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GEOMETRIC ASPECTS OF THE ALLEN-CAHN EQUATION *

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1 Introduction

In these lectures, we are interested in entire solutions in $\mathbb{R}^N$, of the semilinear elliptic equation

$$\Delta u + (1 - u^2)\, u = 0\,, \tag{1.1}$$

known as the Allen-Cahn equation. This problem has its origin in the *gradient theory of phase transitions* [2] where one is interested in critical points of the energy

$$E_\varepsilon(u) := \frac{\varepsilon}{2}\int_M |\nabla^g u|_g^2\, \mathrm{dvol_g} + \frac{1}{4\varepsilon}\int_M (1-u^2)^2\, \mathrm{dvol_g}\,,$$

where (M, g) is a N-dimensional compact Riemannian manifold (with or without boundary). The critical points of E_ε satisfy the Euler-Lagrange equation

$$\varepsilon^2 \Delta_g u + (1 - u^2)\, u = 0\,, \tag{1.2}$$

in M. Working in local coordinates and changing x into x/ε, it is easy to see that (1.1) appears as the limit problem in the blow up analysis of (1.2) as ε tends to 0.

As we will see in these lectures, the space of entire solutions of (1.1) is surprisingly rich and also has an interesting structure.

2 The role of minimal hypersurfaces

The relation between sets of minimal perimeter and critical points of functional of the form E_ε was first established by Modica in [35]. Let us briefly recall the

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main results in this direction : If u_ε is a family of *local minimizers* of E_ε for which

$$\sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon) < +\infty, \tag{2.3}$$

then, up to a subsequence, u_ε converges as ε tends to 0, in L^1 to $\mathbf{1}_\Lambda - \mathbf{1}_{\Lambda^c}$, where $\partial\Lambda$ has minimal perimeter. Here $\mathbf{1}_\Lambda$ (resp. $\mathbf{1}_{\Lambda^c}$) is the characteristic function of the set Λ (resp. $\Lambda^c = M - \Lambda$). Moreover,

$$E_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}}\,\mathcal{H}^{N-1}(\partial\Lambda)\,.$$

For *critical points* of E_ε which satisfy (2.3), a related assertion is proven in [24]. In this case, the convergence of the interface holds with certain integer multiplicity to take into account the possibility of multiple transition layers converging to the same minimal hypersurface.

These results provide a link between solutions of (1.1) and the theory of minimal hypersurfaces which has been widely explored in the literature. For example, solutions concentrating along non-degenerate, minimal hypersurfaces of a compact manifold were found in [37] (see also [27]). Let us describe these results more carefully. We assume that (M, g) is a compact Riemanian N-dimensional manifold without boundary and that $\Gamma \subset M$ is an oriented minimal hypersurface such that $M \setminus \Gamma = M_+ \cup M_-$ and n the unit normal vector field to Γ which is compatible with the orientation points towards M_+ while $-n$ points toward M_- (Γ might have many connected components).

We recall the definition of the Jacobi operator about Γ

$$J_\Gamma := \Delta_{\mathring{g}} + \mathrm{Ric}_g\,(n,n) + |A_\Gamma|^2_{\mathring{g}}\,,$$

where $\Delta_{\mathring{g}}$ is the Laplacian on $(\Gamma, \bar{g})$ for $\mathring{g}$ the induced metric on Γ, Ric_g denotes the Ricci tensor on (M, g) and $|A_\Gamma|^2_{\mathring{g}}$ denotes the square of the norm of the shape operator defined by

$$\mathring{g}(A_\Gamma X, Y) = g(\nabla^g_X Y, n)\,,$$

for all $X, Y \in T\,\Gamma$. We will say that Γ is nondegenerate if J_Γ has trivial kernel.

Then we have the :

Theorem 2.1. *[37] Assume that (M, g) is an N-dimensional compact Riemanian manifold without boundary and $\Gamma \subset M$ is a nondegenerate oriented minimal hypersurface such that $M \setminus \Gamma = M_+ \cup M_-$ and n points toward M_+ while $-n$ points towards M_-. Then, there exists $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$ there exists u_ε, critical point of E_ε, such that u_ε converges uniformly to 1 on compacts subsets of M^+ (resp. to -1 on compacts subsets of M^-) and $E_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}} \mathcal{H}^{N-1}(\Gamma)$, as ε tends to 0.*

As far as multiple transition layers are concerned, given a minimal hypersurface Γ (subject to some additional property on the sign of the potential of the Jacobi operator about Γ, which holds on manifolds with positive Ricci curvature) and given an integer $k \geq 1$, solutions of (1.2) with multiple transitions layers near Γ were built in [15], in such a way that

$$E_\varepsilon(u_\varepsilon) \longrightarrow \frac{k}{\sqrt{2}} \mathcal{H}^{N-1}(\Gamma).$$

More precisely, we have the following difficult result :

Theorem 2.2. *[15] Assume that (M, g) is a N-dimensional compact Riemanian manifold and $\Gamma \subset M$ is a nondegenerate, oriented, connected minimal hypersurface such that $M \setminus \Gamma = M_+ \cup M_-$ and n points toward M_+ and $-n$ points toward M_-. Further assume that*

$$\mathrm{Ric}_g(n, n) + |A_\Gamma|^2_{\mathring{g}} > 0$$

along Γ. Then, there exists a sequence $(\varepsilon_j)_j$ of positive numbers tending to 0 and there exists u_j, critical point of E_{ε_j}, such that u_j converges uniformly to 1 on compacts subsets of M^+ (resp. to $(-1)^k$ on compacts subsets of M^-) and $E_{\varepsilon_j}(u_j) \longrightarrow \frac{k}{\sqrt{2}} \mathcal{H}^{N-1}(\Gamma)$, as j tends to ∞. In particular, u_j has k transition layers close to Γ.

Observe that the former result produces solutions for any ε small while the later only produces solutions for some sequence of ε tending to 0. This is due to some resonance phenomena which arises in the case of multiple interfaces.

3 The de Giorgi's conjecture

In dimension 1, solutions of (1.1) which have finite energy are given by translations of the function H which is the unique solution of the problem

$$H'' + (1 - H^2)\, H = 0, \quad \text{with} \quad H(\pm\infty) = \pm 1 \quad \text{and} \quad H(0) = 0\,. \tag{3.4}$$

In fact, the function H is explicitly given by

$$H(y) = \tanh\left(\frac{y}{\sqrt{2}}\right).$$

Then, for all $\mathbf{a} \in \mathbb{R}^N$ with $|\mathbf{a}| = 1$ and for all $b \in \mathbb{R}$, the function $u(\mathbf{x}) = H(\mathbf{a} \cdot \mathbf{x} + b)$ solves (1.1).

A celebrated conjecture due to de Giorgi asserts that, in dimension $N \leq 8$, these solutions are the only ones which are bounded and monotone in one direction. In other words, if u is a (smooth) bounded solution of (1.1) and if $\partial_{x_N} u > 0$ then $u^{-1}(\lambda)$ is either a hyperplane or the empty set.

