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NONLINEAR EXACT CONTROLLABILITY AND NONLINEAR STABILIZATION OF HYPERBOLIC OR HYPERBOLIC-LIKE EVOLUTION EQUATIONS

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Abstract

The present manuscript provides a comprehensive and detailed version of the content of the lectures delivered by the author at the Winter School on Nonlinear Analysis, Instituto de Matematica/UFRJ, August 3-7, 2009, Rio de Janeiro, Brazil. They encompassed two distinct yet related topics: (1) nonlinear global exact controllability, in the space of finite energy, for a second-order hyperbolic equation with Neumann-boundary control; and(2) sharp uniform decay rates, at the $L_2(\Omega)$-level, of the Schrödinger equation with nonlinear boundary dissipation.

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Contents

Orientation. The present manuscript provides a comprehensive and detailed version of the content of the six one-hour lectures delivered by the author at the Winter School on Nonlinear Analysis, Instituto de Matematica, Universita Federale de Rio de Janeiro, August 3-7, 2009, Rio de Janeiro, Brazil. They encompassed two distinct yet related topics: (1) nonlinear global exact controllability, in the space of finite energy, for a second-order hyperbolic equation with Neumann-boundary control; and (2) sharp uniform decay rates, at the $L_2(\Omega)$-level, of the Schrödinger equation with nonlinear boundary dissipation. Each topic is given a separate treatment.

As made clear in the exposition below, while focused on a specific, 'concrete' dynamics, the strategy employed for the resolution of each problem, as well as the technical apparatus required, are actually applicable *mutatis mutandis* to other hyperbolic or hyperbolic-like dynamics. This is also confirmed by the references mentioned for each problem.

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TOPIC I: Global Exact Controllability on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of Semilinear Wave Equations with Neumann $L_2(0,T; L_2(\Gamma_1))$-Boundary Control[1]

Abstract

We provide four proofs that the semilinear wave equation with globally Lipschitz nonlinear term is exactly controllable on the finite energy space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2(0,T; L_2(\Gamma_1))$-boundary controls exercised in the Neumann B.C., whenever the corresponding linear equation satisfies the same property; and on the same time interval. One proof is based on the global inversion approach applied to the original, controlled problem, which was proposed in [L-T.4]. In contrast, the other three proofs are based, instead, on a uniform continuous observability inequality of the dual uncontrolled problem. Precise links are exhibited, and exploited, among these four approaches. All proofs are essentially operator-theoretic with a PDE-interpretation. A common thread of the four proofs is an analysis of suitable families of collectively compact operators, which then admit uniform inversions in the style of [L-T.4]. This way compactness/uniqueness arguments are entirely dispensed with.

1 Controlled Dynamics: Wave Equation with Neumann Control

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. In Ω we consider the following semilinear problem for the wave equation in the solution $w(t,x)$, with Neumann-boundary control u based on Σ_1:

$$\begin{cases} w_{tt} = \Delta w + f(w) & \text{in } (0,T] \times \Omega \equiv Q; & (1.1a) \\ w(0,\cdot) = w_0, w_t(0,\cdot) = w_1 & \text{in } \Omega; & (1.1b) \\ w|_{\Sigma_0} \equiv 0; \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = u & \text{in } (0,T] \times \Gamma_i \equiv \Sigma_i, \ i = 0,1. & (1.1c) \end{cases}$$

[1]Control Theory of Partial Differential Equations, Volume 242, Lecture Notes in Pure and Applied Mathematics, Taylor and Francis Group, Chapman and Hall/CRC, 2005, Chapter 19, 273–336.

Here $\nu(x)$ is the unit outward normal on $x \in \Gamma$. The standing *assumption* on the non-linearity f is a global Lipschitz condition [Ro.1, p. 108] as follows:

$f : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function with first derivative a.e. which is uniformly bounded on $\mathbb{R}$:

$$|f'(r)| \leq \text{const.} \qquad \text{a.e. } r \in \mathbb{R}. \tag{1.2}$$

Remark 1.1. To streamline the treatment, we shall also *assume* throughout the paper that: $f(0) = 0$, in order not to be bothered by what are essentially benign terms in the analysis. This assumption of convenience $f(0) = 0$ will be maintained throughout this paper, unless otherwise noted.

Exact controllability. Under some geometrical conditions and for all $T >$ some universal $T_0 > 0$, the linear problem (1.1) with $f \equiv 0$ is exactly controllable over $[0, T]$, on the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of controls $u \in L_2(0, T; L_2(\Gamma_1))$ [B-L-R.1], [Lag.1], [Lio.1], [L-T.2], [Ta.1]–[Ta.3], etc. (The most general results are in [B-L-R.1]: one specific such set of conditions is given in (3.6), (3.7) below, for sake of concreteness.) The result below extends the same (global) exact controllability property to the semilinear problem (1.1) subject to (1.2), over the same time interval.

Theorem 1.1. *Let $T > 0$ be a time for which exact controllability of the linear problem with $f \equiv 0$ holds true in the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2(0, T; L_2(\Gamma_1))$-controls u. Let f satisfy assumption (1.2). Then, a similar exact controllability result holds true for the original problem (1.1) for the same time T: for any pair $\{w_0, w_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of initial conditions, and any pair $\{\tilde{w}_0, \tilde{w}_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of target states, there exists a suitable control function $u \in L_2(0, T; L_2(\Gamma_1))$ such that the corresponding solution $\{w, w_t\}$ of problem (1.1) satisfies $\{w(T, \cdot), w_t(T, \cdot)\} = \{\tilde{w}_0, \tilde{w}_1\}$.*

Literature. We confine only to second-order hyperbolic equations, in fact for dim $\Omega \geq 2$. For dim $\Omega = 1$, we refer to [C-K-L.1]. Theorem 1.1 above is new.

In the case of *Dirichlet* boundary control: $w|_{\Sigma_0} \equiv 0$, $w|_{\Sigma_1} = u$, exact controllability of the semilinear wave equation (1.1a) under the same assumption (1.2) on the non-linearity was given in [L-T.4]: in the state space $H_0^1(\Omega) \times L_2(\Omega)$ within the class of $H_0^1(0,T;L_2(\Gamma_1))$-controls; and in the state space $L_2(\Omega) \times H^{-1}(\Omega)$ within the class of $L_2(0,T;L_2(\Gamma_1))$-controls. Hence, by interpolation, in the state space $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$, $0 \leq \gamma \leq 1$, $\gamma \neq \frac{1}{2}$ (resp., in the state space $H_{00}^{\frac{1}{2}}(\Omega) \times [H_{00}^{\frac{1}{2}}(\Omega)]'$ for $\gamma = \frac{1}{2}$) within the class of $H_0^\gamma(0,T;L_2(\Gamma_1))$-controls (resp., within the class of $H_{00}^{\frac{1}{2}}(0,T;L_2(\Gamma_1))$-controls, for $\gamma = \frac{1}{2}$). These topological settings are optimal [L-T.1], [L-L-T.1], [L-T.7], [Lio.1], [L-T.9, Chapter 10, Section 10.5]. Paper [L-T.4] was prompted by paper [Z.1] which used a Schauder fixed-point theorem approach (the approach of using a fixed point theorem is a well-established strategy that goes back to the mid-sixties [H.1] in the case of finite-dimensional systems). This way, however, paper [Z.1] obtains the above exact controllability results with Dirichlet boundary control, but only in the range $0 < \gamma < 1$, and under an *a-priori* assumption of a uniqueness result, for which [R.1] could provide verification, only with control applied on the entire boundary. Since then, appropriate uniqueness results have become available also in the case where the control is applied on a suitable portion of the boundary [K-K.1], [L-T-Z.1], [Zh.1]. The limit cases $\gamma = 0$ and $\gamma = 1$—the most interesting ones—are explicitly excluded from [Z.1], as the requirement $0 < \gamma < 1$ is essential to its treatment. The case $\gamma = 0$ is excluded for lack of compactness in the Schauder fixed-point argument. The case $\gamma = 1$ is excluded as not enough information is available in passing to the limit in the compactness/uniqueness argument [Z.1, Remark 2.6, p. 375]. The same restriction $0 < \gamma < 1$, this time without the additional uniqueness assumption, is present in the result of [Zh.1, Theorem 4.2, p. 831] for the same wave model, as this work falls into [Z.1]. In contrast, the approach of [L-T.4] is based on a *global inversion theorem*, as carried out for an abstract equation under assumptions which are intended to capture dynamical properties of hyperbolic and hyperbolic-like PDE problems (waves, plates, etc.). The aforementioned results for semilinear wave equations with Dirichlet controls are then obtained in [L-T.4] by specializing the abstract results. The global inversion theorem continues to be the carrier of the analysis of the present paper with Neumann boundary control, as well, at least in its Part I. Moreover, Parts II, III, IV—which deal with the dual uncontrolled linearized problem—either proceed through an approach dual to Part I, or else eventually fall into Part I.

Orientation. As explained above, the (global) exact controllability of problem (1.1) with Neumann-boundary control is studied in Part I of this paper (Sections 2–4) by virtue of the *global inversion approach* of [L-T.4], here revisited to fit present circumstances. Proposition 4.1(a) below provides an operator-theoretic characterization for the solution of the (global) exact controllability problem of system (1.1), in terms of the global inversion (homeomorphism) of the map: $u \to [u+\Lambda_T(u)]$. A sufficient condition for this to occur is that the Frechet derivative $[I+\Lambda'_T(u)]$ of this map has the inverse which is uniformly bounded in u,

$$\|[I+\Lambda'_T(u)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} \leq \text{const.}, \ \ \forall u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}. \tag{1.3}$$

This result is contained in (4.18b) of Proposition 4.1(b), indeed in its version (4.64) of Lemma 4.6. On the other hand, Proposition 4.3, Eqn. (4.39a) yields that the operator $[I+\Lambda'_T(u)]$ coincides with the operator $\mathcal{M}_T[y_u]$ modulo an (algebraic) isomorphism as a factor. Here, $\mathcal{M}_T[y_u]$ is the map $\tilde{u} \to z(T) = \mathcal{M}_T[y_u]\tilde{u}$ in (3.15), (4.33), with u as a parameter, where z is the solution of the corresponding linearized problem with I.C. $z_0 = 0$; see (2.18), whose PDE-version is given by (2.37). Thus, the uniform bounded inversion $[I+\Lambda'_T(u)]^{-1}$ is, in turn, equivalent to the condition that the adjoint operator $\mathcal{M}^*_T[y_u]$ *be bounded below uniformly in* u as in (5.5), see Proposition 5.1. The latter condition is then ultimately verified as a *continuous observability inequality of the linearized problem, uniformly with respect to the potential* $q(t,x) \in L_\infty(Q)$ *in a given ball*, where, in the relevant case, $q = f'(w(t,x))$. This is inequality (5.10) for the dual uncontrolled ϕ-problem in (3.9) = (6.3), actually proved as inequality (A.21) of Theorem A.3 in Appendix A. Two different strategies are then pursued here to establish (global) exact controllability of (1.1), with the second strategy receiving three different implementations.

Part I (Sections 2–4), following [L-T.4], extracts some key properties of the linearization system.

A first group of properties consists of structural properties of the operators defining the linearization system: they are labelled here (P.1) = (2.14), (P.2) = (2.27) and (P.3) = (2.34). A second group of properties consists of control-theoretic properties labelled in Section 3 as (C.1) (exact controllability of the linear problem) and (C.2) (an approximate controllability property of the linearized system). Following the strategy of [L-T.4], it is here shown in Part I that, cumulatively, properties (P.1), (P.2), (P.3), (C.1), (C.2) imply the desired uniform bound (1.3) = (4.64), whereby then exact controllability of the semilinear system (1.1) is established via a global inversion theorem, as stated in Theorem 1.1. Part I deals entirely with the *original controlled* problem.

A second strategy centers on establishing the *uniform continuous observability inequality* (5.5) of the linearized problem, involving $\mathcal{M}_T^*[q]$, uniformly in the potential $q \in L_\infty(Q)$ in a fixed finite ball. To this end, three different implementations may be pursued, see Section 5, where we note the following equivalences: (5.5) $\Longleftrightarrow$ (5.10) $\Longleftrightarrow$ (5.11) (Proposition 5.1 and Remark 5.1).

The first such implementation makes up Part II [Sections 6, 7]: unlike Part I, it deals entirely this time with the *dual uncontrolled* problem (3.9) = (6.3) and seeks to prove *directly* the uniform COI (5.11) of the linearized problem by the operator-theoretic properties (Lemma 4.4 and Lemma 4.5) established in Part I.

A second implementation—pursued in Part III (Section 8)—consists again in showing the uniform COI (5.5) or (5.10), however, this time by seeking to prove the *equivalent* inequality (8.4) = (8.7) on traces or their operator representations. But, again, it does so *not* directly, but by proving, in turn, an *equivalent* inequality (8.8) (*equivalence via an isomorphism*), which is precisely the operator-theoretic inequality (8.9), in turn, established via Lemma 4.4 and Lemma 4.5 of Part I.

A third implementation of the second strategy makes up Part IV (Section 9); this also deals entirely with the *dual uncontrolled* problem and seeks likewise to establish the uniform COI (5.5) $\Longleftrightarrow$ (5.10) $\Longleftrightarrow$ (5.11). However, unlike Part III, it achieves this goal by proving the trace inequality (or its operator reformulation) (8.4) = (8.7) this time *directly*, not its *equivalent* (via an isomorphism) version (8.8). Indeed, there are some advantages in pursuing this route. One still falls in the "collectively compact family of operators" mode that characterizes this entire paper (following [L-T.4]) and consequent application of its basic theory [A.1]. However, now only the counterpart of Lemma 4.4 is needed. The counterpart of Lemma 4.5 is entirely dispensed with on this Part IV. In short: Part IV requires only one uniform inversion (as in Lemma 4.4), not two uniform inversions (as in Lemma 4.5). This way, it provides an alternative proof of Theorem 1.1. Fig. 5.1 in Section 5 illustrates graphically the four different paths of the four proofs.

The paper closes by presenting, in Section 10, some comparison comments, also in light of a generalization of the Dirichlet-boundary control case with $L_2(\Sigma_1)$-control (case $\gamma = 0$) for a far more general model than in past literature, based on the extension in the Riemannian setting [T-Y.1], of prior sharp results in [K-K.1], [L-T-Z.1], [Zh.1] in the Euclidean setting.

Some essential, but more standard, background material is collected, sometimes for completeness, in Appendices A-B-C.

Part I: Proof of Theorem 1.1 by the Abstract Global Inversion Approach of [L-T.4] Via the Controlled Problem (Sections 2 through 4)

2 Abstract Formulation of Problem (1.1) and Corresponding Structural Properties

Abstract model. Problem (1.1) can be cast into the following abstract equation with $y(t) = [w(t), w_t(t)]$:

$$\dot{y} = Ay + F(y) + Bu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y \tag{2.1}$$

to be interpreted as specified below, where the relevant spaces and operators are identified.

(i) First, let $\mathcal{A} : L_2(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Omega)$ be the positive, self-adjoint operator defined by $\mathcal{A}h = -\Delta h$, $\mathcal{D}(\mathcal{A}) \equiv \{h \in H^2(\Omega), h|_{\Gamma_0} \equiv \frac{\partial h}{\partial \nu}|_{\Gamma_1} \equiv 0\}$. Then, $-\mathcal{A}$ generates a strongly continuous (s.c.) cosine operator $C(t)$ on $L_2(\Omega)$ with 'sine'-operator $S(t) = \int_0^t C(\tau)d\tau$. With $y(t) = [w(t), w_t(t)]$, the operator A in model (2.1) is given by

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}, \ \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}),$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega) = \{h \in H^1(\Omega), h|_{\Gamma_0} = 0\}. \tag{2.2}$$

Then, A generates the unitary s.c. group e^{At} given by

$$e^{At} = \begin{bmatrix} C(t) & S(t) \\ -\mathcal{A}S(t) & C(t) \end{bmatrix}, \ \|C(t)\|_{L(L_2(\Omega))} + \|\mathcal{A}^{\frac{1}{2}} S(t)\|_{L(L_2(\Omega))} \leq \text{const.}, \ t \in \mathbb{R}, \tag{2.3}$$

on either of the spaces

$$H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \ \text{ or } \ Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]', \tag{2.4}$$

which are topologically equivalent to $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ and $L_2(\Omega) \times [H^1_{\Gamma_0}(\Omega)]'$, respectively, where $[\quad]'$ denotes duality with respect to $L_2(\Omega)$ as a pivot space.

(ii) Next, let N be the Neumann map (harmonic extension of boundary data) defined by

$$Ng = h \Longleftrightarrow \left\{ \Delta h = 0 \text{ in } \Omega;\ h|_{\Gamma_0} = 0,\ \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_1} = g \right\}; \tag{2.5a}$$

$$N:\ \text{continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{3}{4}-\epsilon}),\ \forall\ \epsilon > 0 \tag{2.5b}$$

[L-M.1], [L-T.9]. Then, with $U = L_2(\Gamma_1)$, the operator B in model (2.1) is

$$Bu = \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix},\ A^{-1}Bu = \begin{bmatrix} -Nu \\ 0 \end{bmatrix},\ A^{-1}B \in L(U; H), \tag{2.6}$$

where $\mathcal{A}$ in $\mathcal{A}Nu$ is actually the isomorphic extension, say $L_2(\Omega) \to [\mathcal{D}(\mathcal{A})]'$ of the original operator $\mathcal{A}$ defined above (2.2).

(iii) Finally, the non-linear operator F in model (2.1) and its Frechet derivative $F'[\eta] \in L(Y)$ at the point $\eta \in Y$ are given by

$$F(y) = \begin{bmatrix} 0 \\ f(y_1(\,\cdot\,)) \end{bmatrix},\ F'[\eta](y) = \begin{bmatrix} 0 \\ f'(\eta_1(\,\cdot\,))y_1(\,\cdot\,) \end{bmatrix},\ F(0) = 0, \tag{2.7}$$

$y = [y_1, y_2] \in Y$, $\eta = [\eta_1, \eta_2] \in Y$, with f the function in (1.1a). Because of assumption (1.2) on f, we have that: F is a continuous operator on H or on Y, and, moreover, that

$$\|F'[\eta]\|_{L(H)} + \|F'[\eta]\|_{L(Y)} \leq \text{const., uniformly in } \eta \in Y, \tag{2.8}$$

while $F(0) = 0$ is a consequence of $f(0) = 0$, Remark 1.1.

Additional properties. Instead of the differential version (2.1) on, say, $[\mathcal{D}(A^*)]'$, duality with respect to H as a pivot space, we consider its variation of parameter version

$$y(t) = e^{At}y_0 + (\mathcal{L}u)(t) + (\mathcal{R}Fy)(t),\ y(t) = [w(t), w_t(t)]; \tag{2.9}$$

$$y(T) = e^{AT}y_0 + \mathcal{L}_T u + \mathcal{R}_T Fy. \tag{2.10}$$

Regarding the operators $\mathcal{L}$ and $\mathcal{L}_T$. The following critical regularity property of problem (1.1) with $f \equiv 0$ and $\{w_0, w_1\} = 0$ is known

$$\begin{bmatrix} w_{f\equiv 0}(t) \\ (w_t)_{f\equiv 0}(t) \end{bmatrix} = (\mathcal{L}u)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$$

$$= \begin{bmatrix} \mathcal{A}\displaystyle\int_0^t S(t-\tau)Nu(\tau)d\tau \\ \mathcal{A}\displaystyle\int_0^t C(t-\tau)Nu(\tau)d\tau \end{bmatrix} \tag{2.11a}$$

$$: \text{ continuous } L_2(0,T;L_2(\Gamma_1)) \rightarrow C([0,T]; H^{\alpha}_{\Gamma_0}(\Omega) \times H^{\alpha-1}(\Omega)) \tag{2.11b}$$

$$: \text{ closed } L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}) \rightarrow L_2(0,T;H), \tag{2.11c}$$

$\alpha = \frac{3}{4}$ for Ω a parallelepiped, $\alpha = \frac{2}{3}$ for a general smooth bounded domain [L-T.3], [L-T.7], [Ta.5]. The corresponding result with $\alpha > \frac{3}{4}$ is false for dim $\Omega \geq 2$ [L-T.3]. For the formulas in (2.11a), see [Tr.1], [L-T.1], [L-T.7], [L-T.9]. Moreover, we have

$$H^{\alpha}_{\Gamma_0}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}), \quad H^{\alpha-1}(\Omega) = [H^{1-\alpha}_0(\Omega)]' \equiv [H^{1-\alpha}(\Omega)]' = [\mathcal{D}(\mathcal{A}^{(1-\alpha)/2})]', \tag{2.12}$$

since $H^{\rho}(\Omega) \equiv H^{\rho}_0(\Omega)$, $0 \leq \rho \leq \frac{1}{2}$ [L-M.1]. Thus

$$\begin{bmatrix} w_{f\equiv 0}(T) \\ (w_t)_{f\equiv 0}(T) \end{bmatrix} = \mathcal{L}_T u \equiv (\mathcal{L}u)(T)$$

$$: \text{ continuous } L_2(0,T;L_2(\Gamma_1)) \rightarrow H^{\alpha}_{\Gamma_0}(\Omega) \times H^{\alpha-1}_{\Gamma_0}(\Omega), \tag{2.13a}$$

$$\text{closed } L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \rightarrow H. \tag{2.13b}$$

When viewing $\mathcal{L}_T$ as a closed operator as in (2.13b), we shall consider $\mathcal{D}(\mathcal{L}_T)$ as a Hilbert space endowed with the graph norm.

For the analysis of the semilinear exact controllability problem to be carried out in Section 4, the following property—identified as (P.1)—plays a critical role:

(P.1) (key property of $\mathcal{L}$)

$$\mathcal{L}: \text{ compact } L_2(0,T;L_2(\Gamma_1)) \to \mathcal{E}_T \equiv L_2(0,T;Y);\ Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'. \tag{2.14}$$

Property (2.14) is obtained by application of Aubin's Lemma [A.2], which requires (the weaker version, L_2 in time of) (2.11b), as well as

$$\left(\frac{d}{dt}\mathcal{L}u\right)(t) = \begin{bmatrix} \mathcal{A}\displaystyle\int_0^t C(t-\tau)Nu(\tau)d\tau \\ \mathcal{A}\left[Nu(t) - \mathcal{A}\displaystyle\int_0^t S(t-\tau)Nu(\tau)d\tau\right] \end{bmatrix} \tag{2.15a}$$

$$: \text{ continuous } L_2(0,T;L_2(\Gamma_1))$$

$$\to L_2(0,T;H^{\alpha-1}(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'), \tag{2.15b}$$

see (2.11c), along with compactness in space, from $H^\alpha(\Omega) \times H^{\alpha-1}(\Omega) \to Y$.

