it is easy to see that the normal of the immersion is given by

$$N(u_1, u_2) = \alpha(u_1) \cdot \zeta_0 \cdot \beta(u_2),$$

where $\zeta_0 = N(0,0) \in \mathbb{S}^2$ because $0 = \langle \psi(0,0), N(0,0) \rangle = \langle 1, \zeta_0 \rangle$.

Moreover,

$$\begin{aligned} 0 &= \langle \psi_{u_1}, N \rangle = \langle \alpha'\beta, \alpha\zeta_0\beta \rangle = \langle \alpha', \alpha\zeta_0 \rangle, \\ 0 &= \langle \psi_{u_2}, N \rangle = \langle \alpha\beta', \alpha\zeta_0\beta \rangle = \langle \beta', \zeta_0\beta \rangle. \end{aligned} \tag{26}$$

Each equality in (26) only depends on one asymptotic curve and, in fact, it was observed by Kitagawa [18] that these conditions characterize to the curves in $\mathbb{S}^3$ with torsion ± 1 at the points with non vanishing curvature. More concretely,

Proposition 4. *Let $\zeta_0 \in \mathbb{S}^2 \subseteq \mathbb{S}^3$ and let $\gamma(t)$ be a curve in $\mathbb{S}^3$ parametrized by the arc length such that*

1. *$\langle \gamma', \gamma\zeta_0 \rangle = 0$ then $\gamma(t)$ has torsion 1 at the points with curvature different from 0.*
2. *$\langle \gamma', \zeta_0\gamma \rangle = 0$ then $\gamma(t)$ has torsion -1 at the points with curvature different from 0.*

Proof. *Let us assume $\langle \gamma', \gamma\zeta_0 \rangle = 0$. By differentiating we have:*

$$0 = \langle \nabla_{\gamma'}\gamma', \gamma\zeta_0 \rangle + \langle \gamma', \gamma'\zeta_0 \rangle = \langle \nabla_{\gamma'}\gamma', \gamma\zeta_0 \rangle,$$

where we have used that $\langle \gamma', \gamma'\zeta_0 \rangle = 0$ because $\zeta_0 \in \mathbb{S}^2$. Due to the same reason $\langle \gamma, \gamma\zeta_0 \rangle = 0$.

Hence, $\gamma\zeta_0$ is orthogonal to γ, γ' and $\nabla_{\gamma'}\gamma'$. Thus, if $\nabla_{\gamma'}\gamma' \neq 0$, or equivalently, the curvature of γ does not vanish, then $\gamma\zeta_0$ is the binormal of the curve. Therefore, the normal of γ is $-\gamma'\zeta_0$ and so the torsion is $\tau = -\langle \nabla_{\gamma'}\gamma\zeta_0, -\gamma'\zeta_0 \rangle = \langle \gamma'\zeta_0, \gamma'\zeta_0 \rangle = 1$.

The second case is similar to the previous one.

□

Let γ be a curve satisfying $\langle \gamma', \gamma\zeta_0 \rangle = 0$ then the curve $\overline{\gamma}$ satisfies Condition 2 in Proposition 4. To see that,

$$0 = \langle \gamma', \gamma\zeta_0 \rangle = \langle \overline{\gamma'}, \overline{\gamma\zeta_0} \rangle = \langle \overline{\gamma}', \overline{\zeta_0}\overline{\gamma} \rangle = -\langle \overline{\gamma}', \zeta_0\overline{\gamma} \rangle.$$

Here, we have used $\overline{\zeta_0} = -\zeta_0$ because $\zeta_0 \in \mathbb{S}^2$.

Now, we are ready to enunciate the converse to Theorem 9.

Theorem 10. *Let $\zeta_0 \in \mathbb{S}^2 \subseteq \mathbb{S}^3$. Consider two regular curves $\alpha(u_1)$ and $\beta(u_2)$ parametrized by the arc length satisfying*

$$\langle \alpha', \alpha\zeta_0 \rangle = 0, \qquad \langle \beta', \beta\zeta_0 \rangle = 0,$$

with $\alpha(0) = 1 = \beta(0)$ and

$$\overline{\alpha(u_1)}\, \alpha'(u_1) \neq \pm\, \overline{\beta(u_2)}\, \beta'(u_2) \qquad \textit{for all } u_1, u_2. \tag{27}$$

Then

$$\psi(u_1, u_2) = \alpha(u_1)\overline{\beta(u_2)}$$

is a flat immersion with unit normal $N(u_1, u_2) = \alpha(u_1)\, \zeta_0\, \overline{\beta(u_2)}$.