In dimensions $N = 2, 3$, de Giorgi's conjecture has been proven in [20], [3] and (under some extra assumption) in the remaining dimensions in [40] (see also [17], [19]). When $N = 2$, the monotonicity assumption can even be replaced by a weaker stability assumption [23]. Finally, counterexamples in dimension $N \geq 9$ have recently been built in [14], using the existence of non trivial minimal graphs in higher dimensions.

Let us briefly explain the proof of the de Giorgi conjecture in the two dimensional case, as given by A. Farina in [18].

Theorem 3.1. *[20] The de Giorgi's conjecture is true in dimension $N = 2$.*

Proof. [18] *Assume that u satisfies (1.1). Then*

$$\Delta \nabla u + (1 - 3u^2)\, \nabla u = 0\,. \tag{3.5}$$

It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and write

$$\nabla u = \rho\, e^{i\theta}\,,$$

where ρ and θ are real valued functions. Observe that we implicitly use the fact that $\partial_{x_2} u > 0$ and hence we can choose the function θ to take values in $(0, \pi)$. Elliptic estimates imply that ∇u is bounded and hence so is ρ.

Now, with these notations, (3.5) can be written as

$$\Delta \rho - |\nabla \theta|^2 \, \rho + (1 - 3u^2) \, \rho + i \, (\rho \, \Delta \theta + 2 \, \nabla \rho \cdot \nabla \theta) = 0 \, .$$

In particular, the imaginary part of the left-hand side of the equation is identically equal to 0 and hence

$$\operatorname{div} \left(\rho^2 \, \nabla \theta \right) = 0 \, .$$

As already mentioned, $\theta \in (0, \pi)$ and the next Lemma (Liouville type result) implies that θ is in fact a constant function. Therefore, the unit normal vector to the level lines of u is constant.

□

In order for the proof to be complete, it remains to prove the following :

Lemma 3.1. *Assume that ρ is a positive smooth, bounded function. Further assume that θ is a bounded solution of*

$$\operatorname{div} \left(\rho^2 \, \nabla \theta \right) = 0 \, , \tag{3.6}$$

in $\mathbb{R}^2$. Then θ is the constant function.

Proof. *Let χ be a cutoff function which is identically equal to 1 in the unit ball and identically equal to 0 outside the ball of radius 2. For all $R > 0$ we define*

$$\chi_R = \chi(\cdot / R) \, .$$

Observe that $|\nabla \chi_R| \leq C \, R^{-1}$ for some constant $C > 0$ independent of $R > 0$. We multiply (3.6) by $\chi_R^2 \, \theta$ and integrate the result over $\mathbb{R}^2$ to find

$$\int_{\mathbb{R}^2} |\nabla \theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x = -2 \int_{\mathbb{R}^2} \theta \rho^2 \chi_R \, \nabla \theta \nabla \chi_R \, \mathrm{d}x \, .$$

Now, the integral on the right-hand side it taken over the set of $x \in \mathbb{R}^2$ such that $|x| \in [R, 2R]$. Using Cauchy-Schwarz inequality, we conclude that

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x \;\leq\; 2 \left(\int_{|x|\in[R,2R]} \rho^2 \theta^2 \, |\nabla \chi_R|^2 \, \mathrm{d}x \right)^{1/2} \times \left(\int_{|x|\in[R,2R]} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x \right)^{1/2} .$$

Using the bound $|\nabla \chi_R| \leq C \, R^{-1}$, we conclude that

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x \leq C \left(\int_{|x|\in[R,2R]} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x \right)^{1/2} . \tag{3.7}$$

This in particular implies that

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x \leq C^2 \, ,$$

and hence

$$\lim_{R\to\infty} \int_{x\in[R,2R]} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x = 0 \, .$$

Inserting this information back in (3.7), we conclude that

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \, \rho^2 \, \chi_R^2 \, \mathrm{d}x = 0 \, ,$$

which implies that θ is the constant function.

□

There is also a strong relation between the set of solutions of (1.1) which are monotone in one direction and the stable solutions. We have the following result which holds in any dimension :

Proposition 3.1. *Assume that u is a solution of (1.1) which satisfies $\partial_{x_N} u > 0$. Then u is stable in the sense that*

$$\int_{\mathbb{R}^N} (|\nabla w|^2 + (3u^2 - 1) \, w^2) \, \mathrm{d}x > 0$$

for any function $w \in \mathcal{C}^2(\mathbb{R}^N)$, $w \neq 0$, with compact support in $\mathbb{R}^N$.

Proof. *We set*

$$\phi := \partial_{x_N} u \,.$$

Certainly $\Delta\phi + (1-3u^2)\,\phi = 0$ *in* $\mathbb{R}^N$. *Multiplying this identity by* $w^2\,\phi^{-1}$ *and integrating the result over* $\mathbb{R}^N$, *we conclude that*

$$\int_{\mathbb{R}^N} \left(2\,\phi^{-1}\, w\,\nabla\phi\,\nabla w - w^2\,\phi^{-2}\,|\nabla\phi|^2 + (3u^2-1)\,w^2\right)\,\mathrm{d}x = 0\,.$$

In particular, we can write

$$\int_{\mathbb{R}^N} \left(|\nabla w|^2 + (3u^2-1)\,w^2\right)\,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla w - w\,\phi^{-1}\,\nabla\phi|^2\,\mathrm{d}x = 0\,,$$

and the result follows at once.

□

4 Entire solutions to the Allen-Cahn equation

In view of de Giorgi conjecture, it is natural to study the set of entire solutions of (1.1), namely, solutions which are defined in the entire $\mathbb{R}^N$. The functions $u(\mathbf{x}) = H(\mathbf{a}\cdot\mathbf{x} + b)$ are obvious solutions. In dimension $N = 2$, nontrivial examples (whose nodal set is the union of two perpendicular lines) were built in [8] using the following strategy : For all $R > 0$ we define

$$\Omega_R := \{(x,y)\in\mathbb{R}^2 \,:\, x > |y| \quad \text{and} \quad x^2+y^2 < R^2\}\,.$$

Then one considers the energy

$$E_R(u) := \frac{1}{2}\int_{\Omega_R} |\nabla u|^2\,\mathrm{d}x + \frac{1}{4}\int_{\Omega_R} (1-u^2)^2\,\mathrm{d}x\,.$$

Standard arguments of the calculus of variations implies that, for all $R > 0$, there exists a minimiser $u_R \in H_0^1(\Omega_R)$ which can be assumed to be positive and bounded by 1. This minimiser is a smooth solution of (1.1) in Ω_R which has 0 boundary data and is bounded. It is easy to cook up a test function to show that

$$E_R(u_R) \leq C\,R\,. \tag{4.8}$$

(Just build a function which interpolate smoothly from 0 to 1 in a layer of size 1 around the boundary of Ω_R and which is identically equal to 0 elsewhere. Obviously the energy of this function is then controlled by the length of $\partial\Omega_R$ and hence is less than a constant times R). Since the energy of the trivial solution (identically equal to 0) is

$$E_R(0) = \frac{1}{4}\int_{\Omega_R} dx \geq \bar{C}\, R^2 \,,$$

we conclude that u_R is certainly not identically equal to 0 for R large enough since otherwise we would have the inequalities

$$\bar{C}\, R^2 \leq E_R(0) \leq E_R(u_R) \leq C\, R \,,$$

which does not hold for R large enough.