Regarding the operators $\mathcal{R}$, $\mathcal{R}_T$. We recall from (2.4) that $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ and $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Then

$$(\mathcal{R}g)(t) = \int_0^t e^{A(t-\tau)}g(\tau)d\tau: \quad \text{continuous } L_1(0,T;Y) \to C([0,T];Y); \tag{2.16a}$$

$$\text{continuous } L_1(0,T;H) \to C([0,T];H). \tag{2.16b}$$

$$\mathcal{R}_T g = (\mathcal{R}g)(T): \quad \text{continuous } L_1(0,T;Y) \to Y; \tag{2.17a}$$

$$\text{continuous } L_1(0,T;H) \to H. \tag{2.17b}$$

The linearized problem. The linearized problem corresponding to Eqn. (2.1) is given by

$$\dot{z} = Az + F'[\eta]z + Bu, \quad z(0) = z_0 \in H,\ \eta \in Y; \tag{2.18}$$

$$\begin{aligned} z(t) &= e^{At}z_0 + (\mathcal{L}u)(t) + (\mathcal{K}[\eta]z)(t); && (2.19) \\ z(T) &= e^{AT}z_0 + \mathcal{L}_T u + \mathcal{K}_T[\eta]z && (2.20) \end{aligned}$$

(while its PDE-version will be given in (2.37) below), with $\mathcal{L}$, $\mathcal{L}_T$ defined in (2.11), (2.13). Moreover, $\mathcal{K}[\eta]g$ and $\mathcal{K}_T[\eta]g$, are defined as follows for $\eta = [\eta_1, \eta_2]$, $g = \{g_1, g_2\} \in \mathcal{E}_T \equiv L_2(0,T; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$, via (2.7), (2.3), (2.16):

$$\begin{aligned} \mathcal{K}[\eta] &= \mathcal{R}F'[\eta]; \quad (\mathcal{K}[\eta]g)(t) = \int_0^t e^{A(t-\tau)}F'[\eta(\tau)]g(\tau)d\tau && (2.21) \\ &= \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix} d\tau \\ &= \begin{bmatrix} \displaystyle\int_0^t S(t-\tau)f'(\eta_1(\tau))g_1(\tau)d\tau \\ \displaystyle\int_0^t C(t-\tau)f'(\eta_1(\tau))g_1(\tau)d\tau \end{bmatrix} && (2.22a) \end{aligned}$$

$$: \text{continuous } \mathcal{E}_T \equiv L_2(0,T;Y) \to C([0,T];H) \subset \mathcal{E}_T. \qquad (2.22b)$$

by (1.2) or (2.8). Recalling $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, we obtain via also (1.2) used in (2.22),

$$\begin{aligned} \|\mathcal{K}[\eta]g\|_{\mathcal{E}_T} &\leq \sqrt{T}\|\mathcal{K}[\eta]g\|_{C([0,T];H)} \\ &\leq C\|g_1\|_{L_2(0,T;L_2(\Omega))} \leq C\|g\|_{\mathcal{E}_T}, \text{uniformly in} \\ &\quad \eta \in \mathcal{E}_T \equiv L_2(0,T;L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'); && (2.23) \end{aligned}$$

$$\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta] : \text{ continuous } \mathcal{E}_T \to H, \text{ uniformly in } \eta \in \mathcal{E}_T. \qquad (2.24a)$$

$$\mathcal{K}_T[\eta]g = \int_0^T e^{A(T-t)}F'[\eta(t)]g(t)dt = \int_0^T e^{A(T-t)} \begin{bmatrix} 0 \\ f'(\eta_1(t))g_1(t) \end{bmatrix} dt. \qquad (2.24b)$$

Similarly, from (2.22),

$$\left(\frac{d}{dt}\mathcal{K}[\eta]g\right)(t) = \begin{bmatrix} \displaystyle\int_0^t C(t-\tau)f'(\eta_1(\tau))g_1(\tau)d\tau \\ \displaystyle f'(\eta_1(t))g_1(t) - \mathcal{A}\int_0^t S(t-\tau)f'(\eta_1(\tau))g_1(\tau)d\tau \end{bmatrix}; \tag{2.25}$$

$$\begin{aligned} &\left\|\frac{d}{dt}\mathcal{K}[\eta]g\right\|_{L_2(0,T;L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')} \\ &\qquad \le \quad C\|g_1\|_{L_2(0,T;L_2(\Omega))} \le C\|g\|_{\mathcal{E}_T}, \\ &\qquad\qquad \text{uniformly in } \eta \in \mathcal{E}_T \equiv L_2(0,T;L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'). \end{aligned} \tag{2.26}$$

Again, via Aubin's Lemma [A.2], which uses (2.22b) or (2.23) and (2.26), we obtain the following property of $\mathcal{K}[\eta] \equiv \mathcal{R}F'[\eta]$, which plays a critical role in the subsequent analysis of the semilinear exact controllability problem in Section 4:

(P.2) (key property of $\mathcal{K}[\eta] \equiv \mathcal{R}F'[\eta]$) With $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, not only do we have that

$$\mathcal{K}[\eta] = \mathcal{R}F'[\eta]: \text{ compact } \mathcal{E}_T \to \mathcal{E}_T, \text{ for each } \eta \text{ fixed } \in \mathcal{E}_T \equiv L_2(0,T;Y), \tag{2.27}$$

but, in addition, we have that:

(a)

$$\begin{cases} \text{the family of operators } \{\mathcal{K}[\eta]\}_{\eta\in\mathcal{E}_T} \text{ is collectively compact on the} \\ \text{space } \mathcal{E}_T \equiv L_2(0,T;Y)\text{: that is [A.1, p. 4], the set union } \bigcup_{\eta\in\mathcal{E}_T}\mathcal{K}[\eta] \\ \text{(unit ball of } \mathcal{E}_T) \text{ is a pre-compact set in } \mathcal{E}_T, \end{cases} \tag{2.28}$$

where the union of the image of the unit ball of $\mathcal{E}_T$ under the operator $\mathcal{K}[\eta]$ is taken over all $\eta \in \mathcal{E}_T$. The validity of property (2.28) is, of course, due to the fact that estimates (2.23) and (2.26) are *uniform* in $\eta \in \mathcal{E}_T$.

Moreover, $\mathcal{K}[\eta]$ satisfies the following additional property.

(b) For any sequence $\eta_n \in \mathcal{E}_T \equiv L_2(0,T;Y), Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, we can extract a subsequence η_{n_k} such that

$$\mathcal{K}[\eta_{n_k}] \equiv \mathcal{R}F'[\eta_{n_k}] \to \mathcal{K}^0 \equiv \mathcal{R}F_0 \text{ strongly in } \mathcal{E}_T \equiv L_2(0,T;Y), \tag{2.29a}$$

$$(K^0 g)(t) = (\mathcal{R}F_0 g)(t) = \int_0^t e^{A(t-\tau)} F_0(\tau) g(\tau) d\tau \tag{2.29b}$$

$$= \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ f_0(\tau) g_1(\tau) \end{bmatrix} d\tau, \tag{2.29c}$$

for a suitable operator $F_0 \in L(Y)$, possibly depending on the subsequence. The proof of property (2.29) follows as in [L-T.4, Proposition 3.3(b), p. 129]. In short: given $\eta_n \in \mathcal{E}_T$, assumption (1.2) yields $|f'(\eta_{n,1})| \leq$ const., uniformly in n, so that by Alaoglou's Theorem one can extract a subsequence $\eta_{n_k,1} \in L_2(0,T;L_2(\Omega))$ such that $f'(\eta_{n_k,1}) \to$ some f_0 in $L^\infty(\mathbb{R})$-weak star. One then defines

$$F_0 y \equiv \begin{bmatrix} 0 \\ f_0(\,\cdot\,) y_1 \end{bmatrix}, \ \|F_0 y\|_Y = \|f_0(\,\cdot\,) y_1\|_{[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \leq c\|f_0(\,\cdot\,) y_1\|_{L_2(\Omega)}$$

$$\leq c\|y_1\|_{L_2(\Omega)} \leq c\|y\|_Y. \tag{2.30}$$

Then, one obtains that

$$\begin{bmatrix} \mathcal{A}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{A}^{\frac{1}{2}} \end{bmatrix} \mathcal{K}[\eta_{n_k}] \to \begin{bmatrix} \mathcal{A}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{A}^{\frac{1}{2}} \end{bmatrix} \mathcal{R}F_0 \text{ weakly in } \mathcal{E}_T \equiv L_2(0,T;Y), \tag{2.31}$$

as well as

$$\frac{d\mathcal{K}[\eta_{n_k}]}{dt} \to \frac{d\mathcal{R}F_0}{dt} \text{ weakly in } \mathcal{E}_T \equiv L_2(0,T;Y). \tag{2.32}$$

Then, as a consequence of the weak converge (2.31) (that takes care of the "space"-coordinates) and of (2.32) (that takes care of the "time"-coordinate), as well as of the compactness of $\mathcal{A}^{-\frac{1}{2}}$ on $L_2(\Omega)$, we deduce that the strong convergence (2.29) holds true, as desired, and property (P.2)(b) is established.

Consequence of (P.2)(a)–(b). As a consequence of the collectively compact property (2.28) and of the strong convergence (2.29), it then follows that [A.1, p. 5]

$$\mathcal{K}^0 \text{ is compact on } \mathcal{E}_T;\ \|\mathcal{K}[\eta]\|_{L(\mathcal{E}_T)} \leq \text{const., uniformly in } \eta \in \mathcal{E}_T \equiv L_2(0,T;Y). \tag{2.33}$$

There is one more property this time for the operator $\mathcal{K}_T[\eta] = \mathcal{R}_T F'[\eta]$, that will be needed in the analysis of the exact controllability property of the semilinear model (2.1) or (1.1):

(P.3) (key property of $\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta]$) For each $\eta \in \mathcal{E}_T \equiv L_2(0,T;Y)$ fixed, where $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, we have

$$\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta] : \text{ continuous } \mathcal{E}_T \equiv L_2(0,T;Y) \to H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \tag{2.34}$$

as already noted in (2.23), (2.24) and, moreover: given any sequence $\eta_n \in \mathcal{E}_T$, we can extract a subsequence η_{n_k}, such that the following weak convergence takes place:

$$\begin{cases} \mathcal{K}_T[\eta_{n_k}] \equiv \mathcal{R}_T F'[\eta_{n_k}] \to \text{ some } \mathcal{K}_T^0 \equiv \mathcal{R}_T F_0, \\ \text{weakly from } \mathcal{E}_T \equiv L_2(0,T;Y) \to H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega), \end{cases} \tag{2.35a}$$

$$K_T^0 g = \mathcal{R}_T F_0 g = \int_0^T e^{A(T-t)} F_0(t) g(t) dt = \int_0^T e^{A(T-t)} \begin{bmatrix} 0 \\ f_0(t) g_1(t) \end{bmatrix} dt, \tag{2.35b}$$

for some operator $F_0 \in L(Y)$, possibly depending on the subsequence, which can be taken to be the same that arises in (2.29).

The validity of property (2.35) follows through an argument similar to that yielding property (P.2)(b) = (2.29), see details in [L-T.4, Proposition 3.4, p. 131].

Consequence of (P.3). As a consequence of properties (2.35) and (2.8), the latter being a consequence of the standing assumption (1.2) on f, we obtain

$$\|\mathcal{K}_T[\eta]\|_{L(\mathcal{E}_T;H)} \leq \text{const., uniformly in } \eta \in \mathcal{E}_T. \tag{2.36}$$

In PDE terms the linearized z-problem (2.18) is given by

$$\begin{cases} \zeta_{tt} = \Delta\zeta + f'(\eta_1(\,\cdot\,))\zeta & \text{in } (0,T] \times \Omega \equiv Q; & (2.37\text{a}) \\ \zeta(0,\,\cdot\,) = \zeta_0, \zeta_t(0,\,\cdot\,) = \zeta_1 & \text{in } \Omega; & (2.37\text{b}) \\ \zeta|_{\Sigma_0} \equiv 0; \left.\dfrac{\partial\zeta}{\partial\nu}\right|_{\Sigma_1} = u & \text{in } (0,T] \times \Gamma_i \equiv \Sigma_i,\ i = 0,1, & (2.37\text{c}) \end{cases}$$

$z_0 = \{\zeta_0, \zeta_1\} \in H$, where $\eta_1 \in L_2(0,T;L_2(\Omega))$. Its dual version is precisely the ϕ-problem in (3.9a–c) below with potential $q(t,x) = f'(\eta_1(\,\cdot\,))$, eventually $q(t,x) = f'(w(t,x))$, therefore satisfying only assumption (3.10) below: $q \in L_\infty(Q)$.

3 Control-Theoretic Properties

In this subsection we collect control-theoretic properties enjoyed by system (1.1) (or (2.1)), to be used in the subsequent analysis of the semilinear exact controllability property in Section 4. They are two: (i) exact controllability on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of the linear problem (1.1), or (2.1), with $f \equiv 0$; and (ii) an approximate controllability property of the corresponding linearized abstract problem (2.18), rewritten in PDE-terms in (2.37).

Preliminaries. However, before doing so, we need to address the issue that, in the present Neumann-boundary control case, a pathological feature arises (that has no counterpart in the corresponding Dirichlet-boundary control case): namely, the space $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of finite energy where exact controllability is sought is much smoother (for dim $\Omega \geq 2$) than the space $H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)$, $\alpha = \frac{2}{3}$ or $\frac{3}{4}$, of optimal regularity of the boundary control $\to$ solution operator $\mathcal{L}$ (see (2.11b)) or corresponding operator $\mathcal{L}_T$ (see (2.13a)). In other words, $\mathcal{L}_T$ is *not* continuous from $L_2(0,T;L_2(\Gamma_1))$ to $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, where $H^1_{\Gamma_0}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. However, $\mathcal{L}$ is *closed* as an operator (see (2.11a)):

$$(\mathcal{L}u)(t) \;=\; A\int_0^t e^{A(t-\tau)}A^{-1}Bu(\tau)d\tau,\;\; A^{-1}B \in L(U;H) \tag{3.1a}$$

$$: \;\; L_2(0,T;U) \supset \mathcal{D}(\mathcal{L}) \to C([0,T];H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)) \text{ closed.} \tag{3.1b}$$

(This is so, since A is closed on H, $A^{-1} \in L(H)$, and the integral term in (3.1a) is bounded from $L_2(0,T;U)$ to $C([0,T];H)$ [K.1, p. 164]). In addition, we have with reference to (2.11a) (where the I.C. $\{w_0, w_1\} = 0$):

$$\begin{aligned}
\begin{bmatrix} w_{f\equiv 0}(t) \\ (w_t)_{f\equiv 0}(t) \end{bmatrix} &= (\mathcal{L}u)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau \\
&= \begin{bmatrix} \mathcal{A}\displaystyle\int_0^t S(t-\tau)Nu(\tau)d\tau \\ \mathcal{A}\displaystyle\int_0^t C(t-\tau)Nu(\tau)d\tau \end{bmatrix} && (3.2a) \\
&: \quad \text{continuous } H^1(0,T;L_2(\Gamma_1)) \to C([0,T];H); && (3.2b)
\end{aligned}$$

$H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$. (To see (3.2b), one integrates by parts in t each component in (3.2a) and invokes (2.11b); or else invokes [L-T.5].) Similarly, we have that (see (2.13b))

$$\begin{aligned}
\mathcal{L}_T u \equiv (\mathcal{L}u)(T) &: L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \\
&\qquad \to H = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \text{ is closed;} && (3.3) \\
&: H^1(0,T;L_2(\Gamma_1)) \to H \text{ is continuous.} && (3.4)
\end{aligned}$$

(C.1) (Exact controllability of the linear problem (1.1) ($f \equiv 0$), on the space $H = H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-Neumann boundary controls for $T > T_0$.) This is the property for the closed operator $\mathcal{L}_T$ that

$$\mathcal{L}_T : L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \text{ is surjective } \xrightarrow{\text{onto}} H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega). \quad (3.5a)$$

Equivalently, [T-L.1, p. 235] its corresponding (Hilbert) adjoint is bounded below

$$\|\mathcal{L}_T^* h\|_{L_2(0,T;L_2(\Gamma_1))} \geq C_T \|h\|_H, \quad h \in \mathcal{D}(\mathcal{L}_T^*). \quad (3.5b)$$

It is well known that such property for problem (1.1) with $f \equiv 0$ holds true under some geometrical conditions and for all T sufficiently large [B-L-R.1], [L-T.2], [Lio.1], [Ta.1]–[Ta.3].

One setting of geometrical conditions. For the sake of concreteness we shall make reference to one such setting, even though it is not the most general one. The most general one is in [B-L-R.1]. Let $d : \Omega \Rightarrow \mathbb{R}^+$ be a C^3-strictly convex function and let $h(x) = \nabla d(x)$ be the corresponding conservative vector field, so that, if $\mathcal{H}_d(x)$ and $J_h(x)$ denote, respectively, the Hessian matrix of d [e.g., $d(x) = \|x - x_0\|^2$, $x_0 \in \mathbb{R}^n$, $\rho = 1$] and Jacobian matrix of h, then it holds that

$$\mathcal{H}_d(x) \equiv J_h(x) \geq 2\rho > 0, \quad \forall\, x \in \Omega, \tag{3.6}$$

for some constant $\rho > 0$. Assume further that

$$h(x) \cdot \nu(x) \leq 0 \ \text{ on } \Gamma_0, \ \nu(x) = \text{outward unit normal}. \tag{3.7}$$

Define

$$T_0 = 2 \left(\frac{\max_{x \in \overline{\Omega}} d(x)}{\rho} \right)^{\frac{1}{2}}, \ \rho \text{ as in (3.6)}. \tag{3.8}$$

Then, exact controllability of the linear problem (1.1) with $f \equiv 0$ is *guaranteed to hold true* on $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ *within the class* of $L_2(0, T; L_2(\Gamma_1))$-*controls, under the geometrical conditions* (3.6), (3.7), *for all* $T > T_0$ [L-T.8], [L-T-Y.1]–[L-T-Y.2].

(C.2) Approximate controllability (unique continuation) of the linearized system (2.18) or its PDE-version (2.37) and its limit version involving the limits (2.29) and (2.35). We consider the following homogeneous backward problem:

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{in } (0,T] \times \Omega \equiv Q; & (3.9a) \\ \phi(T, \cdot\,) = \phi_0, \phi_t(T, \cdot\,) = \phi_1 & \text{in } \Omega; & (3.9b) \\ \phi|_{\Sigma_0} \equiv \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } (0,T] \times \Gamma_i \equiv \Sigma_i, & (3.9c) \end{cases}$$

under the standing assumption that

$$\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'; \quad q(t,x) \in L_\infty(Q). \tag{3.10}$$

We begin by recalling a relevant regularity result

Theorem 3.1. *With reference to the ϕ-problem (3.9), with q as in (3.10), we have*

$$\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \Rightarrow \{\phi, \phi_t\} \in C([0,T]; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]') \quad \text{(3.11a)}$$

continuously. More precisely,

$$\|\{\phi, \phi_t\}\|^2_{C([0,T];L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')} \leq C_T(1 + e^{M_q T}\|q\|_{L_\infty(Q)})\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \quad \text{(3.11b)}$$

One may provide several standard proofs of (3.11a): By fixed point (contraction); by the evolution operator $U(t,s)$ as in Appendix B below, etc. In Appendix C, Theorem C.1, we provide, for completeness, a proof which has the advantage of showing the dependence upon q of the constant of continuity, as in (3.11b).

Unique continuation property. We now pass to the relevant unique continuation result, or (by duality) approximate controllability.

Theorem 3.2. *Consider problem (3.9a–b) with $\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ and B.C.*

$$\phi|_\Sigma \equiv 0, \quad \left.\frac{\partial \phi}{\partial \nu}\right|_{\Sigma_1} \equiv 0, \quad \text{(3.12)}$$

in place of (3.9c). Assume the geometrical conditions (3.6), (3.7), and let $T > T_0$ as defined in (3.8). Then, in fact, $\phi_0 = \phi_1 = 0$.

Proof of Theorem 3.2. As in [L-T.4, p. 133–4].

Step 1. First, under the weaker B.C. (3.9c), Theorem 3.1 provides the *a-priori* regularity (3.11a) for $\{\phi, \phi_t\} \in C([0,T]; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$. Using this information, we next boost the regularity using the full strength of the overdetermined B.C. (3.12) and obtain that, for $T > T_0$, we have

$$\infty > \|\phi\|^2_{L_\infty(0,T;L_2(\Omega))} \geq C_T(T - T_0)\|\{\phi, \phi_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)}. \quad \text{(3.13)}$$

In view of (3.13), we actually have that the corresponding solution satisfies

$$\{\phi_0, \phi_1\} \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega), \text{ hence } \{\phi, \phi_t\} \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)), \quad \text{(3.14)}$$

as it follows by multiplying (3.9) by ϕ_t and using the Gronwall inequality (as in (C.11) below). Inequality (3.13) for $T > T_0$ (defined in (3.8)) is already in the literature. In fact, one invokes [L-T.8, Theorem 2.1.2, Eqn. (2.1.10a), p. 222], or [L-T-Y.1, Theorem 3.4, Eqn. (3.15), p. 30] (with forcing term $\equiv 0$), combined with inequality [L-T.8, (2.4.3), p. 237] or [L-T-Y.1, (5.2.1), p. 46], yielding the boundary terms $\overline{BT}|_\Sigma = \frac{1}{2}\int_{\Sigma_0} e^{\tau\varphi}\left(\frac{\partial\phi}{\partial\nu}\right)^2 h\cdot\nu\, d\Sigma_0 \le 0$ under the B.C. (3.12).

Step 2. With the regularity boosted to the $H^1_{\Gamma_0}(\Omega)\times L_2(\Omega)$-level as stated in (3.14), we can now invoke available uniqueness theorem [L-T-Z.1], [Zh.1], [T-Y.1]. Indeed, this uniqueness result follows as a very special case of inequality (10.12) below in a Riemannian setting, taken from [T-Y.1], and previously available in the Euclidean setting in [K-K.1], [L-T-Z.1], and [Zh.1]. However, [R.1], [I.2], [I.3] do not apply, as they require $\Gamma_0 = \emptyset$. □

Approximate controllability of a ζ-problem. The above treatment casts property (C.2) as a unique continuation property of the ϕ-problem (3.9) and (3.12) with I.C. $\{\phi_0,\phi_1\}\in L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. We now justify why this property (C.2) is also, equivalently, an approximate controllability property.

Indeed we note that the ϕ-problem (3.9) is dual to the ζ-problem (2.37) with $f'(\eta_1(t,x))\equiv q(t,x)$, see Appendix A or Appendix B. (Ultimately, $\eta_1(t,x) = w(t,x)$). Then, the uniqueness property in Theorem 3.2 with $q(t,x) = f'(\eta_1(t,x))$ means, by duality, that

$$\text{the map } u\in L_2(0,T;U)\to\mathcal{M}_T[q]u\equiv\{\zeta(T),\zeta_t(T)\} \text{ has dense range in } H, \tag{3.15}$$

where $U = L_2(\Gamma_1)$, $H = H^1_{\Gamma_0}(\Omega)\times L_2(\Omega)$, which is precisely an approximate controllability property of the corresponding controlled ζ-problem, with zero I.C. $\{\zeta_0,\zeta_1\} = 0$.

When referring particularly to the z-problem (2.18) or its PDE-version, the ζ-problem (2.37), both depending on the parameter $\eta = [\eta_1,\eta_2]\in\mathcal{E}_T$, we shall write alternatively $\mathcal{M}_T[\eta]$ in place of $\mathcal{M}_T[q]$ for the map in (3.15) (where, of course, $q(t,x) = f'(\eta_1(t,x)))$. Below, in the analysis of Section 4, we shall obtain an explicit formula for the map $\mathcal{M}_T[\eta]$. It is given by (see Eqn. (4.33))

$$\mathcal{M}_T[\eta] \equiv \mathcal{L}_T + \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L} : L_2(0,T;U) \supset \mathcal{D}(\mathcal{M}_T[\eta]) = \mathcal{D}(\mathcal{L}_T) \to H. \tag{3.16}$$

Similarly, for the limit version given by Eqns. (2.29a) and (2.35a), we have likewise

$$\mathcal{M}_T^0 \equiv \mathcal{L}_T + \mathcal{K}_T^0(I - \mathcal{K}^0)^{-1}\mathcal{L} : \mathcal{D}(\mathcal{M}_T) \to H. \tag{3.17}$$

This corresponds to the ζ-problem (2.37) with potential $q = f_0(\,\cdot\,)$ as obtained in (2.29b) or (2.35b).

Thus, the denseness property (3.15) encompasses, in particular, the following two statements that: for each fixed $\eta \in \mathcal{E}_T$, the map $\mathcal{M}_T[\eta]$ in (3.16), as well as the map $\mathcal{M}_T^0$ in (3.17), have each *range dense* in H.

4 Exact Controllability of the Semilinear Problem (1.1), (1.2) under Properties: (P.1), (P.2), (P.3), (C.1), (C.2). Proof of Theorem 1.1 by the Global Inversion Approach of [L-T.4]

So far, our treatment of the semilinear problem (1.1), under assumption (1.2) on the non-linearity, has re-cast (1.1) as the abstract semilinear Eqn. (2.1) subject to property (2.8) and has extracted several additional key properties of (1.1), which have all been expressed in abstract form: the structural properties (P.1), (P.2), (P.3), as well as the control-theoretic properties (C.1) and (C.2). We shall now use these properties to establish exact controllability of (1.1), through the abstract approach of [L-T.4], based on a global inversion theorem [D.1], [S.1]. We thus make reference to Eqn. (2.1) where $y(t) = \{w(t), w_t(t)\}$ in regard to (1.1).

The present section corresponds to [L-T.4, §2.1] revisited, with the key operator $\mathcal{L}_T$, however, unbounded (but closed) $L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T)$ onto H.