The proof of this theorem is a direct computation, bearing in mind that Equation 27 is equivalent to $\psi_{u_1} \neq \pm\psi_{u_2}$. That is, ψ is an immersion if and only if (27) happens.

6.5 The Kitagawa representation

Kitagawa was able to give a geometric construction that allows the construction of curves with torsion ± 1 without solving the corresponding differential equation. To explain it, let us consider the unit tangent bundle $U\mathbb{S}^2$ of $\mathbb{S}^2$, that can be seen as

$$U\mathbb{S}^2 = \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle x, y \rangle = 0\}.$$

Then, given $\xi_0 \in \mathbb{S}^3$ orthogonal to both $\mathbf{1}$ and $\mathbf{i}$, we can define the map $\pi : \mathbb{S}^3 \longrightarrow U\mathbb{S}^2$ given by

$$\pi(x) = (\mathrm{Ad}(x)\mathbf{i}, \mathrm{Ad}(x)\xi_0)\,.$$

This map is a double cover with $\pi(x) = \pi(-x)$ for every $x \in \mathbb{S}^3$.

With this, the key observation by Kitagawa was the following one: let c denote a regular curve in $\mathbb{S}^2$ with tangent indicatrix $c^* = c'/\|c'\|$, and define $\hat{c} := (c, c^*)$, with values in $U\mathbb{S}^2$. Thus, there is a regular curve a in $\mathbb{S}^3$ (unique up to an initial condition) such that $\pi(a) = \hat{c}$. In particular, $h(a) = c$. Then it is not difficult to show that c' is collinear with $\mathrm{Ad}(a)\xi_0$, and consequently we get

$$\langle a', a\,\xi_0 \rangle = 0. \tag{28}$$

But this means, as we saw above, that the curve a in $\mathbb{S}^3$ has torsion 1 at points with non-vanishing curvature. And conversely, any regular curve a in $\mathbb{S}^3$ verifying (28) can be constructed by the above process.

This construction also tells that $a(u)$ must necessarily be an asymptotic curve of the Hopf cylinder $h^{-1}(c)$, belonging to the *non-trivial* asymptotic family, that is, the one which is not made up by the fibers (great circles) of the Hopf fibration.

Given a curve c in $\mathbb{S}^2$, Kitagawa called the *asymptotic lift* of c any of these non-trivial asymptotic curves of $h^{-1}(c)$. This concept is well defined just taking into account that two such asymptotic curves only differ by a left or right translation in $\mathbb{S}^3$, due to the Bianchi-Spivak results.

Using the above results, Kitagawa [18] was able to give a general method to construct flat surfaces in $\mathbb{S}^3$. Let us expose in detail this method following [14]:

Theorem 11 (Kitagawa's representation)**.** *Let $c_1(u), c_2(v)$ be two regular curves in $\mathbb{S}^2$, with $c_i(0) = \mathbf{i}$, $c_i'(0) = \xi_0$, for some $\xi_0 \in \mathbb{S}^3$ orthogonal to both $\mathbf{1}, \mathbf{i}$, and that verify the condition*

$$k_1(u) \neq k_2(v) \qquad \textit{for all } u, v,$$

where here k_1, k_2 are the geodesic curvatures of $c_1(u)$ and $c_2(v)$, respectively. Let $\pi : \mathbb{S}^3 \longrightarrow U\mathbb{S}^2$ be the double cover given by

$$\pi(x) = (\mathrm{Ad}(x)\mathbf{i}, \mathrm{Ad}(x)\xi_0)\,.$$

Let us consider $a_1(u), a_2(v)$ two curves in $\mathbb{S}^3$ parametrized by arclength and satisfy-

ing $\pi(a_i) = (c_i, c_i'/\|c_i'\|)$*, and define*

$$\begin{array}{rcl} \Phi(u,v) & = & a_1(u)\,\bar{a_2}(v) \\ N(u,v) & = & a_1(u)\,\xi_0\,\bar{a_2}(v). \end{array}$$

on a rectangle R *in the* u,v*-plane. If* Σ *is a simply connected surface and* $\Psi : \Sigma \longrightarrow \Psi(\Sigma) = R$ *is an immersion, then* $f = \Phi \circ \Psi$ *is a flat surface in* $\mathbb{S}^3$ *with unit normal* $N \circ \Psi$*. In that case* Ψ *is a coordinate Tschebyscheff immersion, and the angle function of this flat surface is*

$$\omega(u,v) = \cot^{-1}(k_1(u)) - \cot^{-1}(k_2(v)).$$

Conversely, every analytic flat surface in $\mathbb{S}^3$ *is constructed in this way for some* ξ_0*.*