Elliptic estimates together with Ascoli-Arzela's theorem allow one to prove that, as R tends to ∞, this sequence of minimisers u_R converges (up to a subsequence and uniformly on compacts) to a solution of (1.1) which is defined in the quadrant $\{(x, y) \in \mathbb{R}^2 \,:\, x > |y|\}$ and which vanishes on the boundary of this set. A solution u_2 defined in the entire space is then obtained using odd reflections through the lines $x = \pm y$. The function u_2 is a solution of (1.1), whose 0-level set is the union of the two axis.

Observe that we need to rule out the fact that u_2 is the trivial solution (identically equal to 0). The proof is again by contradiction. If $u_2 \equiv 0$, then for R large enough, the solution u_R would be less that $1/2$ on $\Omega_{\bar{R}}$ (here $\bar{R}$ is fixed large enough, independently of R). However, arguing as above it would be possible to modify the definition of u_R on $\Omega_{\bar{R}}$ while reducing its energy (this would contradict the fact that u_R is a minimiser of the energy).

This construction can easily be generalised to obtain solutions with dihedral symmetry by considering, for $k \geq 3$, the corresponding solution within the sector $\{(r\,\cos\theta, r\sin\theta) \,:\, r > 0\,,\, |\theta| < \frac{\pi}{2k}\}$ and extending it by $2k-1$ consecutive odd reflections to yield an entire solution u_k (we refer to [22] for the details). The zero level set of u_k is constituted outside any ball by $2k$ infinite half lines with dihedral symmetry.

Recently, X. Cabre and J. Terra [4] have obtained a higher dimensional version of this construction (using similar arguments) and they are able to find solutions in $\mathbb{R}^{2m}$ whose zero set is the minimal cone $\{(x,y) \in \mathbb{R}^{2m} \, : \, |x| = |y|\}$.

5 New entire solutions

Recently, there has been some progress on the understanding of the existence of solutions of 1.1 which are defined in the entire space. All these new solutions are conterparts, in the complete noncompact setting of the solutions obtained in [37] and rely on the knowledge of complete noncompact minimal hypersurfaces which are not invariant by dilations. Let us mention two important results along these lines.

There is a rich family of minimal surfaces in $\mathbb{R}^3$ which are complete, embedded and have finite total curvature. Among these surfaces there is the catenoid, Costa's surface [5] and its higher genus analogues, and all k-ended embedded minimal surfaces studied by J. Perez and A. Ros [38]. The main result in [16] asserts that there exists solutions of (1.1) whose nodal set is close to a dilated version of any nondegenerate complete, noncompact minimal surface with finite total curvature. In other words, if one considers the equation with scaling

$$\varepsilon^2 \, \Delta u + u - u^3 = 0 \,, \tag{5.9}$$

then the following result holds :

Theorem 5.1. *[16] Given Γ, a nondegenerate complete, embedded minimal surface with finite total curvature, there exists $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$ there exists u_ε solution of (5.9), such that $u_\varepsilon^{-1}(0)$ converges uniformly on compacts to Γ.*

In this result, nondegeneracy refers to the fact that all bounded Jacobi fields of Γ come from the action of rigid motions on Γ.

Also, thanks to the result of Bombieri-de Giorgi-Giusti, it is known that there exists minimal graphs which are not hyperplanes in dimension $N \geq 9$.

Following similar ideas, it is proved in [14] that one can construct entire solutions of (1.1) whose level sets are not hyperplanes, provided the dimension of the ambient space is $N \geq 9$.

6 Entire solutions in the 2-dimensional case

We assume from now on that the dimension is equal to $N = 2$.

Definition 1. *We say that u, solution of (1.1), has $2k$-ends if, away from a compact set, its nodal set is given by $2k$ connected curves which are asymptotic to $2k$ oriented, disjoint, half lines $\mathbf{a}_j \cdot \mathbf{x} + b_j = 0$, $j = 1, \ldots, 2k$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) and if, along these curves, the solution is asymptotic to either $H(\mathbf{a}_j \cdot \mathbf{x} + b_j)$ or $-H(\mathbf{a}_j \cdot \mathbf{x} + b_j)$.*

Given any $k \geq 1$, one can prove the existence of a wealth of $2k$-ended solutions of (1.1). Moreover, one can show that these solutions belong to a smooth $2k$-parameter family of $2k$-ended solutions of (1.1). To state this result in precise way, we assume that we are given a solution $q_1, \ldots, q_k$ of the *Toda system*

$$c_0\, q_j'' = e^{\sqrt{2}(q_{j-1} - q_j)} - e^{\sqrt{2}(q_j - q_{j+1})}\,, \tag{6.10}$$

for $j = 1, \ldots, k$, where $c_0 = \frac{\sqrt{2}}{24}$ and we agree that

$$q_0 \equiv -\infty \qquad \text{and} \qquad q_{k+1} \equiv +\infty\,.$$

The Toda system (6.10) is a classical example of integrable system which has been extensively studied. It models the dynamics of finitely many mass points on the line under the influence of an exponential potential. We refer to [26] and [36] for the complete description of the theory. Of importance is the fact that solutions of (6.10) can be described (almost explicitly) in terms of $2k$ parameters. Moreover, if $\mathbf{q}$ is a solution of (6.10), then the long term behavior (i.e. long term scattering) of the q_j at $\pm\infty$ is well understood and it is known that, for all $j = 1, \ldots, k$, there exist $a_j^+, b_j^+ \in \mathbb{R}$ and $a_j^-, b_j^- \in \mathbb{R}$, all depending on $q_1, \ldots, q_k$, such that

$$q_j(t) = a_j^\pm \, |t| + b_j^\pm + \mathcal{O}_{\mathcal{C}^\infty(\mathbb{R})}(e^{-\tau_0 \, |t|}) \, , \tag{6.11}$$

as t tends to $\pm\infty$, for some $\tau_0 > 0$. Moreover, $a_{j+1}^\pm > a_j^\pm$ for all $j = 1, \ldots, k-1$.

Given $\varepsilon > 0$, we define the vector valued function $q_{1,\varepsilon}, \ldots, q_{k,\varepsilon}$, whose components are given by

$$q_{j,\varepsilon}(x) := q_j(\varepsilon \, x) - \sqrt{2} \left(j - \frac{k+1}{2} \right) \log \varepsilon \, . \tag{6.12}$$

It is easy to check that the $q_{j,\varepsilon}$ are again solutions of (6.10).

Observe that, according to the description of the asymptotics of the functions q_j, the graphs of the functions $q_{j,\varepsilon}$ are asymptotic to oriented half lines at infinity. In addition, for $\varepsilon > 0$ small enough, these graphs are disjoint and in fact their mutual distance is given by $-\sqrt{2} \, \log \varepsilon + \mathcal{O}(1)$ as ε tends to 0.