Step 1. Let $y_0 \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ and $u \in \mathcal{D}(\mathcal{L})$ defined by (3.1b), where $H^1(0,T;L_2(\Gamma_1)) \subset \mathcal{D}(\mathcal{L}) \subset L_2(0,T;L_2(\Gamma_1))$, see (3.1b), (3.2b). By a fixed point argument, we then obtain that $y(t) \equiv \{w(t), w_t(t)\} \in C([0,T];H)$. In this setting, we return to the variation of parameter formula (2.9), rewritten as $[I - \mathcal{R}F]y_u = e^{A\cdot}y_0 + \mathcal{L}u \in C([0,T];H)$ (we note the dependence of y on u) and obtain the solution

$$y_u(t) = \{[I - \mathcal{R}F]^{-1}[e^{A\cdot}y_0 + \mathcal{L}u]\}(t) \in C([0,T];H), \ u \in \mathcal{D}(\mathcal{L}), \tag{4.1}$$

directly in terms of the data, in particular for $u \in H^1(0,T;L_2(\Gamma_1))$ by (3.2b). In view of (1.2), we have:

$$[I - \mathcal{R}F]^{-1} : \text{ bounded and continuous } C([0,T];H) \to C([0,T];H). \tag{4.2}$$

The validity of (4.2) follows from solving uniquely on successive intervals of suitably small but equal size $[0,T_1], [T_1, 2T_1]$, $T_1 > 0$, etc. the corresponding equation

$$[I - \mathcal{R}F]\xi = \mathcal{X} \in C([0,T];H), \text{ or } \xi(t) - \int_0^t e^{A(t-\tau)}F(\xi(\tau))d\tau = \mathcal{X}(t), \tag{4.3}$$

using the property, from (1.2), that F in (2.7) is globally Lipschitz on H : $\|F(\xi)\|_H \leq \text{const}\|\xi\|_H$.

We return to the variation of parameter formula (2.10) for $t = T$:

$$y_u(T) = e^{AT}y_0 + \mathcal{L}_T u + \mathcal{R}_T F y_u \in H, \ y_0 \in H, \ u \in \mathcal{D}(\mathcal{L}_T), \tag{4.4}$$

in particular, for $u \in H^1(0,T;L_2(\Gamma_1))$. Without loss of generality for our present exact controllability problem, we *can restrict to controls* $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ throughout this section, where

$$\mathcal{N}(\mathcal{L}_T) = \{u \in \mathcal{D}(\mathcal{L}_T) : \mathcal{L}_T u = 0\}; \tag{4.5}$$

a closed subspace of $L_2(0,T;U)$ by the closedness of $\mathcal{L}_T$ in (3.3), and $[\quad]^\perp$ denotes the orthogonal complement in $L_2(0,T;L_2(\Gamma_1))$ to the closed subspace $[\quad]$,

$$L_2(0,T;L_2(\Gamma_1)) = [\mathcal{N}(\mathcal{L}_T)] + [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{4.6}$$

The closed complementary subspaces $\mathcal{N}(\mathcal{L}_T)$ and $[\mathcal{N}(\mathcal{L}_T)]^\perp$ will be *always topologized* by *the inherited* $L_2(0,T;U)$-norm, $U = L_2(\Gamma_1)$. Thus, the *exact controllability problem* under investigation *can be reformulated as follows*: Given $T > 0$, $y_0 \in H$, $y_T \in H$, seek, if possible, a control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0,T;L_2(\Gamma_1))$ such that

$$y_T = y_u(T) = e^{AT}y_0 + \mathcal{L}_T u + \mathcal{R}_T F y_u, \ u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{4.7}$$

Henceforth, $[\mathcal{N}(\mathcal{L}_T)]^\perp$ will be *our control space* and $\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}$, by which we mean the restriction of $\mathcal{L}_T$ on $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, is injective.

Step 2. We now invoke the exact controllability property (C.1) = (3.5) for the linear system, whereby

$$\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp} : \text{closed, injective, surjective from } [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \text{ onto } H. \tag{4.8}$$

By the open mapping theorem, we then obtain that

$$\mathcal{L}_T^\# \equiv \left(\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}\right)^{-1} : \text{bounded, injective, surjective } H \xrightarrow{\text{onto}} [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0,T;U), \tag{4.9}$$

where $\mathcal{L}_T^\#$ is the pseudo-inverse of $\mathcal{L}_T$. Thus, $\mathcal{L}_T^\#$ is an *isomorphism* between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, in the sense of [T-L.1, p. 15]: a one-to-one correspondence between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$. $\mathcal{L}_T^\#$ is bounded in (4.9), but $(\mathcal{L}_T^\#)^{-1} = \mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}$ is not bounded from $L_2(0,T;U)$ to H. It is bounded only if the subspace $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ of $L_2(0,T;L_2(\Gamma_1))$ is topologized by the inherited norm

$$\|u\|_{[\mathcal{N}(\mathcal{L}_T)]^\perp} \equiv \|(\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})u\|_H,$$

which is equivalent to the graph norm in which case $\mathcal{L}_T^{\#}$ becomes a *topological isomorphism* ($\mathcal{L}_T^{\#}$ and $(\mathcal{L}_T^{\#})^{-1}$ both continuous with such topologies) or *linear homeomorphism*. We shall not use this property. An explicit formula for $\mathcal{L}_T^{\#}$ is known, under the present property (C.1) = (3.5). Indeed, by a well-known result in Functional Analysis [T-L.1, p. 235], the surjectivity property (3.5a) for the closed operator $\mathcal{L}_T$ is *equivalent* to the condition that the Hilbert space adjoint: $\mathcal{L}_T^* : H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0,T;L_2(\Gamma_1))$ is bounded below as in (3.5b)

$$\|\mathcal{L}_T^* x\|_{L_2(0,T;L_2(\Gamma_1))} \geq C_T \|x\|_H, \quad \forall\, x \in \mathcal{D}(\mathcal{L}_T^*) \tag{4.10}$$

[sometimes the Banach-space adjoint is more convenient to take].

Remark 4.1. Throughout this paper, $*$ will denote adjoint with respect to the space H of exact controllability. Other adjoints, with respect to other spaces, will be denoted by a different symbol, as documented in Remark 4.3 below.

Inequality (4.10) is the corresponding COI (Continuous Observability Inequality) for the linear problem (1.1) with $f \equiv 0$. Based on (4.10), that is under the exact controllability property (C.1) = (3.5), the minimization problem of steering the origin "O" (rest) to the target $y_T \in H$ while minimizing the $L_2(0,T;U)$-norm leads via a Lagrange multiplier argument [Tr.2, Appendix], [L-T.2, Appendix] to the following explicit expression for $\mathcal{L}_T^{\#}$:

$$\mathcal{L}_T^{\#} = \mathcal{L}_T^*[\mathcal{L}_T\mathcal{L}_T^*]^{-1} : \text{ bounded, injective } H \to [\mathcal{N}(\mathcal{L}_T)]^{\perp} \subset L_2(0,T;L_2(\Gamma_1)) \tag{4.11}$$

($\mathcal{L}_T\mathcal{L}_T^{\#}$ defines an isomorphism from H onto its dual H' by the Lax-Milgram Lemma applied to (4.10)). The pseudo-inverse $\mathcal{L}_T^{\#}$ enjoys the usual properties

$$\mathcal{L}_T\mathcal{L}_T^{\#} = \text{identity on } H; \quad \mathcal{L}_T^{\#}\mathcal{L}_T = \text{identity on } [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T). \tag{4.12}$$

Next, applying $\mathcal{L}_T^{\#}$ on Eqn. (4.7) and using (4.12) (right side), we see that the *exact controllability problem* under investigation *can be reformulated as follows*: Given $T > 0$, $y_0 \in H$, $y_T \in H$, seek, if possible, a control $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0,T;L_2(\Gamma_1))$, such that

$$u + \mathcal{L}_T^{\#}\mathcal{R}_T F(y_u) = \mathcal{L}_T^{\#}(y_T - e^{AT}y_0), \tag{4.13}$$

or, after substituting (4.1) for y_u into (4.13), such that

$$u + \mathcal{L}_T^{\#}\mathcal{R}_T F[I - \mathcal{R}F]^{-1}[e^{A\cdot}y_0 + \mathcal{L}u] = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0]. \tag{4.14}$$

On the other hand, the exact controllability property (C.1) = (3.5) for the linear system ensures that, given y_T and y_0 in H, there exists a control $v_T^0 \in \mathcal{D}(\mathcal{L}_T)$—which without loss of generality we may also take in $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$—such that (recall (4.12) (right side)):

$$\mathcal{L}_T v_T^0 = y_T - e^{AT}y_0, \text{ hence } \mathcal{L}_T^{\#}\mathcal{L}_T v_T^0 = v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0],$$
$$v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T). \tag{4.15}$$

Thus, using (4.15) on the RHS of (4.14), we conclude that: Given $T > 0$, in order to solve the semilinear exact controllability problem steering y_0 at $t = 0$ to y_T at $t = T$ along (1.1) (or (2.1)), we seek a control $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$, such that

$$u + \Lambda_T(u) = v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T) \equiv \text{range}(\mathcal{L}_T^{\#}), \tag{4.16}$$

where we have set, via (4.13), (4.14),

$$\Lambda_T(u) \equiv \mathcal{L}_T^{\#}\mathcal{R}_T F(y_u) = \mathcal{L}_T^{\#}\mathcal{R}_T F[I - \mathcal{R}F]^{-1}[e^{A\cdot}y_0 + \mathcal{L}u] \subset \text{range}(\mathcal{L}_T^{\#})$$
$$= [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T), \tag{4.17}$$

and where $v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0] \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ is, by (4.15) (left), the control steering y_0 at $t = 0$ to y_T at time $t = T$, along the corresponding *linear* dynamics (1.1) with $f \equiv 0$.

[Notice that given $v_T^0 \in \text{range}(\mathcal{L}_T^{\#})$, then if (4.16) is solvable for $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$ (the control space), then by necessity, $u \in \text{range}(\mathcal{L}_T^{\#})$ as well, since $\Lambda_T(u) \in \text{range}(\mathcal{L}_T^{\#})$.]

The converse is also true: Let $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ be a control steering $y_0 \in H$ at $t = 0$ to $y_T \in H$ at $t = T$, along the solution of the semilinear problem (2.1). Then, u satisfies (4.7) and, by (4.12), u satisfies (4.13), (4.14) as well. Then the control $v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ defined by (4.16) has the property that $\mathcal{L}_T v_T^0 = y_T - e^{AT} y_0$, see (4.15), (4.12), and hence steers y_0 at $t = 0$ to the state y_T at $t = T$ along the linear dynamics (1.1) with $f \equiv 0$. Thus, (C.1) = (3.5) is satisfied. We have obtained the following result.

Proposition 4.1. *Assume (1.2).*

(a) Given $T > 0$, $y_0 = \{w_0, w_1\} \in H$, $y_T = \{\tilde{w}_0, \tilde{w}_1\} \in H$, $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, there exists a control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0, T; L_2(\Gamma_1))$ steering y_0 at $t = 0$ to y_T at $t = T$ along the semilinear system (1.1) subject to (1.2) [or the system (2.1) subject to (2.8)] if and only if (C.1) = (3.5) holds true and such steering control u satisfies identity (4.16), where $v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT} y_0] \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ is then the control steering likewise y_0 at $t = 0$ to the target y_T at time $t = T$, along the corresponding linear dynamics (1.1) with $f \equiv 0$ [(2.1) with $F \equiv 0$].

(b) A sufficient condition for part (a) to hold true—that is, the exact controllability of the semilinear problem (1.1) or (2.1)—is the global inversion of the C^1 map $u \to u + \Lambda_T(u) : [\mathcal{N}(\mathcal{L}_T)]^\perp \to$ itself, in (4.16). More precisely, such a map $u \to u + \Lambda_T(u)$ is a homeomorphism (bijective, continuous, with continuous inverse), provided that [D.1, p. 153]:

(b1) its Frechet derivative $I + \Lambda_T'(u)$ has a bounded inverse: $[\mathcal{N}(\mathcal{L}_T)]^\perp \to [\mathcal{N}(\mathcal{L}_T)]^\perp$;

(b2) the norm of such inverse operator grows at most linearly in $\|u\|_{L_2(0,T;L_2(\Gamma_1))}$:

$$\|[I + \Lambda_T'(u)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \leq C\left[1 + \|u\|_{L_2(0,T;L_2(\Gamma_1))}\right], \quad \forall\, u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \tag{4.18a}$$

[$[\mathcal{N}(\mathcal{L}_T)]^\perp$ is topologized by $L_2(0, T; L_2(\Gamma_1))$] a condition a-fortiori *satisfied provided that*

$$\|[I + \Lambda_T'(u)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \leq C, \quad \forall\, u \in [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{4.18b}$$

An explicit expression of $\Lambda'_T(u)$, $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, *is (recall (4.17)):*

$$\begin{aligned}\Lambda'_T(u) \quad &= \quad (\mathcal{L}_T^{\#}\mathcal{R}_T F(y_u))' = \mathcal{L}_T^{\#}\mathcal{K}_T[y_u]y'_u \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[y_u][I-\mathcal{K}[y_u]]^{-1}\mathcal{L} \\ &\quad\; : \textit{compact } L_2(0,T;U) \supset [\mathcal{N}(\mathcal{L}_T)]^\perp \to [\mathcal{N}(\mathcal{L}_T)^\perp;\end{aligned} \tag{4.19a}$$

$$\begin{aligned}[I-\mathcal{K}[y_u]]^{-1} \textit{ bounded on } \quad & C([0,T];H); \textit{ or on } L_2(0,T;H); \\ \textit{or on } \quad & C([0,T];Y); \textit{ or on } L_2(0,T;Y).\end{aligned} \tag{4.19b}$$

Proof. Part (a) was established in Step 2 above.

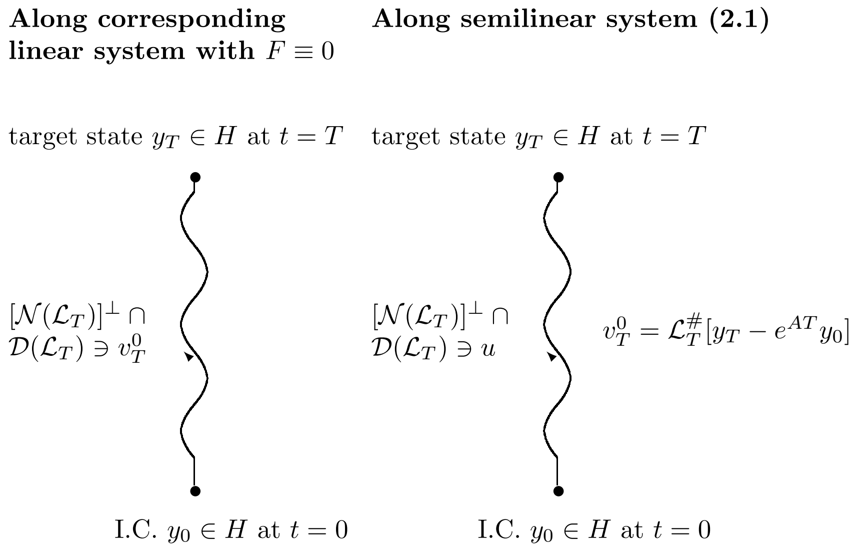


FIG. 4.1: PROPOSITION 4.1(A): GIVEN T, $y_0 \in H$, $y_T \in H$, THERE EXISTS A STEERING CONTROL $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ FOR THE SEMILINEAR SYSTEM (2.1) IF AND ONLY IF (C.1) = (3.5) HOLDS TRUE AND u SATISFIES $u+\Lambda_T(u) = v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0]$, WHERE $v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ IS THE STEERING CONTROL FOR THE CORRESPONDING LINEAR SYSTEM.

Part (b) is nothing but a well-known criterion for global inversion in nonlinear analysis [D.1, Theorem 15.4, p. 153] for (4.18a), and [B.1], [S.1, p. 16] for (4.18b). We now verify (4.19). The operator $\Lambda_T(u) = \mathcal{L}_T^{\#}\mathcal{R}_T F(y_u)$ in (4.17) is

Frechet differentiable on, say, $u \in H^1(0,T;L_2(\Gamma_1))$ and its Frechet derivative is

$$\Lambda_T'(u) = \mathcal{L}_T^\# \mathcal{R}_T F'(y_u) y_u' = \mathcal{L}_T^\# \mathcal{K}_T[y_u] y_u' = \mathcal{L}_T^\# \mathcal{K}_T[y_u][I - \mathcal{K}[y_u]]^{-1}\mathcal{L}, \quad (4.20)$$

recalling (2.24) and anticipating (4.22) below. On the other hand, from (2.9), we obtain by differentiation in $u \in H^1(0,T;U)$,

$$y_u - \mathcal{R}F(y_u) = e^{A\,\cdot}\, y_0 + \mathcal{L}u \in C([0,T];H) \rightarrow y_u' - \mathcal{R}F'(y_u)y_u' = \mathcal{L}, \quad (4.21)$$

or, recalling $\mathcal{R}F'(y_u) = \mathcal{K}[y_u]$ from (2.21) and $U = L_2(\Gamma_1)$, as well as (4.21)

$$\begin{aligned} [I - \mathcal{K}[y_u]]y_u' &= \mathcal{L} : H^1(0,T;U) \rightarrow C([0,T];H) \ \text{ hence} \\ y_u' &= [I - \mathcal{K}[y_u]]^{-1}\mathcal{L} : H^1(0,T;U) \rightarrow C([0,T];H). \quad (4.22) \end{aligned}$$

The inversion in (4.22) satisfying (4.19b) is justified due to assumption (1.2) [or (2.8)] that f is globally Lipschitz, in the same way as the inversion $[I - \mathcal{R}F]^{-1}$ was justified in (4.2), (4.3). Notice the following regularity properties: $\mathcal{L}$: compact $L_2(0,T;U) \rightarrow L_2(0,T;Y)$ by (P.1) = (2.14); $[I - \mathcal{K}[y_u]]^{-1}$ continuous on $L_2(0,T;Y)$, by (4.19b); $\mathcal{K}_T[y_u]$: continuous $L_2(0,T;Y) \rightarrow H$, *a fortiori* from (2.24) (this is property (P.3) in (2.34)); and finally, $\mathcal{L}_T^\#$: continuous H into $[\mathcal{N}(\mathcal{L}_T)]^\perp$, see (4.11). Then, these properties combined in (4.20) yield that $\Lambda_T'(u)$ is compact as an operator $L_2(0,T;U) \supset [\mathcal{N}[\mathcal{L}_T)]^\perp \rightarrow [\mathcal{N}(\mathcal{L}_T)]^\perp$ (with $L_2(0,T;U)$-topology), as claimed in (4.19a). □

Step 3. Here we analyze the boundedness condition for $[I + \Lambda_T'(u)]^{-1}$ in (4.18). Thus, let $v \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \equiv \mathcal{L}_T^\# H$, $\mathcal{L}_T^\#$ injective, see (4.9), so that there exists a unique $h \in H$ such that $v = \mathcal{L}_T^\# h$. Let $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ be a fixed parameter. We seek to solve the problem

$$\tilde{u} + \Lambda_T'(u)\tilde{u} = v = \mathcal{L}_T^\# h \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T), \quad (4.23)$$

for a unique $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, hence necessarily $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, so as to obtain the bound

$$\|\tilde{u}\|_{L_2(0,T;U)} = \|[I + \Lambda_T'(u)]^{-1} v\|_{L_2(0,T;U)} \leq C[1 + \|u\|_{L_2(0,T;U)}]\|v\|_{L_2(0,T;U)}, \quad (4.24)$$

$U = L_2(\Gamma_1)$. To this end, we return to the linearized z-problem in (2.18), with parameter $\eta = y_u$, u the parameter in (4.23), and rewrite (2.18) and (2.19) accordingly with control $\tilde{u}$ as:

$$\dot{z} = Az + F'[y_u]z + B\tilde{u}, \ z(0) = z_0, \quad [I - \mathcal{K}[y_u]]z = e^{A\,\cdot}z_0 + \mathcal{L}\tilde{u} \Rightarrow \tag{4.25}$$

$$z = [I - \mathcal{K}[y_u]]^{-1}[e^{A\,\cdot}z_0 + \mathcal{L}\tilde{u}], \tag{4.26}$$

where the inverse in (4.26) is justified, see (4.19b), under assumption (1.2) [by property (2.8)] as in the case of obtaining (4.2) via (4.3). Evaluating Eqn. (4.25) at $t = T$ yields, see (2.20):

$$z(T) = \mathcal{K}_T[y_u]z + e^{AT}z_0 + \mathcal{L}_T\tilde{u}, \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{4.27}$$

The following result shows the importance of extracting the vector $h \in H$ in solving Eqn. (4.23): it provides an *equivalence* between solving (4.23) for $\tilde{u}$, and exact controllability from the origin of the linearized system via $\tilde{u}$ with target h at $t = T$.

Proposition 4.2. *Assume (1.2) and let $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ be a fixed parameter. Let $v \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ or, equivalently, let $h \in H$ be given, with $v = \mathcal{L}_T^{\#}h$. Then, the following equivalence holds true. Assume that there exists a control $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ which steers the origin $z_0 = 0$ at $t = 0$ to the target $z_T = h \in H$ at $t = T$ along the linearized z-equation (4.25), i.e., via (4.27). Then, that $\tilde{u}$ is a solution of Eqn. (4.23).*

Conversely, let $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ be a solution of Eqn. (4.23). Then, that $\tilde{u}$ is a control which steers the origin $z_0 = 0$ at $t = 0$ to the target $z_T = h \in H$ at $t = T$, along the linearized z-equation (4.25), i.e., via (4.27).

In short: solving Eqn. (4.23) for $\tilde{u}$ as the state h runs over the space H is equivalent to exact controllability of the linearized z-system (4.25), i.e., (4.27) on the space H over the interval $[0, T]$.

Proof. In one direction, let $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ be a control which steers the corresponding solution of the z-system (4.25) from the origin $z_0 = 0$ at

$t = 0$ to the target $z_T = h \in H$ at $t = T$. Then, via (4.27), we have

$$z_T = h = z(T) = \mathcal{K}_T[y_u]z + \mathcal{L}_T\tilde{u}, \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T). \tag{4.28}$$

Apply $\mathcal{L}_T^{\#}$ across (4.28), use $\mathcal{L}_T^{\#}\mathcal{L}_T =$ identity on $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ by (4.12) (right) and obtain

$$\tilde{u} + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u]z = \mathcal{L}_T^{\#}h = v \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T). \tag{4.29}$$

After substituting z in (4.26) with $z_0 = 0$ in (4.29) yields

$$\tilde{u} + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u][I - \mathcal{K}[y_u]]^{-1}\mathcal{L}\tilde{u} = v = \mathcal{L}_T^{\#}h, \tag{4.30}$$

that is, recalling (4.19a), we conclude that

$$\tilde{u} + \Lambda'_T(u)\tilde{u} = v = \mathcal{L}_T^{\#}h \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T), \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T), \tag{4.31}$$

and $\tilde{u}$ satisfies (4.23), as desired.

The above argument is reversible. Conversely, let $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ be a solution of (4.23) = (4.31). Then, recalling (4.19a), such $\tilde{u}$ satisfies (4.30), and hence (4.29) via (4.26). Apply now $\mathcal{L}_T$ across Eqn. (4.29), use twice $\mathcal{L}_T\mathcal{L}_T^{\#} =$ identity on H by (4.12) (left) and obtain

$$\mathcal{L}_T\tilde{u} + \mathcal{K}_T[y_u]z = h. \tag{4.32}$$

Then, in view of (4.27) with $z_0 = 0$, we see that (4.32) says that $z(T) = h$, and then this $\tilde{u}$ steers $z_0 = 0$ at $t = 0$ to $z(T) = h$ at $t = T$ along the z-system, as desired. $\square$

Further properties of the operator $I + \Lambda'_T(u)$ follow next. To the end, we return to (2.20), (4.26) and recall the operator introduced in (3.15), see also (3.16) and (4.19b):

$$\mathcal{M}_T[\eta] : u \to \mathcal{M}_T[\eta]u \equiv z(T) = \mathcal{L}_T u + \mathcal{K}_T[\eta]z = \{\mathcal{L}_T + \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L}\}u; \tag{4.33a}$$

$$L_2(0, T; U) \supset \mathcal{D}(\mathcal{M}_T[\eta]) \equiv [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T) \to H,$$

$$\mathcal{K}_T[\eta](I-\mathcal{K}[\eta])^{-1}\mathcal{L}: \text{ compact } L_2(0,T;U)\to H, \tag{4.33b}$$

see Remark 4.2 below, where $z(T)$ is the solution of (2.20) for $z_0=0$ and $\eta\in Y$, with z given by (4.26) with $z_0=0$.