There are several remarks that should be made regarding this result. First of all, the hypothesis of analyticity for the converse is essential, as there are examples of flat surfaces in $\mathbb{S}^3$ with three mutually non-congruent asymptotic curves [14]. Nevertheless, the converse always works locally, and also for complete flat surfaces with bounded mean curvature. Moreover, in these cases the map Ψ can be assumed to be injective, and thus (u,v) constitute globally defined T-parameters.

6.6 Fundamental global results

M. Spivak raised in [30] several questions regarding the geometry of flat surfaces and flat tori in $\mathbb{S}^3$. It is no surprise that the attempts of answering these questions have produced the basis for the global development of the theory.

The classification of flat tori: this is a problem posed by S.T. Yau [34], that was solved by Kitagawa [18] and Weiner [32] from two different perspectives. In [18] Kitagawa used its representation theorem to prove that *the asymptotic curves of a flat torus in* $\mathbb{S}^3$ *are periodic*, thus answering a question by Spivak. This result showed that any flat tori is generated by the construction process exposed above if the two regular curves c_1, c_2 in $\mathbb{S}^2$ are closed.

An alternative classification was given by Weiner. It can be shown that the generalized Gauss map $\mathcal{G} : \Sigma \longrightarrow G_{2,4} \equiv \mathbb{S}^2 \times \mathbb{S}^2$ into the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ of a flat torus in $\mathbb{S}^3$ is the product of two closed curves $\gamma_1 \times \gamma_2 \subset \mathbb{S}^2 \times \mathbb{S}^2$. In [32] Weiner gave a necessary and sufficient condition for the curves γ_i that describes exactly when $\gamma_1 \times \gamma_2$ is the Gauss map of a flat torus in $\mathbb{S}^3$.

Embedded flat tori in $\mathbb{S}^3$: It is known that a Hopf torus $h^{-1}(c)$ is embedded if and only if its generating curve c in $\mathbb{S}^2$ is embedded. The embeddedness condition for a general flat torus in $\mathbb{S}^3$ was considered in [20] and [9]. More specifically, in [9] it was obtained a structure theorem for embedded flat tori in $\mathbb{S}^3$, in terms of a certain topological condition on the curves c_i in $\mathbb{S}^2$ of the Kitagawa representation theorem. It was also proved in [20, 9] that embedded flat tori in $\mathbb{S}^3$ have antipodal symmetry.

Non-orientable flat surfaces in $\mathbb{S}^3$: In [30] Spivak posed the problem of studying the non-orientable flat surfaces in $\mathbb{S}^3$. In response to this problem, Kitagawa showed in [18] that *any complete flat surface in $\mathbb{S}^3$ is orientable*. Nevertheless, the existence of non-orientable (non-complete) flat surfaces in $\mathbb{S}^3$ was still unclear. In [14] it was shown that *any real analytic flat surface in $\mathbb{S}^3$ is orientable*, which contrasts with the $\mathbb{R}^3$ situation. Moreover, this condition cannot be weakened to smoothness, as there are examples of flat Möbius strips in $\mathbb{S}^3$, constructed in [14].

6.7 Open problems

One of the features of the theory of flat surfaces in $\mathbb{S}^3$ is the existence of very basic questions that have not been answered up to now, and whose solution is likely to be quite complicated. This is one of the main points that make the theory interesting. We shall expose in this last part of the section just the most relevant ones.

Existence of an isometric embedding from $\mathbb{R}^2$ into $\mathbb{S}^3$: this is surely the biggest open problem in the theory. A complete flat surface has the topology of a plane, a cylinder, a torus, a Möbius strip or a Klein bottle. Of these, the two non-orientable cases are impossible if the flat surface is isometrically immersed in $\mathbb{S}^3$, as we saw before. Moreover, it is known that there exist both embedded flat tori (like the Clifford tori) and complete embedded flat cylinders in $\mathbb{S}^3$ [9]. However, the existence of a complete embedded simply connected flat surface in $\mathbb{S}^3$ is unknown. This problem was first posed by Spivak [30] using a slightly different formulation. In [9] it was conjectured that the problem has a negative answer, i.e. *the Euclidean plane* $\mathbb{R}^2$ *cannot be isometrically embedded into* $\mathbb{S}^3$.