It will be convenient to agree that χ^+ (resp. χ^-) is a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $x > 1$ (resp. for $x < -1$) and identically equal to 0 for $x < -1$ (resp. for $x > 1$) and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the 4 dimensional space

$$D := \mathrm{Span}\, \{ x \longmapsto \chi^\pm(x), \; x \longmapsto x \, \chi^\pm(x) \} \, , \tag{6.13}$$

and, for all $\mu \in (0,1)$ and all $\tau \in \mathbb{R}$, we define the space $\mathcal{C}^{2,\mu}_\tau(\mathbb{R})$ of $\mathcal{C}^{2,\mu}$ functions h which satisfy

$$\|h\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R})} := \|(\cosh x)^\tau \, h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty \, .$$

Keeping in mind the above notations, we have the :

Theorem 6.1. *[10] For all $\varepsilon > 0$ sufficiently small, there exists an entire solution u_ε of the Allen-Cahn equation (1.1) whose nodal set is the union of k disjoint curves $\Gamma_{1,\varepsilon}, \ldots, \Gamma_{k,\varepsilon}$ which are the graphs of the functions*

$$x \longmapsto q_{j,\varepsilon}(x) + h_{j,\varepsilon}(\varepsilon \, x) \, ,$$

where the functions $h_{j,\varepsilon} \in \mathcal{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D$ *satisfy*

$$\|h_{j,\varepsilon}\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R})\oplus D} \leq C\,\varepsilon^\alpha\,.$$

for some constants $C, \alpha, \tau, \mu > 0$ *independent of* $\varepsilon > 0$.

In other words, given a solution of the Toda system, we can find a one parameter family of $2k$-ended solutions of (1.1) which depend on a small parameter $\varepsilon > 0$. As ε tends to 0, the nodal sets of the solutions we construct become close to the graphs of the functions $q_{j,\varepsilon}$.

Going through the proof, one can be more precise about the description of the solution u_ε. If $\Gamma \subset \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ which is the graph over the x-axis of some function, we denote by $\mathrm{dist}\,(\cdot, \Gamma)$ the signed distance to Γ which is positive in the upper half of $\mathbb{R}^2 \setminus \Gamma$ and is negative in the lower half of $\mathbb{R}^2 \setminus \Gamma$. Then, we have the :

Proposition 6.1. *The solution of (1.1) provided by Theorem 8.2 satisfies*

$$\|e^{\varepsilon\,\hat\alpha\,|\mathbf{x}|}\,(u_\varepsilon - u^*_\varepsilon)\|_{L^\infty(\mathbb{R}^2)} \leq C\,\varepsilon^{\bar\alpha}\,,$$

for some constants $C, \bar\alpha, \hat\alpha > 0$ *independent of* ε*, where*

$$u^*_\varepsilon := \sum_{j=1}^{k} (-1)^{j+1}\,H\big(\mathrm{dist}(\cdot, \Gamma_{j,\varepsilon})\big) - \frac{1}{2}((-1)^k + 1)\,. \tag{6.14}$$

It is interesting to observe that, when $k \geq 3$, there are solutions of (6.10) whose graphs have no symmetry and our result yields the existence of entire solutions of (1.1) without any symmetry provided the number of ends is larger than or equal to 6.

7 Sketch of the proof of Theorem 8.2

Let us briefly give the main ideas in the proof of Theorem 8.2. We start by considering the linearized operator about H, namely

$$L_0 := \partial_y^2 + 1 - 3\,H^2\,.$$

First, observe that L_0 has a one dimensional kernel spanned by H' since $L_0\,H' = 0$ as can be checked by taking the derivative of $H'' + (1 - H^2)\,H = 0$. Since $H' > 0$ this implies that 0 is the bottom of the spectrum of $-L_0$. In fact more is known and we recall the following result from [1] :

Lemma 7.1. *[1] The spectrum of the operator $-L_0$ is the union of the point spectrum, given by 0 (associated to the eigenfunction H') and $\frac{3}{2}$ (associated to the eigenfunction $H\,\sqrt{H'}$) and the continuous spectrum given by $[2, +\infty)$.*

In particular, for all $\xi \neq 0$, given $f \in L^2(\mathbb{R})$, the problem

$$(L_0 - \xi^2)\,\phi = f, \tag{7.15}$$

is uniquely solvable in $H^1(\mathbb{R})$.

Let us now consider operator

$$L := \partial_x^2 + L_0\,,$$

acting on functions defined in the plane. Obviously, we still have $L\,H' = 0$. The following key result asserts that any bounded solution of $L\,\phi = 0$ is colinear to H'. The proof of this fact follows the method first introduced in [37].

Lemma 7.2. *Let ϕ be a bounded solution of*

$$L\,\phi = 0\,, \tag{7.16}$$

in $\mathbb{R}^2$. Then ϕ is colinear to H'.

Making use of the previous Lemma, we now obtain the existence of a solution of

$$L\,\phi = f\,, \tag{7.17}$$

in $\mathbb{R}^2$. The classification of the bounded solutions of $L\,\phi = 0$ suggests to impose the following orthogonality condition on the function ϕ

$$\int_{\mathbb{R}} \phi(x,\cdot)\,H'\,\mathrm{d}y = 0, \tag{7.18}$$

As far as the existence of solutions of (7.17)-(7.18) is concerned, provided we assume that

$$\int_{\mathbb{R}} f(x,\cdot)\, H'\, \mathrm{d}y = 0\,, \tag{7.19}$$

for all $x \in \mathbb{R}$, we have the following result whose proof relies on the previous analysis :

Proposition 7.1. *Assume that $\sigma \in (0, \sqrt{2})$ is fixed. For all $a \in [0, \frac{1}{\sqrt{2}})$ such that*

$$\sigma^2 + a^2 < 2\,,$$

there exists a constant $C_a > 0$, which depends on a but remains bounded as a tends to 0, *such that, for all f satisfying the orthogonality condition (7.19) and*

$$\|(\cosh x)^a\, (\cosh y)^\sigma\, f\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} < +\infty\,,$$

there exists a unique function ϕ, solution of (7.17)-(7.18), which satisfies

$$\|(\cosh x)^a\, (\cosh y)^\sigma\, \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C_a\, \|(\cosh x)^a\, (\cosh y)^\sigma\, f\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}\,.$$

Let us briefly comment on the orthogonality condition we impose on the function f. Given any function f, with the appropriate decay as in the statement of Proposition 7.1, we want to solve the equation $L\,\phi = f$. We can certainly find a function $x \longmapsto c(x)$ such that $f - c\, H'$ satisfies (7.19). And then, we can apply the result of Proposition 7.1 to solve $L\,\phi = f - c\, H'$. Therefore, it just remains to solve the equation $L\,\psi = c\, H'$, but this is rather easy since it is enough to look for ψ of the form $\psi(x,y) = d(x)\, H'(y)$ in which case the equation reduces to the solvability of the equation $d'' = c$. Observe that, it is not possible to find a solution of this ordinary differential equation which decays exponentially at $\pm\infty$ unless the function c satisfies

$$\int_{\mathbb{R}} c(x)\, \mathrm{d}x = \int_{\mathbb{R}} x\, c(x)\, \mathrm{d}x = 0\,.$$

In fact this solution is explicitly given by

$$d(x) = x \int_{-\infty}^{x} c(z)\, \mathrm{d}z - \int_{-\infty}^{x} z\, c(z)\, \mathrm{d}z\,.$$

Now, if c is bounded by a constant times $(\cosh x)^{-a}$, and satisfies the two conditions above, it is easy to check that d is also bounded by a constant (independent of $a \in (0,1)$) times $a^{-2}\,(\cosh x)^{-a}$. In particular, this solution blows up as a tends to 0. In the proof of the result, we need to invert L on functions spaces corresponding to a tending to 0 and, in order to get a right inverse whose norm does not blow up, is is necessary to impose the restriction (7.19) on the functions f.