Remark 4.2. The operator $\mathcal{M}_T[\eta]$ in (4.33) is given in perturbation form over $\mathcal{L}_T$. Formula (4.33) for $\mathcal{M}_T[\eta]$ has to be interpreted as follows, with $U\equiv L_2(\Gamma_1)$, $Y\equiv L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$:

(i) $\mathcal{L}$: continuous (compact) $L_2(0,T;U)\to\mathcal{E}_T\equiv L_2(0,T;Y)$ as in (2.14);

(ii) $\mathcal{K}[\eta]$: continuous (compact) $\mathcal{E}_T\to\mathcal{E}_T$, as in (**P.2**) = (2.27), hence $(I-\mathcal{K}[\eta])^{-1}$: continuous $\mathcal{E}_T\to\mathcal{E}_T$;

(iii) $\mathcal{K}_T[\eta]$: continuous $\mathcal{E}_T\to H$, as in (2.24) or (2.34) = (**P.3**);

(iv) $\mathcal{L}_T$: closed operator: $L_2(0,T;U)\supset[\mathcal{N}(\mathcal{L}_T)]^\perp\cap\mathcal{D}(\mathcal{L}_T)\to H$, as in (3.3).

Properties (i), (ii), (iii) imply the regularity in (4.33b).

Since $z(T)=\{\zeta(T),\zeta_t(T)\}$, where ζ is the solution of problem (2.37) with I.C. $\{\zeta_0,\zeta_1\}=z_0=0$, for $\eta_1=w(t,x)$, then the last part of Section 3, see (**C.2**), including Eqn. (3.15)–(3.17) says that the maps:

$$\begin{aligned}
\mathcal{M}_T[\eta] &\equiv \mathcal{L}_T+\mathcal{K}_T[\eta](I-\mathcal{K}[\eta])^{-1}\mathcal{L}, \text{ fixed } \eta\in\mathcal{E}_T && (4.34a)\\
\mathcal{M}_T^0 &\equiv \mathcal{L}_T+\mathcal{K}_T^0(I-\mathcal{K}^0)^{-1}\mathcal{L} && (4.34b)\\
&\quad : L_2(0,T;U)\supset[\mathcal{N}(\mathcal{L}_T)]^\perp\cap\mathcal{D}(\mathcal{L}_T)\to H && (4.34c)
\end{aligned}$$

have range dense in H, for each $\eta\in Y$. Equivalently, the Hilbert adjoint operators of (4.34a–b):

$$\mathcal{M}_T^*[\eta]\equiv\mathcal{L}_T^*+\mathcal{L}^{\circledast}(I-\mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]: H\supset\mathcal{D}(\mathcal{L}_T^*)\to L_2(0,T;U); \tag{4.35a}$$

$$(\mathcal{M}_T^0)^*=\mathcal{L}_T^*+\mathcal{L}^{\circledast}[I-(\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*: H\supset\mathcal{D}(\mathcal{L}_T^*)\to L_2(0,T;U) \tag{4.35b}$$

where $\mathcal{L}_T^*=(\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^*$ in our present setting, and where by (4.33b), or Remark 4.3 below

$$\mathcal{L}^{\circledast}(I-\mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta], \qquad \mathcal{L}^{\circledast}[I-(\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*$$
$$: \text{ compact } H\to L_2(0,T;U), \tag{4.35c}$$

are *injective*; that is, they have a trivial null space on H:

$$\mathcal{N}\{\mathcal{M}_T^*[\eta]\} = \mathcal{N}\{\mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\} = \{0\}, \text{ fixed } \eta \in \mathcal{E}_T; \quad (4.36a)$$

$$\mathcal{N}\{(\mathcal{M}_T^0)^*\} = \mathcal{N}\{\mathcal{L}_T^* + \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*\} = \{0\}. \quad (4.36b)$$

by (**C.2**).

Remark 4.3. Consistently with Remarks 4.1 and 4.2, in (4.35), (4.36) we have chosen the following notation for the adjoints:

(i) $\mathcal{L}_T^*$: $H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0,T;U)$ as in Remark 4.1;

(ii) $\mathcal{L}^{\circledast}$: compact $\mathcal{E}_T \to L_2(0,T;U)$ (see also (6.12) below for explicit definition);

(iii) $\mathcal{K}^{\circledast}[\eta]$; $(\mathcal{K}^0)^{\circledast}$: compact $\mathcal{E}_T \to \mathcal{E}_T$; hence, $(I - \mathcal{K}^{\circledast}[\eta])^{-1}$, $[I - (K^0)^{\circledast}]^{-1}$: continuous $\mathcal{E}_T \to \mathcal{E}_T$;

(iv) $\mathcal{K}_T^*[\eta]$; $(\mathcal{K}_T^0)^*$: continuous $H \to \mathcal{E}_T$.

With this premise, we now note a quantitative version of Proposition 4.2. To this end, we set for convenience (recalling (4.19a))

$$\Lambda_T'[\eta] \equiv C_T[\eta] \quad \equiv \quad \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L} : \text{ compact } [\mathcal{N}(\mathcal{L}_T)]^{\perp} \to \text{itself}, \quad (4.37)$$

$$C_T^0[\eta] \quad \equiv \quad \mathcal{L}_T^{\#}\mathcal{K}_T^0(I - \mathcal{K}^0)^{-1}\mathcal{L} : \text{ compact } [\mathcal{N}(\mathcal{L}_T)]^{\perp} \to \text{itself}, \quad (4.38)$$

where $\mathcal{K}_T^0$, $\mathcal{K}^0$ are the limits in (2.29a), (2.35a). ($[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ could be replaced by $L_2(0,T;L_2(\Gamma_1)$.)

Proposition 4.3. *With reference to $\Lambda_T'(u)$ in (4.37) and to (4.34) with $\eta = y_u$, we have*

(a) recalling (4.19a),

$$I + \Lambda_T'(u) = \mathcal{L}_T^{\#}\mathcal{M}_T[y_u] = I + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u](I - \mathcal{K}[y_u])^{-1}\mathcal{L}$$
$$: \textit{continuous } L_2(0,T;U) \supset [\mathcal{N}(\mathcal{L}_T)]^{\perp} \to [\mathcal{N}(\mathcal{L}_T)]^{\perp}, \quad (4.39a)$$

with $\mathcal{L}_T^{\#}$ *isomorphism between* H *and* $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$*, see below (4.9); hence*

$$I + (\Lambda_T'(u))^* = \mathcal{M}_T^*[y_u](\mathcal{L}_T^{\#})^* \\ : \text{continuous } [\mathcal{N}(\mathcal{L}_T)]^\perp \to [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(0,T;U); \tag{4.39b}$$

$$(\Lambda_T'[u])^* = C_T^*[u] \quad = \quad \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[y_u])^{-1}\mathcal{K}_T^*[y_u](\mathcal{L}_T^{\#})^* \tag{4.40a}$$

$$\text{compact } [\mathcal{N}(\mathcal{L}_T)]^\perp \to \text{itself}; \tag{4.40b}$$

by (4.19a).

(b) the bounded operator $I + (\Lambda_T'[u])^* = I + C_T^*[u]$ *is injective on* $[\mathcal{N}(\mathcal{L}_T)]^\perp$*:*

$$\tilde{u} + (\Lambda_T'(u))^*\tilde{u} = \tilde{u} + \mathcal{L}^{\circledast}[I - \mathcal{K}^{\circledast}[y_u]]^{-1}\mathcal{K}_T^*[y_u](\mathcal{L}_T^{\#})^*\tilde{u} = 0;\ \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$$

$$\Rightarrow \tilde{u} = 0 \text{ (for each fixed value of } u \in [\mathcal{N}(\mathcal{L}_T)]^\perp\text{)}, \tag{4.41}$$

so that the original operator $I + \Lambda_T'(u)$ *has dense range as an operator from* $[\mathcal{N}(\mathcal{L}_T)]^\perp$ *into itself. Similarly, the operator* $I + (C_T^0)^*$ *is injective on* $[\mathcal{N}(\mathcal{L}_T)]^\perp$*, where*

$$(C_T^0)^* = \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*(\mathcal{L}_T^{\#})^* : \text{compact } [\mathcal{N}(\mathcal{L}_T)]^\perp \to \text{itself}; \tag{4.42}$$

(c) In fact, more is true: both operators

$$[I + \Lambda_T'(u)]^{-1} \text{ and } [I + (\Lambda_T'(u))^*]^{-1} \text{ exist as bounded operators } \in L([\mathcal{N}(\mathcal{L}_T)]^\perp), \tag{4.43}$$

for each $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$*. Similarly, both operators*

$$[I + C_T^0]^{-1} \text{ and } [I + (C_T^0)^*]^{-1} \text{ exist as bounded operators in } L([\mathcal{N}(\mathcal{L}_T)]^\perp). \tag{4.44}$$

Proof. (a) We apply $\mathcal{L}_T^{\#}$ to identity (4.33) giving $\mathcal{M}_T[\eta]$ on $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ and obtain for $\eta = y_u$:

$$\mathcal{L}_T^{\#}\mathcal{M}_T[y_u] \quad = \quad \mathcal{L}_T^{\#}\mathcal{L}_T + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u](I - \mathcal{K}[y_u])^{-1}\mathcal{L} \tag{4.45}$$

$$= \quad I + \Lambda_T'(u), \tag{4.46}$$

recalling that $\mathcal{L}_T^{\#}\mathcal{L}_T =$ Identity on $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$, via (4.12), as well as the definition (4.19a) = (4.37) for $\Lambda'_T(u)$. Then (4.46), extended to $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$, proves (4.39a) from which (4.39b) follows by duality. For (4.40a), see Remark 4.3 or (4.19a).

(b) In (4.39b), we have that both $(\mathcal{L}_T^{\#})^* = \big((\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^{\perp}})^{-1}\big)^*$ is (plainly) injective from $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ to H and $\mathcal{M}_T^*[y_u]$ is injective on H as noted in (4.36a) by (**C.2**). So $I+(\Lambda'_T(u))^*$ is injective, as claimed in (4.41). Similarly, $[I+(C_T^0)^*]$ is injective, by invoking now that $(\mathcal{M}_T^0)^*$ is injective, as noted in (4.36b).

(c) But $(\Lambda'_T(u))^*$ is compact, since so is $C_T[u] = \Lambda'_T(u)$ by (4.19a) = (4.37). Therefore, injectivity of $[I+(\Lambda'_T(u))^*]$ is equivalent to bounded invertibility of $[I+(\Lambda'_T(u))^*]$, hence to bounded invertibility of $[I+\Lambda'_T(u)]$, and (4.43) follows.

Conclusion (4.44) follows in a similar way, using now that C_T^0 is compact by (4.38). $\square$

$$
\begin{tikzcd}
\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T) \arrow[r, "\mathcal{M}_T[y_u]"] \arrow[dr, "{[I+\Lambda'_T(y_u)]}"'] & h \in H \arrow[d, "\mathcal{L}_T^{\#}:"', "\text{isomorphism}"] \\
 & v \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T); v = \mathcal{L}_T^{\#}h
\end{tikzcd}
$$

FIG. 4.2: PROPOSITIONS 4.2 AND 4.3. GIVEN $v \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$, EQUIVALENTLY, GIVEN $h \in H$, WITH $v = \mathcal{L}_T^{\#}h$, THERE EXISTS $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ SUCH THAT $\tilde{u} + \Lambda'_T(y_u)\tilde{u} = v$ IF AND ONLY IF SUCH $\tilde{u}$ ALSO SATISFIES $\mathcal{M}_T[y_u]\tilde{u} = z(T) = h$, ALONG THE LINEARIZED z-EQUATION (4.25) WITH $z(0) = 0$ (EXACT CONTROLLABILITY OF THE z-PROBLEM). IN SHORT: $I + \Lambda'_T(y_u) = \mathcal{L}_T^{\#}\mathcal{M}_T[y_u]$, AND $[I + \Lambda'_T(y_u)]$ IS SURJECTIVE FROM $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ ONTO ITSELF, IF AND ONLY IF $\mathcal{M}_T[y_u]$ IS SURJECTIVE FROM $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ ONTO H.

Step 4. What remains to be done in order to complete the proof of Theorem 1.1 is to show that the family of inverse operators

$$[I + \Lambda'_T(y_u)]^{-1} = \{I + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u][I - \mathcal{K}[y_u]]^{-1}\mathcal{L}\}^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^{\perp}),$$
$$\forall\, u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}, \quad (4.47)$$

each member of which is in $L([\mathcal{N}(\mathcal{L}_T)]^{\perp})$ by Proposition 4.3(c), Eqn. (4.43), possesses, in fact, a bound which is uniform with respect to $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$; that is,

$$\|[I + \Lambda'_T[y_u]]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} \leq \text{const., uniformly in } u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}. \quad (4.48)$$

Two inversions are involved in (4.47), and both have to be shown to be uniformly bounded in u. It is at this level that properties (P.2) and (P.3) in Section 2 play a critical role, as described below. The first uniform inversion is accomplished in Lemma 4.4 below. The second and conclusive uniform inversion is accomplished in Lemma 4.6 below. To this end, we notice that in order to exploit that $\mathcal{L}$ is compact: $L_2(0,T;U) \to \mathcal{E}_T \equiv L_2(0,T;Y)$, $U = L_2(\Gamma_1)$, $Y = L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, see (2.14), we shall then seek a uniform bound for $[I - \mathcal{K}[\eta]]^{-1}$ in the uniform norm of $L(Y)$, with $\eta \in \mathcal{E}_T$, *a-fortiori* for $\eta = y_u \in C([0,T];H)$, see Step 1.

Lemma 4.4. *Not only do we have $[I - \mathcal{K}[\eta]]^{-1} \in L(\mathcal{E}_T)$, for each $\eta \in \mathcal{E}_T \equiv L_2(0,T;Y)$, $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, as was shown in (4.19b), but moreover:*

(a1)

$$\|(I - \mathcal{K}[\eta])^{-1}\|_{L(\mathcal{E}_T)} \leq \textit{const., uniformly in } \eta \in \mathcal{E}_T. \quad (4.49)$$

(a2) Let η_{n_k} be a subsequence of a given arbitrary sequence $\eta_n \in \mathcal{E}_T$, such that

$$\mathcal{K}[\eta_{n_k}] \to \textit{ some } \mathcal{K}^0 = \mathcal{R}F_0 \textit{ strongly in } \mathcal{E}_T, \quad (4.50)$$

for some $F_0 \in L(Y)$, as guaranteed by property (P.2)(b), Eqn. (2.29), whereby then $\mathcal{K}^0$ is compact on $\mathcal{E}_T$, as noted in (2.33). Then

$$[I - \mathcal{K}^0]^{-1} \in L(\mathcal{E}_T), \textit{ and } (I - \mathcal{K}[\eta_{n_k}])^{-1} \to (I - \mathcal{K}^0)^{-1} \textit{ strongly in } \mathcal{E}_T. \quad (4.51)$$

Proof. (a_2) The proof relies entirely on property (P.2), which then allows one to invoke a standard result [A.1, Theorem 1.6, p. 6] on a family of collectively compact operators. As already noted in (2.33), the operator $\mathcal{K}^0$ is of the form $\mathcal{K}^0 \equiv \mathcal{R}F_0$. This then, as in (4.3) allows one to obtain that $[I - \mathcal{K}^0]^{-1} \in L(\mathcal{E}_T)$, as desired. [One could also use the present property that $\mathcal{K}^0$ is compact, whereby then it suffices to establish that $[I - \mathcal{K}^0]$ is injective: $[I - \mathcal{R}F_0]f = 0 \Rightarrow f = 0$, which is true since this leads by differentiation, via (2.16), to $\dot{f} = Af + F_0 f$, $f(0) = 0 \Rightarrow f \equiv 0$.] Moreover, the strong convergence in (4.50) for the collectively compact family $\mathcal{K}[\eta]$, as noted in (P.2)(b) = (2.29), combined with the existence of $[I - \mathcal{K}^0]^{-1} \in L(\mathcal{E}_T)$ just established implies via [A.1, Theorem 1.6, p. 6] that $[I - \mathcal{K}[\eta_{n_k}]]^{-1} \in L(\mathcal{E}_T)$ for all k sufficiently large, and, moreover, that the strong convergence in (4.51) takes place. Part (a2) is proved and this establishes part (a1) since the sequence $\{\eta_n\}$ was arbitrary. $\square$

Step 5. Let us define the family of operators

$$W_T[\eta] \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1} \tag{4.52a}$$

$$: \text{continuous } \mathcal{E}_T \equiv L_2(0, T; Y) \to [\mathcal{N}(\mathcal{L}_T)]^{\perp} \text{ for each } \eta \in \mathcal{E}_T, \tag{4.52b}$$

so that

$$W_T[\eta]\mathcal{L} \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L} \equiv \Lambda_T'[\eta] \equiv C_T[\eta], \tag{4.53}$$

by (4.19a) or (4.37a), where the indicated regularity property in (4.52b) is obtained by combining (4.49) [but (4.19b) would suffice] with property (2.34) for $\mathcal{K}_T[\eta]$ and property (4.9) for $\mathcal{L}_T^{\#}$. Moreover, we have that: Given any sequence $\eta_n \in \mathcal{E}_T$, we can extract a subsequence η_{n_k} such that

$$W_T[\eta_{n_k}] \to W^0 \equiv \mathcal{L}_T^{\#}\mathcal{K}_T^0(I - \mathcal{K}^0)^{-1} \text{ weakly from } \mathcal{E}_T \text{ to } [\mathcal{N}(\mathcal{L}_T)]^{\perp}. \tag{4.54}$$

This follows by combining property (P.2)(b), Eqn. (2.29a), with property (P.3), Eqn. (2.35a) as well as (4.51) (right) of Lemma 4.4 and the regularity in (4.9) for $\mathcal{L}_T^{\#}$.

The desired conclusion then follows as a specialization of the following result.

Lemma 4.5. *Let Z_i be two Banach spaces, $i = 1, 2$. Let Q be a compact operator $Z_1 \to Z_2$, and let $W(p)$ be a family of bounded operators $Z_2 \to Z_1$ depending on the parameter $p \in \mathcal{P}$, such that*

$$W(p_n) \to W^0 \text{ weakly} \tag{4.55}$$

for any sequence p_n, with $W^0 : Z_2 \to Z_1$ depending on the sequence. Assume further that the operators $[I + W(p)Q]$ and $[I + W^0Q]$ are all injective on Z_1, and hence are boundedly invertible on Z_1, as $W(p)Q$, W^0Q are compact. Then, in fact,

$$\|[I + W(p)Q]^{-1}\|_{L(Z_1)} \leq \text{const., uniformly in } p \in \mathcal{P}, \tag{4.56}$$

and the weak convergence

$$[I + W(p_n)Q]^{-1} \to [I + W^0Q]^{-1} \text{ weakly in } Z_1 \tag{4.57}$$

holds true. Moreover, denoting by $*$ the obvious adjoints,

$$Q^*W^*(p) \text{ is a collectively compact family in } Z_1 \text{ in the parameter } p \in \mathcal{P}; \tag{4.58}$$

$$Q^*W^*(p_n) \to Q^*(W^0)^* \text{ strongly in } Z_1; \tag{4.59}$$

$$[I + Q^*W^*(p_n)]^{-1} \to [I + Q^*(W^0)^*]^{-1} \text{ strongly in } Z_1; \tag{4.60}$$

$$\|[I + Q^*W^*(p_n)]^{-1}\|_{L(Z_2)} \leq \text{const., uniformly in } n. \tag{4.61}$$

Proof. Two proofs—one direct on these operators, one on their adjoints—are given in [L-T.4, Lemma 2.4, p. 123]. □

Specialization to our present problem and completion of proof of Theorem 1.1. We apply Lemma 4.5 with:

$$\begin{aligned} Z_1 &\equiv [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(0, T; L_2(\Gamma_1)); \\ Z_2 &\equiv \mathcal{E}_T \equiv L_2(0, T; Y);\ Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'; \end{aligned} \tag{4.62}$$

$$Q \equiv \mathcal{L} \text{ defined in (2.14)}; W(p) \equiv W_T[\eta] \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}, \tag{4.63}$$

as defined in (4.52). We now verify the required assumptions of Lemma 4.5. First, the required weak convergence (4.55) is assured by the established property (4.54). Here the parameter $p \in \mathcal{P}$ specializes to the parameter $\eta \in \mathcal{E}_T$. Next, via (4.53), injectivity of $[I + W(p)Q] = I + \Lambda'_T[\eta] = I + C_T[\eta]$ and of $I + W^0 Q = I + C^0_T$ is assured by (4.43), (4.44), respectively. Then, conclusion (4.56) of Lemma 4.5 specializes, under the above setting, to the following result.

Lemma 4.6. *Under assumption (1.2), we have*

$$\begin{aligned} & \|[I + \Lambda'_T[\eta]]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \\ & \qquad \equiv \quad \|\{I + \mathcal{L}^{\#}_T \mathcal{K}_T[\eta][I - \mathcal{K}[\eta]]^{-1}\mathcal{L}\}^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \\ & \qquad \leq \quad \textit{const., uniformly in } \eta \in \mathcal{E}_T \equiv L_2(0,T;Y), \end{aligned} \tag{4.64}$$

$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$; *in particular, uniformly in* $\eta = y_u \in C([0,T];H)$, $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, *see Step 1.*

Theorem 1.1 is now proved via Lemma 4.6 via Proposition 4.1(b), followed by Proposition 4.1(a).

5 Two Main Strategies for Exact Controllability of the Semilinear System (1.1) and their Equivalence. Four Implementations

Orientation. Proposition 4.1(a)—depicted in Fig. 4.1—states an *equivalence* between the following two facts: on the one hand, the sought-after exact controllability of the semilinear system (2.1) (subject to (2.8)) [or (1.1) subject to (1.2)] from the initial point $y_0 \in H$ at $t = 0$ to the target state $y_T \in H$ at $t = T$, by means of the steering control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ and, on the other hand, exact controllability of the corresponding *linear* system (2.1) with $F \equiv 0$ [(1.1) with $f \equiv 0$] from the same initial point y_0 at $t = 0$ to the same target state y_T at $t = T$; however, this time, by virtue of the control $v_T^0 = u + \Lambda_T(u) \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$.

Next, Proposition 4.1(b) provides a *sufficient* condition for Proposition 4.1(a) to actually hold true—that is, for the solvability in u of the equation $u + \Lambda_T(u) = v_T^0$—in terms of the uniform invertibility condition (4.18) on the inverse $[I + \Lambda'_T(u)]^{-1}$: more precisely, condition (4.18) implies that the C^1-map $u \to u + \Lambda_T(u)$ is a homeomorphism (bijective, continuous, with continuous inverse) from $[\mathcal{N}(\mathcal{L}_T)]^\perp$ onto itself.

In turn, Proposition 4.2, quantified further in Proposition 4.3 and depicted in Figure 4.2, states in particular that for any parameter $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ *fixed*, consequently with solution $y_u \in C([0,T]; H)$ fixed (see Section 4, Step 1), we have the following equivalence:

$$\begin{cases} \text{the linear map } \tilde{u} \to \tilde{u} + \Lambda'_T(y_u)\tilde{u} \text{ in (4.23) is surjective} \\ \text{from } [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \text{ onto itself,} \end{cases} \tag{5.1}$$

if and only if the z-problem (2.18) with $\eta_1 = y_{u,1} = w$ and $z_0 = 0$ is exactly controllable on H in $[0,T]$: that is,

$$\begin{cases} \text{the closed linear map } \tilde{u} \to \mathcal{M}_T[y_u]\tilde{u} = z(T) \text{ in (4.33) is} \\ \text{surjective from } L_2(0,T;U) \supset [\mathcal{N}[\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \text{ onto } H, \end{cases} \tag{5.2}$$

since $I + \Lambda'_T[y_u] = \mathcal{L}_T^{\#}\mathcal{M}_T[y_u]$ by (4.39a), where $\mathcal{L}_T^{\#}$ in (4.39a) is an isomorphism between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ as noted below (4.9); in turn, by duality [T-L.1, p. 235], if and only if

$$\|u\|_{L_2(\Sigma_1)} \le C'_{Ty_u} \|[I + \Lambda'_T(y_u)]^* u\|_{L_2(\Sigma_1)}, \quad u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(\Sigma_1) \tag{5.3}$$

(duality on (5.1)), $L_2(\Sigma_1) = L_2(0,T; L_2(\Gamma_1))$, and, in turn, if and only if

$$\|h\|_H \le C_{Ty_u} \|\mathcal{M}_T^*(y_u)h\|_{L_2(\Sigma_1)}, \quad h \in \mathcal{D}(\mathcal{M}_T^*(y_u)) = \mathcal{D}(\mathcal{L}_T^*) \subset H \tag{5.4}$$

(duality on (5.2)). The positive constants C'_{Ty_u} and C_{Ty_u} are independent of u and h, respectively. Next, we recall from (4.19a) that we are currently writing $\Lambda'_T(u)$ to really mean $\Lambda'_T(y_u)$, i.e., the dependence of Λ'_T is directly in y_u, rather than u. In fact, more precisely, referring to the operators $\mathcal{K}$

in (2.21), (2.22) and, similarly, $\mathcal{K}_T$—which define Λ'_T—we see that the actual dependence is on $f'(y_{1,u}(t,x)) = f'(w_u(t,x))$, with reference to (2.1) or (1.1), respectively. Because of assumption (1.2), we can therefore consider Λ'_T as, ultimately, depending on a parameter $q \in L_\infty(Q)$, where, at the end, $q(t,x) = f'(w_u(t,x))$, where, moreover, q is confined in a fixed sphere $B(0,r)$ centered at the origin, of radius $r > 0$, in $L_\infty(Q)$. We shall accordingly write $\Lambda'_T(q)$, $q \in B(0,r) \subset L_\infty(Q)$. We next let q run over $B(0,r)$ and show that the corresponding *equivalent* inequalities (5.3) and (5.4)—except, this time, with constants uniform in q—are then *equivalent* to the uniform invertibility condition (4.18b) on the inverse $[I+\Lambda'_T(q)]^{-1}$, which, in turn (by Proposition 4.1(b)) guarantes identity (4.16), and hence by Proposition 4.1(a), implies the exact controllability of the semilinear problem (1.1) or (2.1) over the interval $[0,T]$ on the state space H.