Rigidity of Clifford tori: The rigidity problem is a fundamental topic in submanifold theory. It asks whether two different isometric immersions of a Riemannian manifold M^n into another Riemannian manifold N^{n+p} must necessarily differ just by an isometry of the ambient space N^{n+p}. If this is the case, it is said that M^n is *rigid* in N^{n+p}. As the simplest flat surfaces in $\mathbb{S}^3$ are the Clifford tori $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$, it is quite natural to ask if these tori are rigid in $\mathbb{S}^3$.

This problem has been an attractive one among specialists, and some natural conditions under which Clifford tori are rigid have been achieved (see for instance [7, 19, 20, 21]). Nevertheless, the original rigidity question remains unanswered.

The space of isometric immersions of a flat torus: as we exposed above, the flat tori in $\mathbb{S}^3$ have been classified in [18] in terms of two curves in $\mathbb{S}^2$ satisfying some compatibility conditions, and in [32] in terms of their Gauss maps. However, the following natural classification problem has not been settled: *given an abstract flat torus* $\mathbb{T}$*, which is the space of isometric immersions of* $\mathbb{T}$ *into* $\mathbb{S}^3$*?*

Flat surfaces with singularities: In recent years there has been an increasing interest on surfaces with a certain type of admissible singularities, called *fronts*. It seems an interesting question to investigate how flat fronts in $\mathbb{S}^3$ behave. The first step into this direction was given in [14], where it was shown that flat fronts in $\mathbb{S}^3$ are an important tool in the problem of classifying (regular) isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$.

7 Surfaces with negative constant Gauss curvature.

Let Σ be a surface endowed with a complete Riemannian metric I of negative constant Gauss curvature $K(I) = k < 0$. Thus, if Σ is simply-connected then Σ is isometric to the hyperbolic plane $\mathbb{H}^2(k)$ with the same constant curvature k, from the Cartan-Hadamard theorem.

From Gauss equation, a surface with negative constant Gauss curvature must also have negative constant extrinsic curvature in $\mathbb{R}^3$ and $\mathbb{S}^3$. Moreover, if the Gauss curvature $K(I) < -1$ in $\mathbb{H}^3$ then the extrinsic curvature is also negative.

The following theorem asserts that there exist no complete immersion with negative constant Gauss curvature and negative extrinsic curvature in $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$.

Theorem 12 (Hilbert)**.** *Let Σ be a complete surface with negative constant Gauss curvature $K(I)$. Then, there exists no isometric immersion $f : \Sigma \longrightarrow \mathbb{M}^3(c)$ (with $K(I) < -1$ for $c = -1$).*

Proof. *By using Theorem 7, there exist global parameter (u, v) defined in $\mathbb{R}^2$ such that*

$$\begin{aligned} I &= du^2 + 2\cos\omega\, dudv + dv^2, \\ II &= 2\sqrt{-K}\sin\omega\, dudv, \end{aligned} \tag{29}$$

and $0 < \omega < \pi$.

Now, the Gauss equation gives

$$\omega_{uv} = -(K + c)\,\sin\omega, \tag{30}$$

with $c_0 = -(K + c) > 0$.

Therefore, the theorem will be proved if we show that there is no solution in $\mathbb{R}^2$ to the equation $\omega_{uv} = c_0\,\sin\omega$, with $0 < \omega(u, v) < \pi$.

Since $(\omega_u(u, v))_v$ is positive, one has that the function $\omega_u(u, v)$ is increasing for v. Thus, $\omega_u(u, v) > \omega_u(u, 0)$ for any $v > 0$. Now, by integrating the previous inequality

$$\omega(b, v) - \omega(a, v) = \int_a^b \omega_u(u, v)\, du > \int_a^b \omega_u(u, 0)\, du = \omega(b, 0) - \omega(a, 0) \tag{31}$$

with $a < b$ and $0 < v$.