Let us assume that we are given a solution $q_1, \ldots, q_k$ of the Toda system (6.10), we define

$$q_{j,\varepsilon}(x) := q_j(\varepsilon\, x) - \sqrt{2}\left(j - \frac{k+1}{2}\right)\log \varepsilon\,.$$

With these data at hand, we define the planar curve $\bar\Gamma_j$ to be the image of

$$\gamma_j(x) := (x, q_{j,\varepsilon}(x))\,.$$

and, for each $j = 1, \ldots, k$, we introduce the Fermi coordinates (x_j, y_j) which are associated to the curve $\bar\Gamma_j$. More precisely, we consider the parameterization of a tubular neighborhood of $\bar\Gamma_j$ by X_j

$$X_j(x_j, y_j) := \gamma_j(x_j) + y_j\, n_j(x_j)\,, \tag{7.20}$$

where n_j is the normal vector about $\bar\Gamma_j$ (the curves are assumed to be positively oriented). Observe that the coordinate y_j is nothing but the signed distance to $\bar\Gamma_j$.

The basic idea is to consider the approximate solution which is close to the function $\pm H(\mathrm{dist}(\cdot, \bar\Gamma_j))$ (with alternative signs according to whether j is odd or even). A natural choice would be the function

$$\sum_{j=1}^{k}(-1)^{j+1}\, H(\mathrm{dist}(\cdot, \bar\Gamma_j)) - \frac{1}{2}((-1)^{k+1} + 1)\,. \tag{7.21}$$

The fact that we have to impose the orthogonality condition (7.19) to produce a right inverse of L whose norm does not blow up as the weight parameter

a tends to 0 translates into the fact that, even though the nodal sets of the solutions we will construct are close to the curves $\bar\Gamma_j$ (say in Hausdorff topology), this topology is not refined enough to perform the construction. Hence, in some sense, we need to improve the definition of the nodal sets of the approximate solutions by allowing more flexibility in the definition of the curves $\bar\Gamma_j$. This is the reason why we need to introduce $h_1, \ldots, h_k \in \mathcal{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D$ (here $\tau > 0$) and we define the functions H_j by the identity

$$X_j^* \, H_j(x_j, y_j) := H(y_j - h_j(\varepsilon \, x_j)) \,. \tag{7.22}$$

With these data and notations, we are now in a position to define a multiple-ended approximate solution of (1.1) by the formula

$$\bar u := \sum_{j=1}^k (-1)^{j+1} \, H_j - \frac{1}{2}((-1)^{k+1} + 1) \,.$$

Granted the above notations and definitions, the equation we want to solve reads

$$\Delta(\bar u + \phi) + \bar u + \phi - (\bar u + \phi)^3 = 0 \,, \tag{7.23}$$

for some $\phi \in \mathcal{C}^{2,\mu}(\mathbb{R})$ and $h_1, \ldots, h_k \in \mathcal{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D$. We can then formally rewrite the equation (7.23) as

$$\mathbf{L} \, \phi = Q(\phi),$$

where the linear operator $\mathbf{L}$ is defined by

$$\mathbf{L} := \Delta + 1 - 3 \, \bar u^2 \,,$$

and where the nonlinear operator Q is defined by

$$Q(\phi) := -(\Delta \bar u + (1 - \bar u^2) \, \bar u) + \phi^3 + 3 \, \bar u \, \phi^2 \,. \tag{7.24}$$

We need to study the mapping properties of the linear operator $\mathbf{L}$ and the nonlinear operator Q when defined between appropriate weighted function spaces.

This part is very technical but there is no real difficulty.

We can measure how far the function $\bar{u}$ is from a genuine solution. The key point is the following estimate (ρ_j is a cutoff function which is identically equal to 1 in some tubular neighborhood of width $\frac{\sqrt{2}}{4}\log 1/\varepsilon$ about $\bar{\Gamma}_j$ and identically equal to 0 away from a tubular neighborhood of width $\frac{\sqrt{2}}{2}\log 1/\varepsilon$ about $\bar{\Gamma}_j$)

$$\begin{aligned}\int_{\mathbb{R}}(\Delta\bar{u}+\bar{u}-\bar{u}^3)\,\rho_j\,H_j'\,\mathrm{d}y_j &= -\varepsilon^2\left(c_*\,q_j''+c^*\left(e^{\sqrt{2}\,(q_j-q_{j+1})}-e^{\sqrt{2}(q_{j-1}-q_j)}\right)(\varepsilon\,\cdot)\right.\\ &+\ \mathcal{O}(\varepsilon^{2+\beta}),\end{aligned} \tag{7.25}$$

on any compact of $\mathbb{R}$. Here the constants c^* and c_* are given by

$$c^* := 6\sqrt{2}\int_{\mathbb{R}} e^{\sqrt{2}t}(H')^2\,dt = 12\int_{\mathbb{R}} e^{2t}\,(\cosh t)^{-4}\,dt = 32\,,$$

and

$$c_* := \int_{\mathbb{R}}(H')^2\,dt = \sqrt{2}\int_{\mathbb{R}}(\cosh t)^{-4}\,dt = \frac{4}{3}\sqrt{2}\,.$$

This estimate explains why the functions q_j are required to be solutions of the Toda system (6.10), since, with such a choice, the leading term in (7.25) vanishes.

8 The moduli space of $2k$-ended solutions in dimension $N=2$

We first give a precise description of the solutions we are interested in. This requires some preliminary definitions.