Proposition 5.1. *Let $q \in B(0,r) \subset L_\infty(Q)$, $r > 0$.*

(I) Then, the following inequalities are equivalent:

(i) (uniform COI)

$$\|h\|_H \le C_{Tr}\|\mathcal{M}^*_T(q)h\|_{L_2(\Sigma_1)}, \quad h \in \mathcal{D}(\mathcal{M}^*_T(q)) = \mathcal{D}(\mathcal{L}^*_T) \subset H; \tag{5.5}$$

(ii) (surjectivity of the family of operators $[I+\Lambda'_T(q)]$ from $L_2(0,T;L_2(\Gamma_1)) = L_2(\Sigma_1) \supset [\mathcal{N}(\mathcal{L}_T)]^\perp$ onto itself, "uniformly in $q \in B(0,r)$," see below (5.6))

$$\|u\|_{L_2(\Sigma_1)} \le C'_{Tr}\|[I+\Lambda'_T(q)]^*u\|_{L_2(\Sigma_1)}, \quad u \in [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{5.6}$$

[By the proof of [T-L.1, Theorem 9.4, in particular, the paragraph below (9.1), p. 236], this means that:

$$\overline{[I+\Lambda'_T(q)](S_1)} \supset C_{C'_{Tr}}, \quad \forall\, q \in B(0,r),$$

where $S_m = \{\mu \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(\Sigma_1) :\ \|u\|_{L_2(\Sigma_1)} \le m\}$);

(iii) (condition (4.18b))

$$\|[I + \Lambda_T'(q)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \leq \ const_{Tr}. \tag{5.7}$$

All of the positive constants in (5.5), (5.6), (5.7) depend only on T and the radius $r > 0$ of $B(0,r)$, not on h or u or q.

(II) Accordingly, by Proposition 4.1(b), each of the above conditions (5.5), (5.6), or (5.7) guarantees that the map $u \to u + \Lambda_T(u)$ is surjective from $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ onto itself, that is condition (4.16). Hence, each of the above conditions (5.5), (5.6), (5.7) implies, by Proposition 4.1(a), the exact controllability of the semilinear problem (1.1) or (2.1) on the state space H over the interval $[0,T]$.

Proof. The equivalence (5.5) $\iff$ (5.6) stems, again, from the identity $[I + \Lambda_T'(q)]^* = \mathcal{M}_T^*(q)(\mathcal{L}_T^\#)^*$ in (4.39b). Assume (5.6). For $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, set $h = (\mathcal{L}_T^\#)^* u \in H$, where $(\mathcal{L}_T^\#)^*$ is a bounded operator from $[\mathcal{N}(\mathcal{L}_T)]^\perp$ into H. In fact, Range$(\mathcal{L}_T^\#)^* = (\mathcal{L}_T^\#)^*[\mathcal{N}(\mathcal{L}_T)]^\perp = \mathcal{D}(\mathcal{L}_T^*)$, since

$$(\mathcal{L}_T^\#)^* = \left((\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^{-1}\right)^* = \left((\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^*\right)^{-1} = (\mathcal{L}_T^*)^{-1},$$

in our present setting and notation: $\mathcal{L}_T = \mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}$. Moreover, $(\mathcal{L}_T^\#)^*$ is injective on $[\mathcal{N}(\mathcal{L}_T)]^\perp$, since $\mathcal{L}_T^\#$: H onto $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ by (4.9), hence with range dense in $[\mathcal{N}(\mathcal{L}_T)]^\perp$. In short: $(\mathcal{L}_T^\#)^*$ *is an isomorphism from* $[\mathcal{N}(\mathcal{L}_T)]^\perp$ onto $\mathcal{D}(\mathcal{L}_T^*)$. Hence, $u = ((\mathcal{L}_T^\#)^*)^{-1}h$ and then, by the assumed inequality (5.6) and (4.39b) recalled above, we obtain

$$\frac{1}{\|(\mathcal{L}_T^\#)^*\|}\|h\|_H \leq \|((\mathcal{L}_T^\#)^*)^{-1}h\|_{L_2(\Sigma_1)} \leq C_{Tr}'\|\mathcal{M}_T^*(q)h\|_{L_2(\Sigma_1)}. \tag{5.8}$$

All $h \in \mathcal{D}(\mathcal{L}_T^*) = \mathcal{D}(\mathcal{M}_T^*(q))$, see (4.35a) arise in this way. In conclusion, inequality (5.8) has been shown for all $h \in \mathcal{D}(\mathcal{M}_T^*(q)) = \mathcal{D}(\mathcal{L}_T^*)$. Thus, we have obtained (5.5).

Conversely, assume (5.5). For $h \in \mathcal{D}(\mathcal{M}_T^*(q)) = \mathcal{D}(\mathcal{L}_T^*)$, we set, recalling the above isomorphism on $(\mathcal{L}_T^\#)^*$, $u = ((\mathcal{L}_T^\#)^*)^{-1}h$, or $(\mathcal{L}_T^\#)^*u = h$, so that,

$u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$. Then, (5.5) implies via (4.39b),

$$\begin{aligned} c\|u\|_{L_2(\Sigma_1)} \leq \|(\mathcal{L}_T^{\#})^* u\|_H &\leq C_{Tr}\|\mathcal{M}_T^*(q)(\mathcal{L}_T^{\#})^* u\|_{L_2(\Sigma_1)} \\ &= C_{Tr}\|[I + \Lambda_T'(q)]^* u\|_{L_2(\Sigma_1)}, \end{aligned} \tag{5.9}$$

for all $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, where the inequality at the extreme left is obtained via [T-L.1, p. 235], since $\mathcal{L}_T^{\#}$: H onto $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$. Thus, we have proved (5.6).

Next, the implication (5.7) $\Rightarrow$ (5.6) is obvious, while the reverse implication (5.6) $\Rightarrow$ (5.7) follows via the open mapping theorem, once we know that the surjective operator $[I + \Lambda_T'(q)]$ is also injective for each $q \in L_\infty(Q)$, a fact which is *a-fortiori* true by (4.43) of Proposition 4.3. $\square$

We shall see in Appendix A, Eqn. (A.17), that a reformulation of the dual condition (5.5), may be written *equivalently* in terms of the solution $\phi(t; \Phi_0)$, $\Phi_0 = [\phi_0, \phi_1]$ of problem (3.9) [same as (6.3) below].

Proposition 5.2. *An equivalent reformulation of the sufficient condition (5.5) for exact controllability of the semilinear system (2.1) subject to (2.8) [that is, (1.1) subject to (1.2)], in the state space H, over the interval $[0, T]$ is that: there exists a constant $C_T > 0$, independent of the I.C. and of the potential $q \in B(0, r)$ (a ball of radius $r > 0$ in $L_\infty(Q)$) such that the following inequality holds true for the solutions of problem (3.9)*

$$\|\{\phi_0, \phi_1\}\|_Y^2 \leq C_{Tr} \int_0^T \int_{\Gamma_1} \phi^2 d\Sigma_1, \tag{5.10}$$

where $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$

Remark 5.1. Abstractly, inequality (5.10) can be written in the following form: (5.10) is the explicit version of

$$\|V_0\|_H^2 \leq C_{Tr} \int_0^T \|B^* V(t; V_0)\|_U^2 dt \tag{5.11}$$

where $V(t;V_0)$, $V_0 \in H$, solves the equation (see (B.14)–(B.16) in Appendix B below)

$$V_t = -\mathbb{A}^*(t)V = -A^*V - (F'[y_u])^*V, \quad V(T) = V_0 \in H, \tag{5.12}$$

$F'[y_u] = [0, q(t,x)]^{tr}$, $q(t,x) = f'(w(t,x))$, according to (2.7), with $\eta = y_u$, where the adjoint B^* refers to H, see Eqn. (B.8) of Appendix B.

Proof of Proposition 5.2 and Remark 5.1. This result is contained in the subsequent (independent) Appendix B. Indeed, Eqn. (B.13) gives $\mathcal{M}_T^* h = B^*V(\,\cdot\,;V_0)$, where $V_0 = h$ by (B.12). This way, (5.5) becomes (5.10). Moreover, $V(t;V_0)$ satisfies Eqn. (B.14): $V_t(t;V_0) \equiv -\mathbb{A}^*(t)V(t;V_0)$, where $\mathbb{A}^*(t) \equiv A^* + (F'[q])^*$ by (B.15a). Proposition 5.2 and Remark 5.1 are thus recovered. □

Remark 5.2. Problem (5.12) is dual to the original linearized controlled z-problem (2.18) (with $z_0 = 0$), which defines $\mathcal{M}_T[y_u]$ via (4.33). The PDE-reformulation of (5.12) is given by the ϕ-problem (3.9) = (6.3) [or the v_2-problem (B.17) in Appendix B, where $V(t;V_0) = [v_1(t;V_0), v_2(t;V_0)]^{tr}$ by (B.12), $V_0 = [h_0, h_1] \in H$, $\phi_0 = h_1 \in L_2(\Omega)$, $\phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$.

Two strategies. Four implementations. Thus, on the basis of Proposition 5.1, two strategies arise in order to show exact controllability of the semilinear system (1.1), subject to (1.2):

(a) *Strategy 1*: The first strategy consists in establishing the uniform invertibility condition (5.7) = (4.18b), thus satisfying Proposition 4.1(b) and Proposition 5.1. This strategy was pursued in Part I in the proof of Theorem 1.1 given in Section 4 and based on the *original controlled problem* (1.1) or (2.1). Critical ingredients are Lemmas 4.4 and 4.5. The final step was achieved in Lemma 4.6, Eqn. (4.64). Then, Proposition 4.1(a) (or Proposition 5.1) completes the proof of Theorem 1.1.

(b) *Strategy 2*: A second strategy consists in establishing condition (5.5) of Proposition 5.1 (uniform continuous observability inequality) of the linearized problem uniformly with respect to $L_\infty(Q)$-potentials $q(t,x)$ in a fixed ball of radius r, where then (5.5) is attained for $f'(w(t,x)) = q(t,x)$, and then $F'[y_u] = [0, q(t,x)]^{tr}$ according to (2.7) with $\eta = y_u$. In turn, the uniform COI (5.5) is given explicitly by inequality (5.10) of Proposition 5.2. Thus, strategy 2 centers on the *dual uncontrolled* problem (3.9) = (6.3) in PDE-form ((5.12) in abstract form), rather than the original controlled problem (1.1) or (2.1).

To pursue Strategy 2 centered on the uniform COI (5.5), equivalently (5.10), three different implementations are possible:

(b_1) *Strategy 2, Implementation (i).* Here one seeks to show the uniform COI (5.5), equivalently (5.10), of the linearized problem by a dual analysis of the operator-theoretic approach of Section 4, centered this time on the dual uncontrolled problem (3.9) = (6.3). This implementation is pursued in Part II (Sections 6 through 7). It shows the uniform COI (5.5) = (7.4) *directly*, by virtue of the critical Lemmas 4.4 and 4.5, this time in dual versions, however.

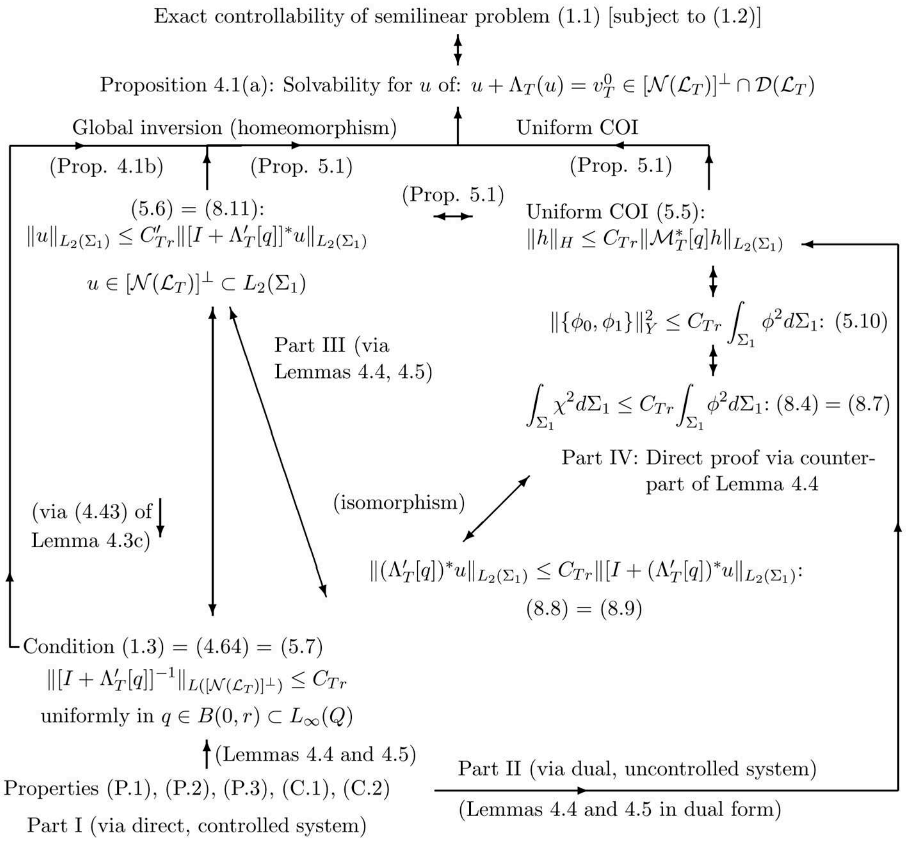
(b_2) *Strategy 2, Implementation (ii).* This is pursued in Part III (Section 8). It is still focused, as in implementation (b_1), on showing the uniform COI (5.5) of the *dual, uncontrolled* problem. However, unlike implementation (b_1), the key point now is that Part III aims at proving the explicit COI (5.10) by seeking to establish the *equivalent* comparison trace inequality (8.4) or its operator-theoretic version (8.7). However, Part III does *not* prove (8.4) = (8.7) *directly.* Rather, Part III seeks to prove the *equivalent* inequality (8.8) (where the equivalence is described by an isomorphism). But inequality (8.8) is precisely the same as inequality (8.9), expressed in terms of quantities of Part I. Thus, inequality (8.9) (= (8.8)) is ultimately established by falling into Part I and invoking Lemmas 4.4 and 4.5 of Part

I.

(b_3) *Strategy 2, Implementation (iii).* This Implementation (iii) constitutes Part IV (Section 9). Here one likewise seeks to establish the uniform COI (5.5), equivalently (5.10), via an analysis of the *dual uncontrolled* problem. However, unlike Implementation (ii), now one seeks to prove the trace inequality (or its operator-theoretic reformulation) (8.4) = (8.7) *directly*, not its *equivalent* (via an isomorphism) version (8.8) as in (b_2). This will result in some advantages described in the "Orientation" of Section 9.

The above strategies and implementations are pictorially condensed in Fig. 5.1.

Remark 5.3. As a matter of fact, Strategy 2 admits a fourth purely PDE-implementation at the level of showing the trace inequality (8.4) by a double compactness/uniqueness argument. This is pursued in another paper [L-T.10].



where $I + (\Lambda'_T[q])^* = \mathcal{M}_T^*[q](\mathcal{L}_T^\#)^*$ [(4.39b)], $\mathcal{L}_T^\#$ isomorphism between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$; $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$

Fig. 5.1: Part I establishes the left column through an operator-theoretic approach on the controlled problem (1.1) or (2.1). Part II establishes the uniform COI (5.5) through a dual operator-theoretic approach on the dual uncontrolled problem (3.9) = (6.3). Part III establishes (8.4) = (8.7) not directly, but via the equivalent (8.8) = (8.9), by invoking (5.6) = (8.11) of Part I. Part IV establishes the uniform COI (5.10) by proving (8.4) = (8.7) directly still through a dual operator-theoretic approach on the dual uncontrolled problem (3.9) = (6.3).

Part II: Proof of Theorem 1.1 through the Uniform COI (5.5): Duality over the Abstract Operator Approach of [L-T.4], via the Dual Uncontrolled Problem (3.9) = (6.3) (Sections 6–7)

6 A Direct Derivation of Formula (4.35a) for the Operator $\mathcal{M}_T^*[\eta]$ via the Dual Problem. PDE-interpretation

The purpose of this section is two-fold. First, we provide a *direct* proof of formula (4.35a) for the operator $\mathcal{M}_T^*[\eta]$, which involves *only the dual linearized* problem (with no control), rather than the original linearized controlled z-(abstract) system in (2.18) or (2.37) in PDE-form. Next, we provide in Proposition 6.7 a PDE-interpretation of the perturbation formula (4.35a). A systematic study of the dual linearized problem will be given in Appendices A, B, and C. The dual linearized problem in question is the ϕ-problem (3.9)= (6.3) below = (A.4) in Appendix A below, with I.C. $\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, see (3.10) (same as the v_2-problem in (B.17) in Appendix B below) and potential $q(t,x) = f'(\eta_1(t,x)) \in L_\infty(Q)$. We shall, accordingly, invoke some results from (the independent) Appendices. By contrast, in Section 4, we first established formula (4.33) for the input (control)-solution map $z(T) = \mathcal{M}_T[\eta]u$ for the controlled linearized z-problem (2.18) with $z_0 = 0$, and next we obtained formula (4.35a) for $\mathcal{M}_T^*[\eta]$ by duality. In short: we seek to avoid the operation of duality from $\mathcal{M}_T[\eta]$ to $\mathcal{M}_T^*[\eta]$ and, instead, derive $\mathcal{M}_T^*[\eta]$ directly.

Derivation of Formula (4.35a). Proposition 6.1 *Let, as in (3.15) or (4.33),*

$$z(T) = \mathcal{M}_T[\eta]u;\ \mathcal{M}_T[\eta] : L_2(0,T;U) \supset \mathcal{D}(\mathcal{M}_T[\eta]) \equiv \mathcal{D}(\mathcal{L}_T) \to H, \tag{6.1}$$

where $\eta \in \mathcal{E}_T \equiv L_2(0,T;Y)$, $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$; $U = L_2(\Gamma_1)$, $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, *and where* z *is the solution of the linearized problem (2.18)* *(given by (2.19)) for I.C.* $z_0 = 0$. *Then, its (Hilbert space) adjoint* $\mathcal{M}_T^*[\eta]$ *is given by*

$$\mathcal{M}_T^*[\eta] = \mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] \tag{6.2a}$$

$$: H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0,T;U), \tag{6.2b}$$

in agreement with formula (4.35a), where the operators on the RHS of (6.2a) are specified in Remark 4.3.

Proof. Here we provide a *direct* proof using the *dual* (uncontrolled) problem.

Step 1. Our starting point is such dual (uncontrolled) problem, which is the ϕ-problem (3.9) [same as (6.3) or (A.4) in Appendix A below] with I.C. $\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, or—which is the same—the v_2-problem (B.17) in Appendix B with I.C. $\{v_2(T), v_{2t}(T)\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, where the potential $q(t,x)$ is throughout specialized to $q(t,x) = f'(\eta_1(t,x)) \in L_\infty(Q)$, $\eta = [\eta_1, \eta_2] \in \mathcal{E}_T$. When convenient, we use the notation of the subsequent Appendix B. Thus, the dual uncontrolled problem is, in the present context, given by

$$\begin{cases} \phi_{tt} = \Delta\phi + f'(\eta_1(t,x))\phi & \text{on } Q; \quad (6.3a) \\ \phi(T) = \phi_0 = h_1 \in L_2(\Omega);\ \phi_t(T) = \phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' & \text{in } \Omega; \quad (6.3b) \\ \phi|_{\Sigma_0} \equiv 0,\ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{on } \Sigma_i \quad (6.3c) \end{cases}$$

(see (B.17) in Appendix B) with $[h_0, h_1] \in H = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$.

In (the independent) Appendices A and B, it is proved in (A.12), or (B.13)–(B.17) that, then, the operator $\mathcal{M}_T^*[\eta]$ is given by

$$\{\mathcal{M}_T^*[\eta]V_0\}(t) = B^*V(t;V_0),\ V_0 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \in H,$$

$$V(t;V_0) = \begin{bmatrix} v_1(t;V_0) \\ v_2(t,V_0) \end{bmatrix} \tag{6.4a}$$

$$v_2(t;V_0) = \phi(t;\Phi_0); \quad v_1(t;V_0) = -\mathcal{A}^{-1}\phi_t(t;\Phi_0);\ \Phi_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \text{ in } Y, \tag{6.4b}$$

where the second component $v_2(t;V_0)$ of $V(t;V_0)$ is precisely the solution of problem (6.3). Here B^* is defined in Remark B.1 (the symbol $*$ refers to duality with respect to H, as stipulated in Remark 4.1), while $V(t;V_0)$ is the solution of (B.14) (with $q(t,x)=f'(\eta_1(t,x))$, rewritten here as

$$V_t(t;V_0) = -\mathbb{A}^*(t)V(t;V_0) = AV(t;V_0) + \begin{bmatrix} 0 & -\mathcal{A}^{-1}[f'(\eta_1(\,\cdot\,))\,\cdot\,] \\ 0 & 0 \end{bmatrix} V(t;V_0); \tag{6.5a}$$

$$V(t;V_0) = \begin{bmatrix} v_1(t;V_0) \\ v_2(t;V_0) \end{bmatrix} = \begin{bmatrix} -\mathcal{A}^{-1}\phi_t(t;\Phi_0) \\ \phi(t;\Phi_0) \end{bmatrix};\ V(T;V_0) = V_0 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \in H;$$

$$\Phi_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} h_1 \\ -\mathcal{A}h_0 \end{bmatrix} \in Y, \tag{6.5b}$$

recalling (B.15a), $-A^* = A$, and (B.12). More explicitly, we have that the solution $V(t;V_0)$ of (6.5a) is given by

$$V(t;V_0) \ = \ e^{A(t-T)}V_0 + \int_T^t e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau \tag{6.6}$$

$$= \ e^{A^*(T-t)}V_0 + \int_t^T e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau, \tag{6.7}$$

since $A^* = -A$ in H (again, $*$ denotes adjoint in H). Applying B^* to (6.7) yields, after recalling (6.4):

$$(\mathcal{M}_T^*[\eta]V_0)(t) \ = \ B^*V(t;V_0) = B^*e^{A^*(T-t)}V_0 \tag{6.8}$$

$$+ \ \int_t^T B^*e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau, \tag{6.9}$$

or recalling the usual dualities [L-T.9],

$$\mathcal{M}_T^*[\eta]V_0 = \mathcal{L}_T^*V_0 + \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\,\cdot\,))v_2(\,\cdot\,;V_0)] \\ 0 \end{bmatrix}, V_0 \in H, \tag{6.10}$$

where $\mathcal{L}_T^* : H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0,T;U)$ as defined in Remark 4.3, while $\mathcal{L}^*$ is defined by

$$(\mathcal{L}u,h)_{L_2(0,T;H)} = (u,\mathcal{L}^*h)_{L_2(0,T;U)},\ \forall\ u \in \mathcal{D}(\mathcal{L}),\ h \in \mathcal{D}(\mathcal{L}^*); \tag{6.11a}$$

$$\mathcal{L}^* : L_2(0,T;H) \supset \mathcal{D}(\mathcal{L}^*) \to L_2(0,T;U), \tag{6.11b}$$

consistently with the agreed-upon notation in Remark 4.1, where $*$ denotes adjoint from H.