Since ω_u can not vanish identically, up to a translation, we can assume that $\omega_u(0, 0) \neq 0$. Moreover, $\omega(-u, -v)$ also satisfies the same PDE (30). So, we can assume $\omega_u(0, 0) > 0$, replacing $\omega(u, v)$ by $\omega(-u, -v)$ if necessary.

Now, let us consider three real numbers u_1, u_2, u_3 such that $0 < u_1 < u_2 < u_3$ and $\omega_u(u, 0) > 0$ for all $u \in [0, u_3]$. And let us define

$$\varepsilon = \min\{\omega(u_3, 0) - \omega(u_2, 0), \omega(u_1, 0) - \omega(0, 0)\} > 0.$$

By using (31), we have $\omega(u_1, v) - \omega(0, v) > \varepsilon$ *and* $\omega(u_3, v) - \omega(u_2, v) > \varepsilon$ *for all* $v > 0$. *Hence,*

$$\varepsilon < \omega(u, v) < \pi - \varepsilon \qquad \text{if } u \in [u_1, u_2],\ v \geq 0,$$

because $0 < \omega(u, v) < \pi$.

Thus, by integrating in the rectangle $[u_1, u_2] \times [0, v]$ *we have*

$$\omega(u_2, v) - \omega(u_1, v) - \omega(u_2, 0) + \omega(u_1, 0) = \int_0^v \int_{u_1}^{u_2} \omega_{uv}\, dudv =$$
$$= c_0 \int_0^v \int_{u_1}^{u_2} \sin\omega\, dudv > c_0 \int_0^v \int_{u_1}^{u_2} \sin\varepsilon\, dudv = c_0(u_2 - u_1)v \sin\varepsilon.$$

So, $\omega(u_2, v) - \omega(u_1, v) > \omega(u_2, 0) - \omega(u_1, 0) + c_0(u_2 - u_1)v \sin\varepsilon$. *Therefore,* $\omega(u_2, v) - \omega(u_1, v)$ *goes to infinity when* $v \to \infty$, *which contradicts* $0 < \omega(u, v) < \pi$.

□

The Hilbert theorem asserts that $\mathbb{H}^2$ can not be isometrically immersed in $\mathbb{R}^2$. It is classically known that $\mathbb{H}^n$ can be isometrically immersed in $\mathbb{R}^{2n}$ but it can not be isometrically immersed in $\mathbb{R}^{2n-2}$. It is an important old problem if $\mathbb{H}^n$ can be isometrically immersed in $\mathbb{R}^{2n-1}$, that is, if the Hilbert theorem can be generalized for $n > 2$.

In order to finish the study of complete surfaces in $\mathbb{M}^3(c)$ with negative constant Gauss curvature, we need to consider the case $K(I) \in [-1, 0[$ for surfaces in the hyperbolic 3-space.

Let us consider the model of $\mathbb{H}^3$ given by (6). Then the map Ψ given by

$$\begin{array}{ccc} \mathbb{H}^3 & \longrightarrow & B(0,1) \subseteq \mathbb{R}^3 \\ (x_0, x_1, x_2, x_3) & \longrightarrow & \frac{1}{1+x_0}(x_1, x_2, x_3) \end{array}$$

is totally geodesic, where $B(0, 1)$ denotes the open unit ball of $\mathbb{R}^3$. That is, the image of a geodesic in $\mathbb{H}^3$ is a segment in $B(0, 1) \subseteq \mathbb{R}^3$.

Thus, if Σ is a surface in $\mathbb{H}^3$ with constant Gauss curvature -1 then $\Psi(\Sigma) \subset \mathbb{R}^3$ is a Euclidean flat surface. Therefore, its study can be developed from the local study of flat surfaces in $\mathbb{R}^3$. It should be mentioned that there is a large family of complete surfaces in $\mathbb{H}^3$ with $K(I) = -1$ with the topology of a compact surface minus a disk (see [30]).

The case of complete surfaces in $\mathbb{H}^3$ with constant Gauss curvature $K(I) \in (-1, 0)$ has been amply studied. It was proved by Rosenberg and Spruck [29] that given a smooth curve Γ in the ideal boundary of $\mathbb{H}^3$ and a constant $k \in (-1, 0)$, there exists a embedded complete surface Σ with Gauss curvature $K(I) = k$ whose boundary at infinity is Γ. Other outstanding results about complete surfaces solving a Plateau problem at infinity were also given by Labourie [22].

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