We will denote by $\perp$ the rotation of angle $\pi/2$ in $\mathbb{R}^2$. At the heart of the description of the nodal set of the solutions is the set Λ of *oriented affine lines* in $\mathbb{R}^2$. Any element $\lambda\in\Lambda$ can be uniquely written as

$$\lambda := r\,\mathsf{e}^{\perp}+\mathbb{R}\,\mathsf{e}\,,$$

for some $r \in \mathbb{R}$ and some unit vector $\mathtt{e} \in S^1$. Observe that Λ is diffeomorphic to $\mathbb{R} \times S^1$ and writing $\mathtt{e} = (\cos\theta, \sin\theta)$ we get local coordinates (r, θ) in Λ. There is also a natural symplectic structure on Λ, which in these local coordinates, is given by

$$\omega = dr \wedge d\theta\,.$$

For all k, we denote by Λ^k the set of k oriented affine lines in $\mathbb{R}^2$. This set is diffeomorphic to $\mathbb{R}^k \times (S^1)^k$ and there exists a natural symplectic structure on Λ^k which, in local coordinates $(r_1, \ldots, r_k, \theta_1, \ldots, \theta_k)$, can be written as

$$\omega_k = dr_1 \wedge d\theta_1 + \ldots + dr_k \wedge d\theta_k\,. \tag{8.26}$$

We have the definition :

Definition 2. *A set of k oriented affine lines $\lambda_1, \ldots, \lambda_k \in \Lambda$ is said to be* ordered *if each λ_j can be written as*

$$\lambda_j := r_j\, \mathtt{e}_j^{\perp} + \mathbb{R}\, \mathtt{e}_j\,,$$

for some $r_j \in \mathbb{R}$ and some unit vector $\mathtt{e}_j \in S^1$ which can be written as $\mathtt{e}_j = (\cos\theta_j, \sin\theta_j)$ with

$$\theta_1 < \theta_2 < \ldots < \theta_k < 2\,\pi + \theta_1\,.$$

We will denote by Λ^k_{ord} the set of k ordered, oriented affine lines.

Assume that we are given k ordered oriented affine lines $\lambda_j := r_j\, \mathtt{e}_j^{\perp} + \mathbb{R}\, \mathtt{e}_j$, for $j = 1, \ldots, k$. It is easy to check that for all $R > 0$ large enough, there exists $s_j \in \mathbb{R}$ such that : The points $\mathtt{x}_j := r_j\, \mathtt{e}_j^{\perp} + s_j\, \mathtt{e}_j$ belong to a circle ∂B_R, with $R > 0$, the half lines

$$\lambda_j^+ := \mathtt{x}_j + \mathbb{R}^+\, \mathtt{e}_j\,, \tag{8.27}$$

are disjoint and included in $\mathbb{R}^2 \setminus B_R$ and the infimum of the distance between two distinct half lines λ_i^+ and λ_j^+ is larger than 4.

Given $k \geq 1$ and

$$(\lambda_1, \ldots, \lambda_{2k}) \in \Lambda^{2k}_{ord}\,,$$

we write $\lambda_j^+ = x_j + \mathbb{R}^+ \, \mathtt{e}_j$ and we define

$$u_{\lambda_1,\ldots,\lambda_{2k}}(\mathtt{x}) := \sum_{j=1}^{2k} (-1)^j \, \mathbb{I}_j(\mathtt{x}) \, u_0((\mathtt{x} - \mathtt{x}_j) \cdot \mathtt{e}_j^\perp) \, .$$

Here $\mathbb{I}_j$ is a partition of unity subordinate to the choice of the half lines $\lambda_1^+, \ldots, \lambda_{2k}^+$ and is designed in such a way that $\mathbb{I}_j$ is identicaly equal to 1 close to λ_j^+ and away from a compact.

Observe that, by construction the function $u_{\lambda_1,\ldots,\lambda_{2k}}$ is, away from a compact, asymptotic to one of the model solutions $\pm u_0$, whose nodal set are the half lines $\lambda_1^+, \ldots, \lambda_{2k}^+$. A simple computation shows that $u_{\lambda_1,\ldots,\lambda_{2k}}$ is not far from being a solution of (1.1) in the sense that $\Delta \, u_{\lambda_1,\ldots,\lambda_{2k}} + (1 - u^2_{\lambda_1,\ldots,\lambda_{2k}}) \, u_{\lambda_1,\ldots,\lambda_{2k}}$ is a function which decays exponentially to 0 at infinity.

We are interested in solutions which are asymptotic to $u_{\lambda_1,\ldots,\lambda_{2k}}$ for some choice of $(\lambda_1, \ldots, \lambda_{2k}) \in \Lambda^{2k}_{ord}$. More precisely, we have the :

Definition 3. *We will denote by $\mathfrak{M}_{2k}$ the set of solutions u of (1.1) which can be written as*

$$u - u_{\lambda_1,\ldots,\lambda_{2k}} \in e^{\delta |\mathtt{x}|} \, W^{2,2}(\mathbb{R}^2) \, , \tag{8.28}$$

for some choice of $(\lambda_1, \ldots, \lambda_{2k}) \in \Lambda^{2k}_{ord}$ and some $\delta < 0$ (close enough to 0).

The existence of the heteroclinic solution u_0 shows that $\mathfrak{M}_2$ is non empty. Indeed, it is enough to consider

$$u(\mathtt{x}) = u_0(\mathtt{x} \cdot \mathtt{e}^\perp - r) \, ,$$

for any unit vector field $\mathtt{e}$ and any $r \in \mathbb{R}$. As we already discussed in the introduction, the result of Theorem 8.2 provides solutions of (1.1) whose nodal set decomposes into k nearly parallel lines, very far from each other, this result (and the existence result in [8]) show that $\mathfrak{M}_{2k} \neq \emptyset$ for any $k \geq 1$.

Before, we can state the main result, we have to introduce the notion of *non-degenerate solution* in this context.

Definition 4. *A function $u \in \mathfrak{M}_{2k}$ is said to be* non-degenerate *if the linearized operator*

$$-\Delta - 1 + 3u^2,$$

is injective in the space $e^{\delta |\mathbf{x}|}\, \mathbb{L}^2(\mathbb{R}^2)$, for some $\delta < 0$.

With these definitions, one has the following :

Theorem 8.1. *Assume that $u \in \mathfrak{M}_{2k}$ is non-degenerate. Then, close to u, $\mathfrak{M}_{2k}$ is a smooth manifold of dimension $2k$.*

We define the mapping $\mathfrak{P} : \mathfrak{M}_{2k} \longrightarrow \Lambda^{2k}$ by

$$\mathfrak{P}(u) = (\lambda_1, \ldots, \lambda_{2k}),$$

if $u - u_{\lambda_1, \ldots, \lambda_{2k}} \in e^{\delta |\mathbf{x}|}\, W^{2,2}(\mathbb{R}^2)$, for some $\delta < 0$. Geometrically the meaning of the mapping $\mathfrak{P}$ should be clear : for any solution $u \in \mathfrak{M}_{2k}$, $\mathfrak{P}(u) \in \Lambda^{2k}$ correspond to the choice of $2k$ oriented affine lines that determine the asymptotics of the nodal set of u at infinity.

Near any nondegenerate elements of $\mathfrak{M}_{2k}$, we also have some information about the mapping $\mathfrak{P}$.

Theorem 8.2. *Assume that $u \in \mathfrak{M}_{2k}$ is non-degenerate. Then, there exists an open neighborhood of u in $\mathfrak{M}_{2k}$ whose image by $\mathfrak{P}$ is a Lagrangian submanifold of Λ^{2k} for the symplectic structure defined in (8.26).*

One can show that there is in reality less *freedom* than what might be initially expected in selecting the asymptotes for the nodal sets of the solutions of (1.1). Indeed, at regular points of $\mathfrak{M}_{2k}$, the image of $\mathfrak{P}$ is a $2k$ dimensional submanifold of Λ which is $4k$ dimensional.