Step 2. In contrast with (6.11), we now consider the operator $\mathcal{L}$ as in (2.14): continuous (compact) $L_2(0,T;U) \to \mathcal{E}_T \equiv L_2(0,T;Y)$ and, as in Remark 4.3, let $\mathcal{L}^{\circledast}$, be the adjoint defined by

$$(\mathcal{L}u, y)_{L_2(0,T;Y)} = (u, \mathcal{L}^{\circledast} y)_{L_2(0,T;U)}, \ \forall\, u \in L_2(0,T;U),\ y \in L_2(0,T;Y); \tag{6.12a}$$

$$\mathcal{L}^{\circledast} : \text{continuous (compact)} : L_2(0,T;Y) \to L_2(0,T;U). \tag{6.12b}$$

Lemma 6.2. *With reference to (6.11) and (6.12), we have*

$$\mathcal{L}^{\circledast} y = \mathcal{L}^{\circledast} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1} y_1 \\ \mathcal{A}^{-1} y_2 \end{bmatrix},\ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'. \tag{6.13}$$

Proof. We have that

$$\begin{bmatrix} \mathcal{A}^{-\frac{1}{2}} & 0 \\ 0 & \mathcal{A}^{-\frac{1}{2}} \end{bmatrix} : \text{isomorphism } Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \overset{\text{onto}}{\to} H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega), \tag{6.14}$$

hence via (6.14) and (6.11a)

$$\begin{aligned} (\mathcal{L}u, y)_{L_2(0,T;Y)} &= \left(\mathcal{L}u, \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{A}^{-1} \end{bmatrix} y \right)_{L_2(0,T;H)} \\ &= \left(u, \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{A}^{-1} \end{bmatrix} y \right)_{L_2(0,T;U)}, \end{aligned} \tag{6.15}$$

and (6.13) follows by comparing (6.12a) with (6.15). ☐

Applying identity (6.13) of Lemma 6.2 to (6.10), we obtain

$$\mathcal{M}_T^*[\eta] V_0 = \mathcal{L}_T^* V_0 + \mathcal{L}^{\circledast} \begin{bmatrix} f'(\eta_1(\,\cdot\,)) v_2(\,\cdot\,; V_0) \\ 0 \end{bmatrix}, V_0 = [h_0, h_1] \in H. \tag{6.16}$$

Step 3. Lemma 6.3 *With reference to the operator $\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta] \in L(\mathcal{E}_T; H)$ in (2.24), we have $\mathcal{K}_T^*[\eta] \in L(H; \mathcal{E}_T)$ where*

$$(\mathcal{K}_T^*[\eta]h)(t) = \begin{bmatrix} f'(\eta_1(t))\{e^{A^*(T-t)}h\}_2 \\ 0 \end{bmatrix}, \quad h \in H, \tag{6.17a}$$

where $e^{A^*(T-t)} = e^{A(t-t)}$ ($*$ adjoint in H, see Remark 4.1) and recalling (2.3)

$$\{e^{A^*(T-t)}h\}_2 = \{e^{A(t-T)}h\}_2 = -\mathcal{A}S(t-T)h_1 + C(t-T)h_2 \tag{6.17b}$$

is the second component, on the space $L_2(\Omega)$, of $e^{A^*(T-t)}h$ on H.

Proof. Let $g \in \mathcal{E}_T$, $h \in H$, $\eta \in \mathcal{E}_T$. With $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, we return to (2.24b) and compute

$$\begin{aligned} (\mathcal{K}_T[\eta]g, h)_H &= (\mathcal{R}_T F'[\eta]g, h)_H = (g, \mathcal{K}_T^*[\eta]h)_{\mathcal{E}_T} \qquad (6.18) \\ &= \left(\int_0^T e^{A(T-t)} \begin{bmatrix} 0 \\ f'(\eta_1(t))g_1(t) \end{bmatrix} dt, h \right)_H \\ &= \int_0^T \left(\begin{bmatrix} 0 \\ f'(\eta_1(t))g_1(t) \end{bmatrix}, e^{A^*(T-t)}h \right)_H dt \\ &= \int_0^T \Big(g_1(t), f'(\eta_1(t))\{e^{A^*(T-t)}h\}_2 \Big)_{L_2(\Omega)} dt \\ &= \int_0^T \left(\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, \begin{bmatrix} f'(\eta_1(t)) \left\{e^{A^*(T-t)}h\right\}_2 \\ 0 \end{bmatrix} \right)_Y dt; \qquad (6.19) \end{aligned}$$

$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Thus, comparing (6.18) (RHS) with (6.19), we obtain (6.17a). □

Step 4. We return to the operator $\mathcal{K}[\eta] = \mathcal{R}F'[\eta]$: continuous (compact) $\mathcal{E}_T \to \mathcal{E}_T$ as noted in (2.27), and let $\mathcal{K}^{\circledast}[\eta] \in L(\mathcal{E}_T)$ be its adjoint, defined in Remark 4.3(iii).

Lemma 6.4. *For the adjoint $\mathcal{K}^{\circledast}[\eta]$, we have for $e = [e_1, e_2] \in \mathcal{E}_T$:*

$$\left(\mathcal{K}^{\circledast}[\eta]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}\right)(t) = \begin{bmatrix} f'(\eta_1(t))\displaystyle\int_t^T \mathcal{A}^{-1}\{e^{A(t-\sigma)}e(\sigma)\}_2 d\sigma \\ 0 \end{bmatrix} \tag{6.20a}$$

$$= \begin{bmatrix} f'(\eta_1(t))\displaystyle\int_t^T [S(\sigma - t)e_1(\sigma) + \mathcal{C}(t-\sigma)\mathcal{A}^{-1}e_2(\sigma)]d\sigma \\ 0 \end{bmatrix}. \tag{6.20b}$$

Proof. For $g = [g_1, g_2] \in \mathcal{E}_T$, $e = [e_1, e_2] \in \mathcal{E}_T$, we compute with $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, recalling (2.22):

$$(\mathcal{K}[\eta]g, e)_{L_2(0,T;Y)}$$

$$\equiv (g, \mathcal{K}^{\circledast}[\eta]e)_{L_2(0,T;Y)} \tag{6.21}$$

$$= \int_0^T \left(\int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix} d\tau, \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}\right)_Y dt \tag{6.22}$$

$$= \int_0^T \int_0^t \left(\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix}, e^{A^{\circledast}(t-\tau)}e(t)\right)_Y d\tau dt \tag{6.23}$$

(changing the order of integration and using skew-adjointness $A^{\circledast} = -A$ on Y)

$$= \int_0^T \int_\tau^T \left(\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix}, e^{A(\tau-t)}e(t)\right)_Y dt d\tau \tag{6.24}$$

$$= \int_0^T \left(\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix}, \int_\tau^T e^{A(\tau-t)}e(t)dt\right)_Y d\tau \tag{6.25}$$

$$= \int_0^T \left(f'(\eta_1(\tau))g_1(\tau), \int_\tau^T \mathcal{A}^{-1}\{e^{A(\tau-t)}e(t)\}_2 dt\right)_{L_2(\Omega)} d\tau \tag{6.26}$$

$$= \int_0^T \left(\begin{bmatrix} g_1(\tau) \\ g_2(\tau) \end{bmatrix}, \begin{bmatrix} f'(\eta_1(\tau)) \int_\tau^T \mathcal{A}^{-1}\{e^{A(\tau-t)}e(t)\}_2 dt \\ 0 \end{bmatrix} \right)_Y d\tau. \tag{6.27}$$

Thus, comparing the RHS of (6.21) with (6.27) yields (6.20a), from which (6.20b) readily follows via (2.3). ☐

Step 5. Lemma 6.5. *Let $V_0 \in H$, $\eta \in \mathcal{E}_T$. With reference to $\mathcal{K}_T^*[\eta]$ in (6.17a) and $\mathcal{K}^{\circledast}[\eta]$ in (6.20a), we have:*

(i)

$$[I - \mathcal{K}^{\circledast}[\eta]] \begin{bmatrix} f'(\eta_1(\,\cdot\,))v_2(\,\cdot\,;V_0) \\ 0 \end{bmatrix} = \mathcal{K}_T^*[\eta]V_0; \tag{6.28}$$

(ii)

$$\begin{bmatrix} f'(\eta_1(\,\cdot\,))v_2(\,\cdot\,;V_0) \\ 0 \end{bmatrix} = [I - \mathcal{K}^{\circledast}[\eta]]^{-1}\mathcal{K}_T^*[\eta]V_0. \tag{6.29}$$

Proof. (i) We return to identity (6.7), in H, extract the second component in $L_2(\Omega)$, multiply this by $f'(\eta_1(t))$ and obtain since $V = [v_1, v_2]^{tr}$, $v_2 = \phi$; see (6.4a–b):

$$\begin{aligned} &f'(\eta_1(t))v_2(t;V_0) \\ &\quad = f'(\eta_1(t))\{e^{A^*(T-t)}V_0\}_2 \\ &\qquad + f'(\eta_1(t)) \int_t^T \left\{ e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} \right\}_2 d\tau. \end{aligned} \tag{6.30}$$

But, recalling (2.3), we have with $e^{A^*(\tau-t)} = e^{A(t-\tau)}$, since $S(\,\cdot\,)$ is odd:

$$\begin{aligned} &\left\{ e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} \right\}_2 \\ &\qquad = -\mathcal{A}S(t-\tau)\mathcal{A}^{-1}f'(\eta_1(\tau))v_2(\tau;V_0) \\ &\qquad = S(\tau-t)f'(\eta_1(\tau))v_2(\tau;V_0). \end{aligned} \tag{6.31}$$

Then, (6.30), augmented by a null second component, becomes, by recalling (6.17a), (6.31), and (6.20b):

$$\begin{aligned}
&\begin{bmatrix} f'(\eta_1(t))v_2(t;V_0) \\ 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} f'(\eta_1(t))\{e^{A^*(T-t)}V_0\}_2 \\ 0 \end{bmatrix} \\
&\qquad\quad + \begin{bmatrix} f'(\eta_1(t)) \int_t^T S(\tau - t) f'(\eta_1(\tau))v_2(\tau;V_0)d\tau \\ 0 \end{bmatrix} \qquad (6.32) \\
&\qquad = (\mathcal{K}_T^*[\eta]V_0)(t) + \left(\mathcal{K}^{\circledast}[\eta] \begin{bmatrix} f'(\eta_1(\,\cdot\,))v_2(\,\cdot\,;V_0) \\ 0 \end{bmatrix}\right)(t). \qquad (6.33)
\end{aligned}$$

Then, (6.33) establishes (6.28) and part (i) is proved.

(ii) Part (ii) follows from (6.28) as in obtaining (4.2) from (4.3).

Step 6. Substituting (6.29) into the RHS of (6.16) yields identity (6.2a) = (4.35a), as desired.

The following result is contained in the proof of Proposition 6.1.

Corollary 6.6. *The following identities hold true:*

$$\begin{aligned}
&\int_T^t B^* e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau \\
&\quad = \int_t^T B^* e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau \qquad (6.34) \\
&\quad = \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\,\cdot\,))v_2(\,\cdot\,;V_0)] \\ 0 \end{bmatrix} = \mathcal{L}^{\circledast} \begin{bmatrix} [f'(\eta_1(\,\cdot\,))v_2(\,\cdot\,;V_0)] \\ 0 \end{bmatrix} \\
&\quad = \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]V_0. \qquad (6.35)
\end{aligned}$$

Proof. We refer to, and trace, (6.6), (6.9), (6.10), (6.13), (6.16), (6.29). $\square$

PDE-interpretation of formula (6.2a) = (4.35). The following result provides a PDE-interpretation of the perturbation formula (6.2a) = (4.35) for $\mathcal{M}_T^*[\eta]$. To this end, we decompose the ϕ-problem (6.3), $\phi(t;\Phi_0) = v_2(t;V_0)$, as

$$\phi = \psi + \chi, \tag{6.36}$$

where ψ and χ are solutions of the following problems

$$\begin{cases} \psi_{tt} &= \Delta\psi \\ \psi(T,\,\cdot\,) &= \phi_0;\ \psi_t(T,\,\cdot\,) = \phi_1 \\ \psi|_{\Sigma_0} &\equiv 0;\ \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 \end{cases} \quad \begin{cases} \chi_{tt} &= \Delta\chi + f'(\eta_1(t,x))\phi & \text{in } Q; & (6.37a) \\ \chi(T,\,\cdot\,) &= 0;\ \chi_t(T,\,\cdot\,) = 0 & \text{in } \Omega; & (6.37b) \\ \chi|_{\Sigma_0} &\equiv 0;\ \left.\dfrac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i, & (6.37c) \end{cases}$$

with $\phi_0 = h_1 \in L_2(\Omega)$, $\phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$; $V_0 \equiv \{h_0, h_1\} \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, hence $\{\psi(T,\,\cdot\,), \psi_t(T,\,\cdot\,)\} = \{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Explicitly, ψ and χ are given by:

$$\psi(t) \equiv C(t-T)h_1 + S(t-T)(-\mathcal{A}h_0) = \{e^{A^*(T-t)}V_0\}_2,\ V_0 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \in H, \tag{6.38}$$

recalling, in the last step, (6.17b), (or (2.3)), and via the variation of parameter formula,

$$\chi(t) = \int_T^t S(t-\tau)f'(\eta_1(\tau))v_2(\tau;V_0)d\tau \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))$$
$$v_2(t;V_0) \equiv \phi(t;\Phi_0) \in C([0,T];L_2(\Omega)). \tag{6.39}$$

[We notice that via (6.38) and (6.39), the *decomposition* (6.36) *already appeared in the top line of identity* (6.32), stripped by the common factor $f'(\eta_1(t))$ across.]

Proposition 6.7. *With reference to problem (6.37) and identity (6.2a) = (4.35a): $\mathcal{M}_T^*[\eta] = \mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]$, we have:*

(i)

$$\{\mathcal{L}_T^* V_0\}(t) = B^* e^{A^*(T-t)} V_0 = [\{e^{A^*(T-t)} V_0\}_2]_{\Sigma_1} \equiv \psi(t)|_{\Sigma_1}; \quad V_0 \in \mathcal{D}(\mathcal{L}_T^*); \tag{6.40}$$

(ii) For $V_0 \in H$:

$$\{\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] V_0\}(t)$$

$$= \int_T^t B^* e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau; V_0) \\ 0 \end{bmatrix} d\tau \tag{6.41}$$

$$= \int_T^t \left[\left\{ e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau; V_0) \\ 0 \end{bmatrix} \right\}_2 \right]_{\Sigma_1} \tag{6.42}$$

$$= \int_T^t [S(t-\tau) f'(\eta_1(\tau)) v_2(\tau; V_0)]_{\Sigma_1} d\tau = \chi(t)_{\Sigma_1} \in C([0,T]; H^{\frac{1}{2}}(\Gamma)); \tag{6.43}$$

(iii) For $V_0 \in \mathcal{D}(\mathcal{L}_T^*)$:

$$\begin{aligned} \{\mathcal{M}_T^*[\eta] V_0\}(t) &= \{\mathcal{L}_T^* V_0\}(t) + \{\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] V_0\}(t) \\ &\equiv \psi(t; \Phi_0)|_{\Sigma_1} + \chi(t)|_{\Sigma_1} = v_2(t; V_0)|_{\Sigma_1} = \phi(t; \Phi_0)|_{\Sigma_1}. \end{aligned} \tag{6.44}$$

Proof. The first identity in (6.40) is the usual duality formula, see (6.8), (6.9). Next, we recall (see (9.8b)) that (#): $B^*\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = b|_{\Sigma_1}$. This formula and (6.38) then justify the last two identities in (6.40).

Identity (6.41) was already noted in Corollary 6.6, Eqns. (6.34), (6.35). Next, we use the above formula (#) for B^* to get (6.42). Next, we recall (6.31) as well as $A^* = -A$ to obtain the LHS of (6.43). Finally, we invoke (6.39) to obtain the RHS of (6.43). ◻

7 Direct Proof of the Uniform COI (5.5) Via Lemmas 4.4 and 4.5 (in Dual Versions) [Thus Using Properties (P.1), (P.2), (P.3), (C.1), (C.2)]

Orientation. In Section 5 we have extracted two basic strategies for proving Theorem 1.1, and, moreover, we have shown their equivalence in Proposition 5.1. Strategy #1 consists of showing the uniform invertibility condition (4.18b) = (5.7) for the operator $[I + \Lambda'_T(q)]^{-1}$. Strategy #2 consists of showing the uniform COI (5.5); that is, the uniform lower bound for the operator $\mathcal{M}^*_T[q]$. The equivalence between these two strategies was proved in Proposition 5.1 and rests on two facts: (a) the basic relationship (4.39b): $\mathcal{M}^*_T[q](\mathcal{L}^{\#}_T)^* = [I + \Lambda'_T(q)]^*$, between the relevant operators, and (b) the property that $(\mathcal{L}^{\#}_T)$: the state target space H onto the control space $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$. So far, however, it was the uniform invertibility condition (5.7) involving $[I+\Lambda'_T(q)]^{-1}$—not the uniform COI (5.10) involving $\mathcal{M}^*_T(q)$—that has been established, in Section 4, Lemma 4.6, by virtue of Lemma 4.4 and Lemma 4.5, on the basis of the structural properties (P.1), (P.2), (P.3) and the control properties (C.1) and (C.2).

In contrast, the *goal* of the present section is to prove the uniform COI (5.5) *directly*, on the basis of Lemmas 4.4 and 4.5 in dual versions.

We begin the present proof by returning to (4.35a) and its limit case (4.35b). We rewrite these formulas for convenience:

$$\mathcal{M}^*_T[\eta] \equiv \mathcal{L}^*_T + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}^*_T[\eta] : H \supset \mathcal{D}(\mathcal{L}^*_T) \to L_2(0, T; U); \tag{7.1a}$$

$$(\mathcal{M}^0_T)^* = \mathcal{L}^*_T + \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}^0_T)^* : H \supset \mathcal{D}(\mathcal{L}^*_T) \to L_2(0, T; U); \tag{7.1b}$$

We recall from (4.8a) that we are throughout considering $\mathcal{L}_T$ as an injective closed surjective operator from $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ onto H (via (C.1) = (3.5a)). Hence, the range $\mathcal{L}^*_T H$ is dense in $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ in the $L_2(0, T; U)$-topology. But, by (C.1) in its equivalent version (3.5b), $(\mathcal{L}^*_T)^{-1}$ is bounded on $\mathcal{L}^*_T H$. Hence,

$(\mathcal{L}_T^*)^{-1}$ is bounded on $[\mathcal{N}(\mathcal{L}_T)]^\perp$: $(\mathcal{L}_T^*)^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^\perp; H)$. With reference to (7.1), since the range of $\mathcal{L}^{\circledast}$ need not be contained in $[\mathcal{N}(\mathcal{L}_T)]^\perp$, we now *extend* $(\mathcal{L}_T^*)^{-1}$ *by zero on* $\mathcal{N}(\mathcal{L}_T)$:

$$(\mathcal{L}_T^*)^{-1} \equiv 0 \text{ on } [\mathcal{N}(\mathcal{L}_T)], \text{ so that } (\mathcal{L}_T^*)^{-1} \in L(L_2(0,T;U);H). \tag{7.2}$$

[A similar extension will be used in (8.12).] Next, we apply such (extended) $(\mathcal{L}_T^*)^{-1}$ across (7.1a) and (7.1b), thus obtaining

$$\{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\}h = (\mathcal{L}_T^*)^{-1}\mathcal{M}_T^*[\eta]h, \quad h \in \mathcal{D}(\mathcal{L}_T^*) \subset H; \tag{7.3a}$$

$$\{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*\}h = (\mathcal{L}_T^*)^{-1}(\mathcal{M}_T^0)^*h, \quad h \in \mathcal{D}(\mathcal{L}_T^*) \subset H. \tag{7.3b}$$

Our present *goal* is to show the uniform COI (5.5), rewritten here in terms of the parameter $\eta \in \mathcal{E}_T$ rather than the parameter (potential) $q \in L_\infty(Q)$ [eventually, $q(t,x) = f'(\eta_1(t,x))$] as

$$\|h\|_H \le C_T\|\mathcal{M}_T^*[\eta]h\|_{L_2(0,T;U)}, \ \forall\ h \in \mathcal{D}(\mathcal{L}_T^*) \text{ uniformly in } \eta \in \mathcal{E}_T, \tag{7.4}$$

for some $C_T > 0$ independent of h and η. Thus, starting from (7.3a), in order to obtain the desired uniform estimate (7.4), *we need to justify* the following two steps in progression:

(i) first, inversion for each $\eta \in \mathcal{E}_T$,

$$h = \{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\}^{-1}(\mathcal{L}_T^*)^{-1}\mathcal{M}_T^*[\eta]h, \quad h \in \mathcal{D}(\mathcal{L}_T^*), \tag{7.5a}$$

as well as (we shall see below) its limit version

$$h = \{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*\}^{-1}(\mathcal{L}_T^*)^{-1}(\mathcal{M}_T^0)^*h, \quad h \in \mathcal{D}(\mathcal{L}_T^*); \tag{7.5b}$$

(ii) second, uniform bound in $\eta \in \mathcal{E}_T$ of the inverse of (7.5a):

$$\begin{aligned} \|h\|_H \quad \le \quad & \|\{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\}^{-1}\|_{L(H)} \\ & \|(\mathcal{L}_T^*)^{-1}\|_{L(L_2(0,T;U);H))}\|\mathcal{M}_T^*[\eta]h\|_{L_2(0,T;U)} \end{aligned} \tag{7.6}$$

$$\leq \quad C_T \|\mathcal{M}_T^*[\eta]h\|_{L_2(0,T;U)}, \ \forall \ h \in \mathcal{D}(\mathcal{L}_T^*) \subset H. \tag{7.7}$$

The inversion occurring in (7.5a) is the present counterpart of the inversion in (4.47) of Section 4. As there, it actually involves a double inversion, one internal and one external, each of which is required to be uniform in $\eta \in \mathcal{E}_T$, in order to obtain (7.7) from (7.6), by use also of (7.2).

Justification of inversion occurring in (7.5a) for each $\eta \in \mathcal{E}_T$ and its limit version (7.5b): These two inversions are justified as in Proposition 4.3(c) in asserting the inversion (4.43) for $[I + \Lambda_T'(u)]^{-1}$ and its limit version (4.44). Namely, they rely on the compactness property of the perturbations of the identity, that is

$$(\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] : \text{ compact } H \to H; \tag{7.8a}$$

$$(\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^* : \text{ compact } H \to H, \tag{7.8b}$$

which are readily derived from Remark 4.3 (in the following order: (iv), (iii), (ii)), as well as property(7.2) for the (extended) operator $(\mathcal{L}_T^*)^{-1}$. Properties (7.8a–b) are the counterpart versions of properties (4.40a–b). In both scenarios, critical is the property of compactness of $\mathcal{L}^{\circledast}$ as in Remark 4.3(ii), due to (P.1) = (2.14).

Justification of uniform inversion from (7.6) to (7.7): To this end, we first invoke Lemma 4.4, in a version dual to (4.49), to assert that

$$\|(I - \mathcal{K}^{\circledast}[\eta])^{-1}\|_{L(\mathcal{E}_T)} \leq \text{const., uniformly in } \eta \in \mathcal{E}_T, \tag{7.9}$$

and next we invoke Lemma 4.5 (in dual version) with, now

$$Z_1 \equiv H, \ Z_2 \equiv \mathcal{E}_T; \ Q^* \equiv (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast} : \text{ compact } Z_2 \equiv \mathcal{E}_T \to Z_1 \equiv H; \tag{7.10}$$

$$W_T[\eta] \equiv \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1} : \text{ continuous } Z_2 \equiv \mathcal{E}_T \to Z_1 \equiv H; \tag{7.11}$$

$$Q^* W_T^*[\eta] \equiv (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] \tag{7.12}$$

[counterpart of the setting: (4.52), (4.53), (4.62), (4.63) in Section 4]. As to the verification of the assumptions of Lemma 4.5 in the present setting, we note that:

(a) the required weak convergence (see (4.55))

$$W_T[\eta_{n_k}] \equiv \mathcal{K}_T[\eta_{n_k}](I-\mathcal{K}[\eta_{n_k}])^{-1} \to W^0 \equiv \mathcal{K}_T^0(I-\mathcal{K}^0)^{-1} \text{ weakly from } \mathcal{E}_T \text{ to } H \tag{7.13}$$

—equivalent to weak convergence of corresponding adjoints—follows as in the case of (4.54): by virtue of combining property (P.2)(b), Eqn. (2.29a) with property (P.3), Eqn. (2.35a);

(b) while the injectivity of

$$I + Q^* W_T^*[\eta] = I + (\mathcal{L}_T^*)^{-1} \mathcal{L}^{\circledast} (I - \mathcal{K}^{\circledast}[\eta])^{-1} \mathcal{K}_T^*[\eta],$$

and its limit version

$$I + Q^*(W^0)^* = I + (\mathcal{L}_T^*)^{-1} \mathcal{L}^{\circledast} [I - (\mathcal{K}^0)^{\circledast}]^{-1} (\mathcal{K}_T^0)^*,$$

were already verified in part (i), see (7.5a–b), by virtue of properties (7.8a–b).

Thus, Lemma 4.5 (*in dual version*) applies and yields the counterpart of the uniform bound (4.56),

$$\begin{aligned} \|\{I + Q^* W_T^*[\eta]\}^{-1}\|_{L(H)} &= \{I + (\mathcal{L}_T^*)^{-1} \mathcal{L}^{\circledast} (I - \mathcal{K}^{\circledast}[\eta])^{-1} \mathcal{K}_T^*[\eta]\}^{-1}\|_{L(H)} \\ &\leq \text{const}_T, \text{ uniformly in } \eta \in \mathcal{E}_T. \end{aligned} \tag{7.14}$$

Then (7.14), along with the boundedness in (7.2), justifies the passage from (7.6) to (7.7), as desired.

Thus, the desired uniform COI (7.4) has been established here *directly.* This then yields an alternative proof of Theorem 1.1, via Proposition 5.1.

Part III: Proof of Theorem 1.1 through the Uniform COI (5.5) = (5.10) via the Dual Uncontrolled Problem (3.9) = (6.3) (Section 8)

8 A Comparison of Traces of χ and ϕ in $L_2(\Sigma_1)$ by Falling into Part I

According to the analysis of Sections 4 and 5, one strategy available for proving the exact controllability result of Theorem 1.1 on the semilinear problem (1.1) consists in establishing the following Uniform COI (see (5.10)) for problem (3.9) = (6.3): there exists a constant $C_{Tr} > 0$, such that

$$C_{Tr} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \le \int_{\Sigma_1} \phi^2 d\Sigma_1, \tag{8.1}$$

uniformly in $q(t,x) = f'(\eta_1(t,x)) \in B(0,r) \subset L_\infty(Q)$, where $B(0,r)$ is a ball centered at the origin of radius $r > 0$ in $L_\infty(Q)$ (ultimately $\eta_1(t,x) = w(t,x)$) where C_{Tr} depends on T and r, but not on the I.C. or on q running in $B(0,r)$. In order to prove (8.1), we return to the splitting $\phi = \psi + \chi$ in (6.36), where ψ and χ are solutions of the problems in (6.37). Next, by duality on assumption (C.1) (in the setting described by (3.7), (3.8)), with reference to the ψ-problem in (6.37), we obtain its COI by invoking [L-T.2, Theorem 3.1, p. 271 and Theorem 2.1, p. 254]: there exists a constant $c_T > 0$ independent on the I.C. such that

$$\begin{aligned} c_T \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} &\le \int_{\Sigma_1} \psi^2 d\Sigma_1 = \int_{\Sigma_1} (\phi - \chi)^2 d\Sigma_1 && (8.2) \\ &\le 2\int_{\Sigma_1} \phi^2 d\Sigma_1 + 2\int_{\Sigma_1} \chi^2 d\Sigma_1. && (8.3) \end{aligned}$$

Thus, in order to establish the desired COI (8.1), it suffices—and indeed necessary as well: see Remark 8.1 below—to show that: there exists a constant $\text{const}_{Tr} > 0$ depending on T and r, but independent on the I.C. or on $q \in B(0,r)$, such that

$$\int_{\Sigma_1} \chi^2 d\Sigma_1 \le \text{const}_{Tr} \int_{\Sigma_1} \phi^2 d\Sigma_1, \tag{8.4}$$

uniformly on $q \in B(0,r) \subset L_\infty(Q)$.

A *direct* operator-theoretic proof of inequality (8.4), or its operator version (8.7) below, will be given in Section 9 below. Here, in contrast, we provide an alternative operator-theoretic proof of inequality (8.4) by actually proving its *equivalent* version (8.8) = (8.9) below, involving the quantity $[I + (\Lambda'_T[q])^*]$ that played a critical role in the analysis of Part I. This way, we shall fall into the technical apparatus of Part I, and, in particular, appeal to Lemmas 4.4 and 4.5 to complete the present proof.

Proof of (8.4). By Proposition 6.7, Eqns. (6.44), (6.43), we have that

$$\phi(t;\Phi_0)|_{\Sigma_1} = v_2(t;V_0)|_{\Sigma_1} \equiv \{\mathcal{M}^*_T[q]V_0\}(t),$$
$$V_0 = \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} \in H,\ \Phi_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \in Y; \tag{8.5}$$
$$\chi(t)|_{\Sigma_1} \equiv \{\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}^*_T[q]V_0\}(t). \tag{8.6}$$

Thus, proving inequality (8.4) is, via (8.5), (8.6), reformulated as proving that

$$\|\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}^*_T[q]V_0\|_{L_2(\Sigma_1)} \le \text{const}_{Tr}\|\mathcal{M}^*_T[q]V_0\|_{L_2(\Sigma_1)},$$
$$V_0 \in \mathcal{D}(\mathcal{L}^*_T) \subset H, \tag{8.7}$$

uniformly in $q \in B(0,r)$ and in $V_0 \in \mathcal{D}(\mathcal{L}^*_T)$. Moreover, we have seen in the proof of Proposition 5.1, that $(\mathcal{L}^{\#}_T)^*$ is an isomorphism from $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ onto $\mathcal{D}(\mathcal{L}^*_T)$. Then inequality (8.7) is, in turn, equivalent to the following inequality: There exists a constant $C'_{Tr} > 0$ such that

$$\|\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}^*_T[q](\mathcal{L}^{\#}_T)^*u\|_{L_2(\Sigma_1)} \le C'_{Tr}\|\mathcal{M}^*_T[q](\mathcal{L}^{\#}_T)^*u\|_{L_2(\Sigma_1)}, \tag{8.8}$$

uniformly in $q \in B(0,r)$ and $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \subset L_2(0,T;L_2(\Gamma_1))$. But, by (4.40a) and (4.39b), we can rewrite inequality (8.8) as

$$\|(\Lambda'_T[q])^*u\|_{L_2(\Sigma_1)} \le C'_{Tr}\|[I + (\Lambda'_T[q])^*]u\|_{L_2(\Sigma_1)}, \tag{8.9}$$

uniformly in $q \in B(0,r)$ and $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \subset L_2(\Sigma_1)$. We now establish (8.9), thereby completing the proof of (8.7), hence of (8.4). We write for $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$:

$$\begin{aligned}\|(\Lambda_T'[q])^* u\|_{L_2(\Sigma_1)} &= \|(I+\Lambda_T'[q])^* u - u\|_{L_2(\Sigma_1)} \\ &\leq \|(I+\Lambda_T'[q])^* u\|_{L_2(\Sigma_1)} + \|u\|_{L_2(\Sigma_1)}. \qquad (8.10)\end{aligned}$$

Moreover, from Section 4, Eqn. (4.64) we have the inequality (same as (5.6))

$$\|u\|_{L_2(\Sigma_1)} \leq \text{const}_{Tr} \|(I+\Lambda_T'[q])^* u\|_{L_2(\Sigma_1)}, \qquad (8.11)$$

uniformly in $q \in B(0,r)$ and $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$. Substituting (8.11) into the RHS of (8.10) yields (8.9), as desired. Indeed, inequality (8.11) is guaranteed by inequality (4.64) of Lemma 4.6 (which was proved by virtue of properties (P.1), (P.2), (P.3), (C.1), (C.2) ultimately via Lemmas 4.4 and 4.5. Thus, inequality (5.10) = (8.1) is established, and this, in turn, proves Theorem 1.1, by virtue of Proposition 5.2. □

Remark 8.1. We have seen above that: inequality (8.4), used in (8.3), implies inequality (8.1). Conversely, here we show that: *inequality* (8.1) *implies inequality* (8.4).

In fact, we return to (6.39) and estimate via trace theory:

$$\begin{aligned}\|\chi\|_{L_2(\Sigma_1)} &\leq C\|\chi\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac12}))} \leq c_T \|\phi(\,\cdot\,;\Phi_0)\|_{L_2(0,T;L_2(\Omega))} \qquad (8.12)\\ &\leq C_{Tr}\|\{\phi_0,\phi_1\}\|_{Y=L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac12})]'}, \qquad (8.13)\end{aligned}$$

uniformly in $q \in B(0,r)$, where in the last step we have invoked the regularity result (3.11b) of Theorem 3.1. Thus, if (8.1) holds true, then (8.13) shows that (8.4) holds true as well.

Remark 8.2. We have seen above that: inequality (8.11), used in (8.10), implies inequality (8.9). Conversely, *inequality* (8.9) *implies inequality* (8.11), as it follows from

$$\|u\| = \|(I+\Lambda_T'[q])^* u - (\Lambda_T'[q])^* u\| \leq \|(I+\Lambda_T'[q])^* u\| + \|(\Lambda_T'[q])^* u\|, \quad (8.14)$$

by substituting (8.9) in the RHS of (8.14).

Part IV: Another Proof of Theorem 1.1 via the Uniform COI (5.10) = (8.1); by showing (8.4) = (8.7) directly (Section 9)

The present Part IV provides an alternative proof of the exact controllability Theorem 1.1, by again establishing the uniform COI (5.10) = (8.1) for the dual uncontrolled ϕ-problem (3.9) = (6.3). In common with Part III, this goal is achieved by proving the trace inequality (8.4) = (8.7). Unlike Part III, however, the present proof of (8.4) = (8.7) is *direct*, while Section 8 of Part III established the *equivalent* inequality (8.8) = (8.9). The present direct proof manages to use only one inversion—the counterpart of Lemma 4.4. By contrast, the double inversion of Lemma 4.5 is now entirely dispensed with. The present analysis still relies critically on suitable families of collectively compact operators, as in the context of Part I, Section 4, up to Lemma 4.4.

9 Direct Operator-Theoretic Proof of Trace Inequality (8.4) = (8.7)

Orientation. In the present section, we seek to prove *directly* the uniform inequality (8.4):

$$\int_{\Sigma_1} \chi^2 d\Sigma_1 \leq \text{const}_{Tr} \int_{\Sigma_1} \phi^2 d\Sigma_1, \tag{9.1}$$

rewritten in operator form, via (6.43) = (8.5) and (6.44) = (8.6) (this was already done in (8.7)) as:

$$\|\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}_T^*[q]V_0\|_{L_2(\Sigma_1)} \leq \text{const}_{Tr}\|\mathcal{M}_T^*[q]V_0\|_{L_2(\Sigma_1)}, \tag{9.2}$$

uniformly in $q \in B(0, r)$ [ball centered at the origin of radius $r > 0$ in $L_\infty(Q)$, see (A.20)] and $V_0 \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, where ultimately $q(t, x) = f'(\eta_1(t, x))$. As we have seen in Section 8, inequality (9.1) = (9.2) is *equivalent* to the uniform COI (5.5), which in turn yields the exact controllability result of Theorem 1.1 for the semilinear problem (1.1), see Fig. 5.1, by Proposition 5.1.

In Section 8, we have proved *not* inequality (9.1) = (9.2) directly, but rather its *equivalent* formulation (8.8) = (8.9) via the isomorphism $(\mathcal{L}_T^{\#})^*$ between $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ and $\mathcal{D}(\mathcal{L}_T^*)$, which connects (9.2) with (8.8) = (8.9). This strategy had the advantage of making the present task of showing (9.1) = (9.2) fall into the setting of Section 4, when it came to establish the required inequality (8.11). In fact, to show (8.11), all we needed in Section 8 was to invoke inequality (4.64) of Lemma 4.6 which was proved via Lemmas 4.4 and 4.5.

By contrast, in this section we are going to prove inequality (9.1) = (9.2) *directly*, without passing through the aforementioned isomorphism $(\mathcal{L}_T^{\#})^*$. This way, we shall provide a treatment which is different from, but partially parallel to, that of Section 4: more precisely, it will use the results on collectively compact families of operators in [A.1] behind Lemma 4.4, however, in a different topological setting.

The counterpart of Lemma 4.5 will now no longer be needed. Thus, only one uniform inversion is encountered, the counterpart of Lemma 4.4. The second uniform inversion of Lemma 4.5 in Part I is now entirely dispensed with.

Step 1. We return to Eqn. (6.44) for η [$= q$] equal to zero, whereby we obtain

$$\{\mathcal{M}_T^*[q=0]V_0\}(t) = \{\mathcal{L}_T^* V_0\}(t) = \psi(t;\Phi_0)|_{\Sigma_1}, \quad V_0 = \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix}, \tag{9.3}$$

where $\psi(t;\Phi_0)$, $\Phi_0 = [\phi_0, \phi_1] \in Y$, solves problem (6.37a-b-c) on the LHS. Formula (9.3) (unperturbed case with $q \equiv 0$) is well known [L-T.2, Eqn. (3.15a), p. 273]. As in (8.2), invoking [L-T.2, Theorem 3.1, p. 271 and Theorem 2.1, p. 254], we have that under the setting described, e.g., by (3.6)–(3.8), the corresponding COI for the ψ-problem holds true: there exists a constant $c_T > 0$, independent on the I.C., such that on $\mathcal{D}(\mathcal{L}_T^*)$:

$$\begin{aligned} \left\| \mathcal{L}_T^* \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} \right\|^2_{L_2(\Sigma_1)} &\equiv \int_0^T \int_{\Gamma_1} \psi^2 d\Sigma_1 \geq c_T \|\{\phi_0, \phi_1\}\|_Y^2 \\ &\equiv \|\{-\mathcal{A}^{-1}\phi_1, \phi_0\}\|_H^2, \end{aligned} \tag{9.4}$$

$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$.

The COI (9.4) says that the operator $\mathcal{L}_T^*$ is boundedly invertible on its range (Range $\mathcal{L}_T^*) = \mathcal{L}_T^* H$ in H. Since $\psi(t;\Phi_0)$ is in its range by (9.3), we conclude that

$$\begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} = (\mathcal{L}_T^*)^{-1}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \in H. \tag{9.5}$$

Hence, recalling from (2.2) A: topological isomorphism $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ onto $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, we obtain

$$\Phi_0 \equiv \begin{bmatrix} \phi_0 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} = A(\mathcal{L}_T^*)^{-1}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \in Y, \tag{9.6}$$

which gives the reconstruction map—from the Dirichlet trace on Σ_1 to its I.C.—of the solution to the ψ-problem (6.37). From (6.37) and (9.6), it follows via (2.3) that

$$\begin{aligned} \begin{bmatrix} \psi(t;\Phi_0) \\ \psi_t(t;\Phi_0) \end{bmatrix} &= e^{A(t-T)} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \\ &= \begin{bmatrix} C(t-T) & S(t-T) \\ -\mathcal{A}S(t-T) & C(t-T) \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \\ &= e^{A(t-T)} A(\mathcal{L}_T^*)^{-1}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \qquad (9.7a) \\ &\equiv \{\mathcal{O}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1})\}(t) \in C[0,T];Y), \qquad (9.7b) \end{aligned}$$

gives the reconstruction (or 'observation' map) $\mathcal{O} \equiv \mathcal{O}_T$ from the Dirichlet trace on Σ_1 to the solution ψ of (6.37) in the interior.

To extract the first component $\psi(t;\Phi_0)$ from identity (9.7), we shall write

$$\begin{aligned} \psi(\,\cdot\,;\Phi_0) &= \mathcal{O}_1(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) = \{\mathcal{O}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1})\}_1 \\ &= [C(\,\cdot\,-T) \quad S(\,\cdot\,-T)]A(\mathcal{L}_T^*)^{-1}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}). \qquad (9.8) \end{aligned}$$

We recall from (4.8) that we are throughout considering $\mathcal{L}_T$ as an injective closed operator from $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ onto H (via (C.1) = (3.5)). Hence, the range $\mathcal{L}_T^* H$ is dense in $[\mathcal{N}(\mathcal{L}_T)]^\perp$. But, by (9.4), $(\mathcal{L}_T^*)^{-1}$ is bounded on $\mathcal{L}_T^* H$, as noted above (9.5). Hence $(\mathcal{L}_T^*)^{-1}$ is bounded on $[\mathcal{N}(\mathcal{L}_T)]^\perp$. Hence

$$(\mathcal{L}_T^*)^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^\perp; H); \quad A(\mathcal{L}_T^*)^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^\perp; Y); \tag{9.9}$$

$$\mathcal{O} \equiv \mathcal{O}_T \equiv e^{A(\,\cdot\,-T)} A(\mathcal{L}_T^*)^{-1} : \text{ continuous } [\mathcal{N}(\mathcal{L}_T)]^\perp \to C([0,T]; Y); \tag{9.10}$$

$$\begin{aligned} \mathcal{O}_1 \equiv \mathcal{O}_{1,T} \equiv (\mathcal{O}_T)_1 \quad &= \quad [C(\,\cdot\,-T) \quad S(\,\cdot\,-T)] A(\mathcal{L}_T^*)^{-1} \\ &\quad : \text{ continuous } [\mathcal{N}(\mathcal{L}_T)]^\perp \to C([0,T]; L_2(\Omega)); \end{aligned} \tag{9.11}$$

We next extend $\mathcal{O}$ (and $\mathcal{O}_1$) by zero on $\mathcal{N}(\mathcal{L}_T)$, so that

$$\mathcal{O}_{\text{ext}} = \begin{cases} \mathcal{O} & \text{on } [\mathcal{N}(\mathcal{L}_T)]^\perp \\ 0 & \text{on } \mathcal{N}(\mathcal{L}_T) \end{cases}; \qquad \mathcal{O}_{1,\text{ext}} = \begin{cases} \mathcal{O}_1 & \text{on } [\mathcal{N}(\mathcal{L}_T)]^\perp \\ 0 & \text{on } \mathcal{N}(\mathcal{L}_T) \end{cases}. \tag{9.12}$$

(A similar extension was performed in (7.2) of Section 7.) This way, since by (9.3) $\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1} \in \mathcal{L}_T^* H \subset [\mathcal{N}(\mathcal{L}_T)]^\perp$, we have $\mathcal{O}_{1,\text{ext}}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) = \mathcal{O}_1(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1})$, given by (9.8), but we can also apply $\mathcal{O}_{1,\text{ext}}$ to both $\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}$ and $\chi(\,\cdot\,)|_{\Sigma_1}$.

Step 2. Lemma 9.1. *In the notation of (9.10)–(9.12), and with reference to $\phi = \psi + \chi$ in (6.36), (6.37), we have:*

$$\begin{aligned} \chi(t) - \int_t^T S(\tau - t) q(\tau) \{\chi(\tau) + [\mathcal{O}_{1,\text{ext}}(\gamma_0 \chi(\,\cdot\,))](\tau)\} d\tau \\ = \int_t^T S(\tau - t) q(\tau) [\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1})](\tau) d\tau, \end{aligned} \tag{9.13}$$

where γ_0 denotes the Dirichlet trace operator, $\gamma_0 : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$, $s > \frac{1}{2}$.

Proof. We return to (6.39) with $S(\,\cdot\,)$ odd and with $q(\tau) = f'(\eta_1(\tau))$, $v_2(t; V_0) \equiv \phi(t; \Phi_0)$ and obtain by using $\phi = \psi + \chi$ in (6.36), the extended reconstruction map $\mathcal{O}_{1,\text{ext}}$, which coincides with $\mathcal{O}_1$ on $\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}$, given by (9.8), (9.12):

$$
\begin{aligned}
\chi(t) &= \int_T^t S(t-\tau)q(\tau)\phi(\tau;\Phi_0)d\tau \\
&= \int_t^T S(\tau-t)q(\tau)\psi(\tau;\Phi_0)d\tau + \int_t^T S(\tau-t)q(\tau)\chi(\tau)d\tau \qquad (9.14)\\
&\text{(by (9.8))}\\
&= \int_t^T S(\tau-t)\mathcal{O}_{1,\text{ext}}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1})(\tau) + \int_t^T S(\tau-t)q(\tau)\chi(\tau)d\tau \\
&= \int_t^T S(\tau-t)q(\tau)\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1})(\tau)d\tau + \int_t^T S(\tau-t)q(\tau)\chi(\tau)d\tau \qquad (9.15)\\
&\quad - \int_t^T S(\tau-t)q(\tau)\mathcal{O}_{1,\text{ext}}(\chi(\,\cdot\,)|_{\Sigma_1})(\tau)d\tau, \qquad (9.16)
\end{aligned}
$$

and (9.16) establishes (9.13). □

Step 3. Next, for $v \in L_2(0,T;L_2(\Omega))$, and $q \in L_\infty(Q)$ as usual, we introduce the operator

$$(\mathcal{T}[q]v)(t) = \int_t^T S(\tau-t)q(\tau)v(\tau)d\tau \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})), \qquad (9.17)$$

continuously, see (2.3). By comparison with (2.22) (where $f'(\eta_1(\tau))$ is replaced now by $q(\tau)$), we see that: *the operator $\mathcal{T}[q]$ is the $L_2(0,T;L_2(\Omega))$-adjoint of the first component of the operator $\mathcal{K}[q]$ in (2.22).* Then, in view of (9.17), we can rewrite identity (9.13) for $\chi \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))$, see (C.13), as

$$[I - \mathcal{T}[q] + \mathcal{T}[q]\mathcal{O}_{1,\text{ext}}\gamma_0]\chi = \mathcal{T}[q]\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}). \qquad (9.18)$$

Step 4. To fit into the setting of the operator $\mathcal{K}[\eta]$ in (2.22), we complement (9.17) by introducing the notation

$$(\mathcal{T}[\eta_1]v)(t) = \int_t^T S(\tau-t)f'(\eta_1(\tau))v(\tau)d\tau \qquad (9.19a)$$

$$: \text{continuous } L_1(0,T;L_2(\Omega)) \to C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})) \qquad (9.19b)$$

(via the regularity (2.3)) with η_1 (parameter) $\in L_2(0,T;L_2(\Omega))$, and with f subject to (1.2). We next collect relevant properties of the operator $\mathcal{T}[\eta_1]$ [cf., property (P.2) ((2.27)–(2.29)) for the operator $\mathcal{K}[\eta]$ in (2.22)].

Proposition 9.2. *With reference to $\mathcal{T}[\eta_1]$ in (9.19), the following properties hold true:*

Not only do we have that for any $\epsilon > 0$,

$$\mathcal{T}[\eta_1] : \text{compact } L_2(0,T;L_2(\Omega)) \to L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)) \text{ for each } \eta_1 \in L_2(0,T;L_2(\Omega)), \tag{9.20}$$

but, in addition, we have

(a)

$$\begin{cases} \text{with } \mathcal{T}[\eta_1] \text{ viewed as in (9.20), the family of operators } \{\mathcal{T}[\eta_1]\}_{\eta_1 \in L_2(0,T;L_2(\Omega))} \text{ is collectively com-} \\ \text{pact: that is [A.1, p. 4], the set union } \bigcup_{\eta_1} \mathcal{T}[\eta_1] \\ \text{(unit ball of } L_2(0,T;L_2(\Omega)) \text{ is a pre-compact set of} \\ L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)), \end{cases} \tag{9.21}$$

where the union of the image of the unit ball of $L_2(0,T;L_2(\Omega))$ under the operator $\mathcal{T}[\eta_1]$ is taken over all $\eta_1 \in L_2(0,T;L_2(\Omega))$. Moreover, $\mathcal{T}[\eta_1]$ satisfies the following additional property.