Observe that the image of $\mathfrak{P}$ is further naturally constrained. In fact, if $u \in \mathfrak{M}_{2k}$ and

$$\mathfrak{P}(u) = (\lambda_1, \ldots, \lambda_{2k}),$$

with

$$\lambda_j = r_j\, \mathsf{e}_j^{\perp} + \mathbb{R}\, \mathsf{e}_j .$$

it follows from [22] that

$$\sum_{j=1}^{2k} \mathtt{e}_j = 0\,. \tag{8.29}$$

Moreover, pushing further the analysis in [22], we can also prove that

$$\sum_{j=1}^{2k} r_j = 0\,. \tag{8.30}$$

9 Standing waves for the nonlinear Schröedinger equation

We are now interested in the the classical semilinear elliptic problem

$$\Delta u - u + u^p = 0, \quad u > 0, \quad \text{in } \mathbb{R}^N \tag{9.31}$$

where $p > 1$. Equation (9.31) arises for instance as the standing-wave problem for the standard nonlinear Schrödinger equation

$$i\psi_t = \Delta_y \psi + |\psi|^{p-1}\psi,$$

corresponding to that of solutions of the form $\psi(\cdot, t) = u(\cdot)\, e^{-it}$. It also arises in nonlinear models in Turing's biological theory of pattern formation such as the Gray-Scott or Gierer-Meinhardt systems. The solutions of (9.31) which decay to zero at infinity are well understood. Problem (9.31) has a radially symmetric solution $w_N(y)$ which approaches 0 at infinity provided that $1 < p < \frac{N+2}{N-2}$. This solution is unique [28], and actually any positive solution to (9.31) which vanishes at infinity must be radially symmetric around some point [21]. Not much is known about solutions of this equation in the entire space which do not tend to 0 at infinity (while they are all known to be bounded, see [39]). For example, the solution w_N of (9.31) in $\mathbb{R}^N$ induces a solution in $\mathbb{R}^{N+1}$ which only depends on N variables. This solution vanishes asymptotically in all but one variable.

For simplicity, we now restrict ourselves to the case $N = 2$. We denote by K_0 the unique positive solution of

$$K_0'' - K_0 + K_0^p = 0,$$

with $K_0'(0) = 0$ and $\lim_\infty K_0(y) = 0$. This solution corresponds, in the phase plane, to a homoclinic orbit for the equilibrium 0.

Using a classical bifurcation argument [7], we can define a family of solutions K_δ of equation (9.31) which are symmetric with respect to both the x and y axis and which are 1-periodic in the x direction. Namely, $K_\delta(x, y) := K_\delta(x + t, y)$, for some $t \in \mathbb{R}$ depending on δ. To do so, we consider problem (9.31) with t-periodic conditions in x and we regard $t > 0$ as a bifurcation parameter. The linearized operator about K_0 is given by is

$$L = \partial_x^2 + \partial_y^2 - 1 + p\, K_0^{p-1}\,.$$

We consider the operator

$$L_0 = \partial_y^2 - 1 + p\, K_0^{p-1}\,.$$

Recall that L_0 has a one dimensional kernel spanned by K_0' since $L_0\, K_0' = 0$ as can be checked by taking the derivative of $K_0'' - K_0 + K_0^p = 0$. Since K_0' changes sign it can be associated to the least eigenvalue of $-L_0$ and in fact we have the following result which describes the spectrum of $-L_0$ [1] :

Lemma 9.1. *[1] The spectrum of the operator $-L_0$ is the union of the point spectrum, given by $\lambda_1 := -\frac{1}{4}(p-1)(p+3)$ with associated eigenfunction*

$$Z := K_0^{(p+1)/2}\,,$$

$\lambda_2 := 0$ which is associated to the eigenfunction K_0', and the continuous spectrum given by $[1, +\infty)$.

This result implies that the operator L (acting on functions which are t periodic in the x variable) has an eigenfunction

$$Z(y)\, \cos\left(\frac{2\pi}{t} x\right),$$

which corresponds to the eigenvalue

$$-\lambda_1 + \left(\frac{2\pi}{t}\right)^2$$

This eigenvalue crosses 0 precisely when $t = \frac{2\pi}{\sqrt{\lambda_1}}$. Crandall-Rabinowitz's bifurcation Theorem can then be applied to yield the existence of a continuum of solutions bifurcating from K_0 at this value of t. The bifurcating branch of solutions is denoted by K_δ where $\delta = 0$ corresponds to K_0. The functions K_δ is periodic in x with period $t_\delta = \frac{2\pi}{\sqrt{\lambda_1}} + \mathcal{O}(\delta)$. We also have the asymptotic expansion

$$K_\delta(x, y) = K_0(y) + \delta Z(y) \cos(\sqrt{\lambda_1} x) + \mathcal{O}(\delta^2 \, e^{-|y|}).$$

We refer to the functions K_δ in what follows as *Dancer solutions*.

Paralleling what is done for the Allen-Cahn equation it is possible to construct a new type of solutions of (9.31) in $\mathbb{R}^2$ which have multiple ends along which they are asymptotic to copies of K_{δ_j}, for some appropriate choice of the parameters δ_j(depending on j). We refer to [13] for precise statements.

10 Correspondence with geometric problems

One of the striking features of the above existence results which are purely PDE results, is that its counterparts can be found in geometric frameworks. Indeed, there are many examples where correspondence between solutions of (9.31) and those of some geometric problem can be drawn. To illustrate this, we will concentrate on what is perhaps the most spectacular one: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the study of entire solutions of (9.31). For simplicity we will restrict our attention to constant mean curvature surfaces in $\mathbb{R}^3$ which have embedded coplanar ends. In the following we will draw parallels between these geometric objects and families of solutions of (9.31).

Embedded constant mean curvature surfaces of revolution were found by Delaunay in the mid 19th century [9]. They constitute a smooth 1-parameter family of singly periodic surfaces D_τ, for $\tau \in (0,1]$, which interpolate between the cylinder $D_1 = S^1(1) \times \mathbb{R}$ and the singular surface $D_0 := \lim_{\tau\to 0} D_\tau$, which is the union of an infinitely many spheres of radius $1/2$ centered at each of the points $(0,0,n)$ as $n \in \mathbb{Z}$. The Delaunay surface D_τ can be parametrized by

$$X_\tau(x,y) = (\varphi(x)\,\cos y, \varphi(x)\,\sin y, \psi(x)) \in D_\tau \subset \mathbb{R}^3,$$

for $(x,y) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. Here the function φ is smooth solution of

$$(\varphi')^2 + \left(\frac{\varphi^2+\tau}{2}\right)^2 = \varphi^2,$$

and the function ψ is defined by

$$\psi' = \frac{\varphi^2+\tau}{2}.$$

As already mentioned, when $\tau = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as the cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way the single bump-line solution of (9.31) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between

$$\boxed{\begin{array}{c}\text{The cylinder}\\ D_1 = S^1 \times \mathbb{R}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The single bump-line}\\ (x,y) \longmapsto K_0(y)\end{array}}.$$

Let us denote by w_0 the unique entire, radially symmetric, decaying solution of (9.31). Inspection of the other end of the Delaunay family, namely when the parameter τ tends to 0, suggests the correspondence between

$$\boxed{\begin{array}{c}\text{The sphere}\\ S^1(1/2)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The radially symmetric solution}\\ (x,y) \longmapsto w_0(\sqrt{x^2+y^2})\end{array}}.$$

To justify this correspondence, let us observe that on the one hand, as the parameter τ tends to 0, the surfaces D_τ resemble a sequence of spheres of radius $1/2$ arranged along the x_3-axis which are connected together by small catenoidal necks. On the other hand an analogous solution of (9.31) can be built as follows. Let $S_R = \mathbb{R} \times (0, R)$ and consider a least energy (mountain pass) solution in $H^1(S_R)$ for the the energy

$$\frac{1}{2}\int_{S_R} |\nabla u|^2 + \frac{1}{2}\int_{S_R} u^2 - \frac{1}{p+1}\int_{S_R} u^{p+1},$$

for large $R > 0$, which we may assume to be even in y and with maximum located at the origin. For R very large, this solution, which satisfies zero Neumann boundary conditions, resembles half of the unique radial, decaying solution w_0 of (9.31). Extension by successive even reflections in x variable yields a solution to (9.31) which resembles a periodic array of radially symmetric solutions of (9.31), with a very large period, along the x-axis. While this is not known, these solutions may be understood as a limit of the branch solutions constructed by Dancer.

More generally, there is a natural correspondence between

$$\boxed{\begin{array}{c}\text{Delaunay surfaces}\\ D_\tau\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Dancer solutions}\\ (x,y) \longmapsto K_\delta(x,y)\end{array}}$$

To give further credit to this correspondence, let us recall that the Jacobi operator about the cylinder D_1 corresponds to the linearized mean curvature operator when nearby surfaces are considered as normal graphs over D_1. In the above parameterization, the Jacobi operator reads

$$J_1 = \frac{1}{\varphi^2}\left(\partial_x^2 + \partial_y^2 + 1\right).$$

In this geometric context, it plays the role of the linear operator defined in section 2 which is the linearization of (9.31) about the single bump-line solution w. Hence we have the correspondence

$$\boxed{\begin{array}{c}\text{The Jacobi operator}\\ J_1 = \frac{1}{\varphi^2}\left(\partial_x^2 + \partial_y^2 + 1\right)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The linearized operator}\\ L = \partial_x^2 + \partial_y^2 - 1 + p\,K_0^{p-1}\end{array}}$$

In our construction, the polynomially bounded kernel of the linearized operator L plays a crucial role. Similarly, the polynomially bounded kernel of the Jacobi operator J_1 has some geometric interpretation. Let us recall that we only consider surfaces whose ends are coplanar, the Jacobi fields associated to the action of rigid motions are then given by

$$(x, y) \longmapsto \cos x \qquad \text{and} \qquad (x, y) \longmapsto y \cos x,$$

which correspond respectively to the action of translation and the action of the rotation of the axis of the Delaunay surface D_1. Clearly, these Jacobi fields are the counterpart of the elements of the kernel of L which are given by

$$(x, y) \longmapsto \partial_y w(y) \qquad \text{and} \qquad (x, y) \longmapsto x\, \partial_y K_0(y),$$

since the latter are also generated using the invariance of the problem with respect to the same kind of rigid motions.

Two additional Jacobi fields associated to J_1 are given by

$$(x, y) \longmapsto \cos y \qquad \text{and} \qquad (x, y) \longmapsto \sin y,$$

which are associated to the existence of the family D_τ as τ is close to 1, as can be easily seen using a bifurcation analysis, in a similar way that the functions

$$(x, y) \longmapsto Z(y)\, \cos(\sqrt{\lambda_1} x) \qquad \text{and} \qquad (x, y) \longmapsto Z(y)\, \sin(\sqrt{\lambda_1} x),$$

are associated to the existence of Dancer solutions when the parameter δ is close to 0. These two bifurcation results have their origin in the fact that we have the correspondence between

$$\boxed{\begin{array}{c}\text{The ground state 1 of}\\ \partial_y^2 + 1\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The first eigenfunction } Z \text{ of}\\ \partial_y^2 - 1 + p\, K_0^{p-1}\end{array}}$$

both of them associated to negative eigenvalues. The fact that the least eigenvalue of these operators is negative is precisely the reason why a bifurcation analysis can be performed and gives rise to the existence of Delaunay surfaces close to D_1 or Dancer's solutions close to the bump-line w.

With these analogies in mind, we can now *translate* our main result into the constant mean curvature surface framework. The result of Theorem 8.2 corresponds to the connected sum of finitely many copies of the cylinder $S^1(1) \times \mathbb{R}$

which have a common plane of symmetry. The connected sum construction is performed by inserting small catenoidal necks between two consecutive cylinders and this can be done in such a way that the ends of the resulting surface are coplanar. Such a result, in the context of constant mean curvature surfaces, follows at once from [32]. It is observed that, once the connected sum is performed the ends of the cylinder have to be slightly bent and moreover, the ends cannot be kept asymptotic to the ends of right cylinders but have to be asymptotic to Delaunay ends with parameters close to 1, in agreement with the result of Theorem 8.2.

However there is a major difference. The Toda system which governs the level sets has found no analogy in the constant mean curvature surfaces. This is mainly due to the strong interactions in the elliptic equations.

Another (older) construction of complete noncompact constant mean curvature surfaces was performed by N. Kapouleas [25] (see also [31]) starting with finitely many halves of Delaunay surfaces with parameter τ close to 0 which are connected to a central sphere. The corresponding solutions of (9.31) have recently been constructed by A. Malchiodi in [29].

It is well known that the story of complete constant mean curvature surfaces in $\mathbb{R}^3$ parallels that of complete locally conformally flat metrics with constant, positive scalar curvature. Therefore, it is not surprising that there should be a correspondence between these objects in conformal geometry and solutions of (9.31). For example, Delaunay surfaces and Dancer solutions should now be replaced by Fowler solutions which correspond to constant scalar curvature metrics on the cylinder $\mathbb{R} \times S^{N-1}$ which are conformal to the product metric $dt^2 + g_{S^{N-1}}$, when $N \geq 3$. These are given by

$$v^{\frac{4}{N-2}}(dt^2 + g_{S^{N-1}}),$$

where $t \longmapsto v(t)$ is a smooth positive solution of

$$(v')^2 - v^2 + \tfrac{N-2}{N}\, v^{\frac{2N}{N-2}} = -\tfrac{2}{N}\, \tau^2 .$$

When $\tau = 1$ and $v \equiv 1$ the solution is a straight cylinder while as τ tends to 0 the metrics converge on compacts to the round metric on the unit sphere. The connected sum construction for such Fowler type metrics was performed by R. Mazzeo, D. Pollack and K. Uhlenbeck [34] (where it is called the dipole construction). N. Kapouleas' construction mentioned above was initially performed by R. Schoen [41] (see also R. Mazzeo and F. Pacard [31]).

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