(b) For any sequence $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, we can extract a subsequence η_{1n_k} such that

$$\mathcal{T}[\eta_{1n_k}] \to \mathcal{T}^0 \text{ strongly, from } L_2(0,T;L_2(\Omega)) \text{ to } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)), \tag{9.22}$$

where $\mathcal{T}^0$ is given by

$$\begin{aligned} (\mathcal{T}^0 v)(t) &= \int_t^T S(\tau - t) f_0(\tau) v(\tau) d\tau \\ &: \text{ continuous } L_1(0,T;L_2(\Omega)) \to C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})), \end{aligned} \tag{9.23}$$

for some suitable $f_0 \in L_\infty(\mathbb{R})$, possibly depending on the subsequence.

Proof. The proof is identical to that of property (P.2) for the operator $\mathcal{K}[\eta]$ in (2.22), except on a different topological setting. We shall use Aubin's Lemma [A.2]. From (9.19) we readily estimate via (2.3) and (1.2),

$$
\begin{aligned}
&\|\mathcal{T}[\eta_1]v\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \\
&\quad\leq \sqrt{T}\|\mathcal{T}[\eta_1]v\|_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \\
&\quad\leq C_T\|v\|_{L_2(0,T;L_2(\Omega))}, \text{ uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \qquad (9.24)
\end{aligned}
$$

Next, for the time derivative of $\mathcal{T}[\eta_1]$ in (9.19):

$$
\left(\frac{d}{dt}\mathcal{T}[\eta_1]v\right)(t) = -\int_t^T C(\tau - t)f'(\eta_1(\tau))v(\tau)d\tau, \qquad (9.25)
$$

we similarly estimate via (2.3), (1.2),

$$
\begin{aligned}
&\left\|\frac{d}{dt}\mathcal{T}[\eta_1]v\right\|_{L_2(0,T;L_2(\Omega))} \\
&\quad\leq C_T\|v\|_{L_2(0,T;L_2(\Omega))}, \text{ uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \qquad (9.26)
\end{aligned}
$$

Since the injection $H^1_{\Gamma_0}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow H^{1-\epsilon}_{\Gamma_0}(\Omega)$, $\epsilon > 0$, is compact, Aubin's Lemma [A.2] applies and yields (9.20) and, in fact, (9.21), since the estimates (9.24) and (9.26) are *uniform* in $\eta_1 \in L_2(0,T;L_2(\Omega))$.

To prove part (b), we proceed as in (2.29) through (2.32). Given $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, we have $|f'(\eta_{1n})| \leq$ const by (1.2), so that by Alaoglou's Theorem we can extract a subsequence η_{1n_k} such that $f'(\eta_{1n_k}) \to$ some f_0 in $L_\infty(\mathbb{R})$-weak star. One then defines the operator $\mathcal{T}^0$ as in (9.23). We further obtain as in [L-T.4, Proposition 3.3(b), p. 129] that

$$
\mathcal{A}^{\frac{1}{2}}\mathcal{T}[\eta_{1n_k}] \to \mathcal{A}^{\frac{1}{2}}\mathcal{T}^0 \text{ weakly in } L_2(0,T;L_2(\Omega)), \qquad (9.27)
$$

as well as

$$
\frac{d}{dt}\mathcal{T}[\eta_{1n_k}] \to \frac{d}{dt}\mathcal{T}^0 \text{ weakly in } L_2(0,T;L_2(\Omega)). \qquad (9.28)
$$

Then, as a consequence of the weak convergence (9.27) (that takes care of the "space" coordinates) and of (9.28) (that takes care of the "time" coordinate), along with the compactness of $\mathcal{A}^{-\epsilon}$ from $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ to $\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})$, we deduce the strong convergence (9.22). Proposition 9.2 is proved. $\square$

Consequences of Proposition 9.2. As a consequence of the collectively compact property (9.21) and of the strong convergence (9.22) [or else again by Aubin's Lemma applied to $\mathcal{T}^0$], we obtain that

$$\mathcal{T}^0 : \text{ compact } L_2(0,T;L_2(\Omega)) \to L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)). \tag{9.29}$$

Moreover, we have that the integral operator $\mathcal{T}^0$ satisfies

$$[I-\mathcal{T}^0]^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)),\ \epsilon > 0. \tag{9.30}$$

In fact, it suffices by (9.29) to establish that $[I-\mathcal{T}^0]$ is injective: $[I-\mathcal{T}^0]v = 0 \Rightarrow v = 0$, which is true since this leads by differentiating via (9.23) twice that $\ddot{v}= \mathcal{A}v - f_0 v$, $v(T) = \dot{v}(T) = 0$, hence $v = 0$. [Or else, one proceeds as in obtaining (4.2) from (4.3)]. The counterpart of Lemma 4.4 is

Proposition 9.3. *Not only do we have*

$$(I-\mathcal{T}[\eta_1])^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)),\ \epsilon > 0. \tag{9.31}$$

for each $\eta_1 \in L_2(0,T;L_2(\Omega))$ fixed, but moreover,

(a_1)

$$\|(I-\mathcal{T}[\eta_1])^{-1}\|_{L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \le const_\epsilon,\ \text{uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \tag{9.32}$$

(a_2) Let η_{1n_k} be a subsequence of a given arbitrary sequence $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, such that

$$\mathcal{T}[\eta_{1n_k}] \to \mathcal{T}^0 \text{ (given by (9.23)) strongly in } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)), \tag{9.33}$$

as guaranteed a-fortiori *by (9.22), whereby then $\mathcal{T}^0$ is compact on*

$$L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)),$$

as a-fortiori *guaranteed by Proposition 8.2(b), Eqn. (9.29). Then, not only $(I-\mathcal{T}^0)^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$ as already noted in (9.30), but moreover,*

$$(I-\mathcal{T}[\eta_{1n_k}])^{-1} \to (I-\mathcal{T}^0)^{-1} \text{ strongly in } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)). \tag{9.34}$$

Proof. Same as the proof of Lemma 4.4. The proof relies on Proposition 8.2 (counterpart of property (P.2), (2.27)–(2.29)), which then allows the application of a standard result [A.1, Theorem 1.6, p. 6] on the family of collectively compact operators $\{\mathcal{T}[\eta_1]\}$ on $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$, *a-fortiori* from (9.21). Indeed, the strong convergence (9.33) guaranteed *a-fortiori* by (9.22), once combined with the existence of $[I-\mathcal{T}^0]^{-1}$ already established in (9.30) implies via [A.1, Theorem 1.6, p. 6] that

$$(I-\mathcal{T}^0[\eta_{1n_k}])^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))), \tag{9.35}$$

for all k sufficiently large, and moreover, that the strong convergence in (9.34) takes place.

Part (a$_2$) is proved and this establishes part (a$_1$) since the sequence $\{\eta_{1n}\}$ was arbitrary. $\square$

We next introduce the family of operators

$$\Pi[\eta_1] \equiv \mathcal{T}[\eta_1](I+\mathcal{O}_{1,\text{ext}}\gamma_0), \ \eta_1 \in L_2(0,T;L_2(\Omega)) \tag{9.36a}$$

$$: \text{continuous } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)) \to C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})), \ \epsilon>0, \tag{9.36b}$$

where $\mathcal{O}_{1,\text{ext}}$ is defined in (9.12) and γ_0 is the Dirichlet trace operator introduced below (9.13).

The regularity in (9.36b) of $\Pi[\eta_1]$ is justified as follows. First, the Dirichlet trace operator is continuous, in fact, compact, from $H^{1-\epsilon}_{\Gamma_0}(\Omega)$ to $L_2(\Gamma)$. Next, by (9.11), $\mathcal{O}_{1,\text{ext}}$ is continuous from $L_2(0,T;U) \to C([0,T];L_2(\Omega))$, $U=L_2(\Gamma_1)$, so that

$$\mathcal{O}_{1,\text{ext}}\gamma_0 : \text{ continuous } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)) \to C([0,T];L_2(\Omega)). \tag{9.37}$$

Finally, (9.37) combined with (9.19b) yields (9.36b).

Proposition 9.4. *The family of operators $\{\Pi[\eta_1]\}$ in (9.36) has the same properties on the space $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$ as the family of operators $\{\mathcal{T}[\eta_1]\}$. Namely: not only*

(i)

$$\Pi[\eta_1]: \text{ compact } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)) \to \text{ itself}; \tag{9.38}$$

but, in addition,

(ii)

$$\begin{cases} \{\Pi[\eta_1]\}_{\eta_1 \in L_2(0,T;L_2(\Omega))} \text{ is a collectively compact family} \\ \text{of operators on } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)). \end{cases} \tag{9.39}$$

(iii) For any sequence $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, we can extract a subsequence η_{1n_k} such that

$$\Pi[\eta_{1n_k}] \to \Pi^0 \equiv \mathcal{T}^0(I + \mathcal{O}_{1,\text{ext}}\gamma_0) \text{ strongly in } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)), \tag{9.40}$$

with $\mathcal{T}^0$ (depending on the subsequence) that can be taken to be the operator in (9.23).

(iv)

$$\Pi^0: \text{ compact } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)) \to \text{itself}. \tag{9.41}$$

(v)

$$[I - \Pi^0]^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)). \tag{9.42}$$

(vi)

$$(I - \Pi[\eta_1])^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)), \text{ for each fixed } \eta_1 \in L_2(0,T;L_2(\Omega)). \tag{9.43}$$

(vii) Let η_{1n_k} be a subsequence of a given arbitrary sequence

$$\eta_{1n} \in L_2(0,T;L_2(\Omega)),$$

so that the strong convergence (9.40) takes place, as guaranteed by part (iii). Then, not only (9.42) holds true, but, moreover,

$$(I - \Pi[\eta_{1n_k}])^{-1} \to (I - \Pi^0)^{-1} \text{ strongly in } L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)). \tag{9.44}$$

(viii) Finally, the following uniform bound holds true:

$$\|(I - \Pi[\eta_1])^{-1}\|_{L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \leq const_\epsilon, \text{ uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \tag{9.45}$$

Proof. It suffices to invoke the regularity property (9.37) in the definition of $\Pi[\eta_1]$ in (9.36a) which yielded (9.36b). Then all the arguments employed for the operator family $\{\mathcal{T}[\eta_1]\}$ in Propositions 9.2 and 9.3 carry over verbatim to the family $\{\Pi[\eta_1]\}$ in (9.36). $\square$

Step 4. We now prove the desired estimate (9.1).

Proposition 9.5. *With reference to the solutions χ and ϕ of problems (6.37) (RHS) and (6.3), with $q(t,x)$ replaced by $f'(\eta_1(t,x))$, the following inequality holds true: there exists a constant $C_T > 0$, independent of the I.C. $\Phi_0 \in Y$ and of $\eta_1 \in L_2(0,T;L_2(\Omega))$, such that for $\epsilon > 0$,*

$$\|\chi|_{\Sigma_1}\|_{L_2(\Sigma_1)} \leq c\|\chi\|_{L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \leq C_T\|\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\|_{L_2(\Sigma_1)}. \tag{9.46}$$

Proof. By virtue of the definition (9.36), we return to Eqn. (9.18) and rewrite it (we are replacing q with $f'(\eta_1)$) as

$$(I - \Pi[\eta_1])\chi = \mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})), \tag{9.47}$$

for each $\eta_1 \in L_2(0,T;L_2(\Omega))$. By Proposition 9.4(vi), Eqn. (9.43), we can perform the inversion in (9.47) in $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$, thus obtaining

$$\chi = (I - \Pi[\eta_1])^{-1}\mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \in L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)). \tag{9.48}$$

In (9.47), we now recall: the critical uniform bound (9.44) of Proposition 9.4(viii) for $(I - \Pi[\eta_1])^{-1}$ as well as the uniform bound

$$\begin{aligned} &\|\mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}\|_{L(L_2(\Sigma_1);C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \\ &\qquad \leq \quad \text{const}_T, \text{ uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)), \end{aligned} \tag{9.49}$$

which follows by combining the regularity of $\mathcal{O}_{1,\text{ext}}$ in (9.11) with the uniform estimate (9.24) for $\mathcal{T}[\eta_1]$. Setting for notational convenience

$$X \equiv L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)),$$

we then obtain from (9.48) via (9.45) and (9.49),

$$
\begin{aligned}
&\|\chi\|_{L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \\
&\quad \leq \quad \|(I-\Pi[\eta_1])^{-1}_{L(X)}\|\mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}\|_{L(L_2(\Sigma_1);X)}\|\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\|_{L_2(\Sigma_1)} \quad (9.50) \\
&\quad \leq \quad C_T\|\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\|_{L_2(\Sigma_1)}, \quad (9.51)
\end{aligned}
$$

uniformly in $\eta_1 \in L_2(0,T;L_2(\Omega))$. This proves the RHS inequality in (9.46). The LHS inequality in (9.46) is a direct application of trace theory. Proposition 9.5 is proved. □

Finally, inequality (9.1) is then a re-reading of inequality (9.45) with $q(t,x) = f'(\eta_1(t,x)) \in L_\infty(Q)$. Inequality (9.1) is thus proved. As a consequence, Theorem 1.1 is established via Fig. 5.1.

10 Comparison between the Two Strategies: Part I vs. Parts II, Part III, and Part IV

Section 5—condensed in Fig. 5.1—summarizes the approach of the present paper (which is closely based on the original abstract approach in [L-T.4]) in the study of exact controllability of semilinear problems at least in its Part I. The necessary and sufficient condition of Proposition 4.1(a) is then proposed to be tested via either one of two different sufficient conditions (which are, in fact, related by Propositions 4.2 and 4.3): both *with operator-theoretic emphasis*: one which seeks the uniform global invertibility (1.3) = (4.64) = (5.7) through an analysis of the controlled problem (Part I); and one which seeks the uniform COI (5.10) through an analysis of the dual uncontrolled problem (3.9) = (6.3) = (8.4) (Parts II, Part III, and Part IV). Refer to Propositions 5.1 and 5.2. In this section, we perform a comparison between these two basic routes on a few different dynamics.

Case 1. Problem (1.1). Semilinear wave equation with Neumann boundary control $u \in L_2(0,T;L_2(\Gamma_1))$ on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$. The (global) exact controllability result of Theorem 1.1 was obtained in this paper in four ways: through the abstract operator approach of either Part I (Sections 2 through 4), or else Part II (Sections 6 and 7), or else Part III (Section 8) or else Part IV (Section 9). A comparison between the two approaches may be in order. The common thread is represented by uniform operator inversions, which are obtained by identifying suitably collectively compact families of operators, whereby then the basic theory thereof [A.1] may be applied.

Case 2. Second-order hyperbolic equations on a Riemannian manifold with Dirichlet boundary control.

Model. Let M be a finite-dimensional Riemannian manifold with metric $g(\,\cdot\,,\,\cdot\,) = \langle\,\cdot\,,\,\cdot\,\rangle$ and squared norm $|X|^2 = g(X,X)$. Let Ω be an open bounded connected compact set of M with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Let ν denote the outward unit normal field along the boundary Γ. Further, we denote by Δ_g the Laplace (Laplace-Beltrami) operator on the manifold M, and by D the Levi-Civita connection on M. In this section, we consider the following Riemannian wave equation with energy level (H^1)-terms on Ω:

$$\begin{cases} w_{tt} = \Delta_g w + F(w) + f(w) & \text{in } Q = (0,T] \times \Omega; & (10.1\text{a}) \\ w(0,\,\cdot\,) = w_0, \ w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega; & (10.1\text{b}) \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0 = (0,T] \times \Gamma_0; & (10.1\text{c}) \\ w|_{\Sigma_1} = u & \text{in } \Sigma_1 = (0,T] \times \Gamma_1. & (10.1\text{d}) \end{cases}$$

The function F is assumed to be of the following form:

$$F(w) = \langle P(t,x), Dw \rangle + p_1(t,x) w_t + p_0(t,x) w, \tag{10.2a}$$

where $Dw = \nabla_g w$ is a vector field, $Dw \in \mathcal{X}(M) =$ the set of all vector fields on M; p_0, p_1 are functions on $[0,T] \times \Omega$; finally, $P(t,\quad)$ is a vector field on M, $t > 0$. We assume

$$p_0 \in L_{n+1}(Q); \quad p_1 \in L_\infty(Q), \ P \in L_\infty(0,T;\Lambda), \tag{10.2b}$$

[Heb.1]. Thus, *a-fortiori*, there exists a constant $C_T > 0$ such that

$$|F(w)|^2 \leq C_T\{|Dw|^2 + w_t^2 + w^2\}, \quad x, t \in Q, \text{ a.e.} \tag{10.2c}$$

Two vertical bars $|\cdot|$ denote the norm in the tensor space T_x or its completion $L^2(\Omega, T)$ [Heb.1]. The nonlinear function $f(r)$ satisfies the same assumption (1.2) as in the Neumann problem (1.1).

Special case in $\mathbb{R}^n$. A relevant special case on a bounded open domain Ω of $\mathbb{R}^n$ is given when $\mathcal{A}$ is an elliptic second-order differential operator in $\mathbb{R}^n$:

$$\mathcal{A}w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial w}{\partial x_j}\right), \quad x = [x_1, \ldots, x_n], \tag{10.3a}$$

with real coefficients $a_{ij} = a_{ji}$ of class C^1, satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2, \quad x \in \Omega, \tag{10.3b}$$

for some positive $a > 0$. Accordingly, we may obtain that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0, \ \forall\, x \in \mathbb{R}^n, \quad \xi = [\xi_1, \xi_2, \ldots, \xi_n] \in \mathbb{R}^n, \ \xi \neq 0. \tag{10.3c}$$

Let $A(x)$ and $G(x)$ be the $n \times n$ coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)), \quad i, j = 1, \ldots, n; \quad x \in \mathbb{R}^n, \tag{10.4}$$

both positive definite for any $x \in \mathbb{R}^n$, by assumption (10.3c).

Riemannian metric. Let $\mathbb{R}^n$ be endowed with the usual topology and $x = [x_1, x_2, \ldots, x_n]$ be the natural coordinate system. For each $x \in \mathbb{R}^n$, define the inner product and the norm on the tangent space $\mathbb{R}^n_x \equiv \mathbb{R}^n$ by

$$g(X, Y) = \langle X, Y\rangle_g = \sum_{i,j=1}^{n} g_{ij}(x)\alpha_i\beta_j; \tag{10.5a}$$

$$|X|_g = \{\langle X, X\rangle_g\}^{\frac{1}{2}}; \quad X = \sum_{i=1}^{n}\alpha_i\frac{\partial}{\partial x_i}; \quad Y = \sum_{i=1}^{n}\beta_i\,\frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{10.5b}$$

Then, $(\mathbb{R}^n, g)$ is a Riemannian manifold with Riemannian metric g, denoted by $g = \sum_{ij=1}^n g_{ij} dx_i dx_j$. We have

$$\mathcal{A}w = \Delta_g w + Dw. \tag{10.6}$$

Thus, the case of a variable coefficient operator $\mathcal{A}$ as in (10.3a) in $\mathbb{R}^n$ can be included in the Riemannian wave equation model (10.1a).

One could also start with a differential operator such as $\mathcal{A}$ of the form (10.3a) except this time as defined on M and, by a similar change of metric using (10.4), end up with the canonical form (10.1a).

Main assumption. As in [T-Y.1], we assume that there exists a function $d : \bar{\Omega} \Rightarrow \mathbb{R}$ of class C^3 that is strictly convex in the metric g. This means that the Hessian $D^2 d$ (a two-order tensor) satisfies $D^2 d(X, X) > 0$, $\forall\ x \in \bar{\Omega}$, $\forall\ X \in M_x$. By translation, we can always make $d(x)$ positive on $\bar{\Omega}$. Moreover, since $\bar{\Omega}$ is compact, we can always obtain that

$$D^2 d(X, X) \equiv \langle D_x(Dd), X\rangle_g \geq 2\rho |X|_g^2,\ \forall\ x \in \Omega,\ \forall\ X \in M_x, \tag{10.7}$$

for some $\rho > 0$, $\min_{\bar{\Omega}} d(x) > 0$. Accordingly, we define the time T_0 by

$$T_0 = 2\left(\frac{\max_{\bar{\Omega}} d(x)}{\rho}\right)^{\frac{1}{2}}, \quad \rho \text{ as in (10.7).} \tag{10.8}$$

Remark 10.1. Several approaches are available to provide a metric $g = \sum g_{ij} dx_i dx_j$—hence coefficients $a_{ij}(x)$ via (10.4)—which satisfies the Main Assumption (10.7). See [G-L-L-T.1, Section 4] for a review.

Approaches include: the Riemannian distance function (Theorem of R. E. Greene and H. Wu); the Hessian comparison theorem; curvature flow methods; etc.

To simplify the exposition, we shall also *assume* that $d(x)$ has no critical point on $\bar{\Omega}$

$$\inf_{x \in \Omega} |Dd| > 0, \tag{10.9}$$

though this assumption can eventually be dispensed with [T-Y.1, Section 10]. We define the following subset of the boundary $\Gamma = \partial\Omega$:

$$\Gamma_0 = \left\{ x \in \Gamma : \langle Dd(x), \nu(x) \rangle = \frac{\partial d(x)}{\partial \nu} \leq 0 \right\}; \ \Gamma_1 = \Gamma \setminus \Gamma_0. \tag{10.10}$$

Uniform Continuous Observability Inequality of the corresponding linearized system. We shall state below the required "uniform continuous observability inequality" for the following "dual" problem corresponding to the linearized version of problem (10.1):

$$\begin{cases} \phi_{tt} = \Delta_g \phi + \langle \mathcal{Q}(t,x), D\phi \rangle + q_1(t,x)\phi_t + q_0(t,x)\phi & \text{in } Q; \quad (10.11a) \\ \phi(T,\,\cdot\,) = \phi_0, \ \phi_t(T,\,\cdot\,) = \phi_1 & \text{in } \Omega; \quad (10.11b) \\ \phi|_\Sigma \equiv 0 & \text{in } \Sigma, \quad (10.11c) \end{cases}$$

where $\mathcal{Q}$, q_1, q_0 satisfy regularity assumptions as P, p_1, p_0, in (10.2b), respectively. The term $f'(w(t,x)) \in L_\infty(Q)$ which arises in the linearization of Eqn. (10.1a) with f subject to (1.2) is incorporated in the term q_0.

The following result provides the key sought-after version, as it applies to the present case (10.1), of the "uniform observability property" required by Eqn. (5.10).

Theorem 10.1. *Assume hypotheses (10.7), (10.9) on the strictly convex function $d(x)$. Let $T > T_0$, where T_0 is defined in (10.8). Then, with reference to (10.11), the following continuous observability inequality holds true: There exists a constant $C(r) > 0$, such that*

$$\|\{\phi_0, \phi_1\}\|^2_{H^1_0(\Omega)\times L_2(\Omega)} \leq C(r) \int_0^T \int_{\Gamma_1} \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma_1, \tag{10.12}$$

where $C(r)$ is given by

$$\begin{gathered} C(r) = ce^{cr^2} \textit{ for some } c = C(T,\Omega); \\ r = \|q_0\|_{L_{n+1}(Q)} + \|q_1\|_{L_\infty(Q)} + \|\mathcal{Q}\|_{L_\infty(0,T;\Lambda)}. \end{gathered} \tag{10.13}$$

Theorem 10.1 was proved in [T-Y.1, Theorem 8.1, p. 357]. It generalized previous versions with constant coefficient principal part on a Euclidean domain [K-K.1], [Zh.1], and also [L-T-Z.1].

Semilinear system. As already noted before its statement, Theorem 10.1 provides the present version of the sufficient condition (5.10) (Uniform Continuous Observability Inequality for the linearized problem corresponding to the original semilinear system (10.1), in the present Dirichlet boundary control case). The technical work that goes into proving estimate (10.12)—and its predecessors in the Euclidean setting in [K-K.1], [L-T-Z.1], [Zh.1] (based on pointwise Carleman estimates with no lower-order terms)—replaces entirely the approaches of Part II and Part III in the Dirichlet versions. Estimate (10.12) *is* the uniform COI (5.5) in our present Dirichlet case. Therefore, by the counterpart of Proposition 5.1 or 5.2, as applied to the present Dirichlet problem (10.1) (where $\Lambda_T'(q)$ is still compact), we obtain the desired (global) exact controllability of system (10.1).

Theorem 10.2. *Let (1.2) be fulfilled for the function f in Eqn. (10.1a). Assume hypotheses (10.7), (10.9) on the strictly convex function $d(x)$. Let $T > T_0$, where T_0 is defined in (10.8). Then problem (10.1) is (globally) exactly controllable on the state space $H \equiv L_2(\Omega) \times H^{-1}(\Omega)$ within the class of $L_2(0,T;L_2(\Gamma_1))$-controls. Explicitly: given $T > 0$, the I.C. $\{w_0, w_1\} \in H$, the target state $\{\tilde{w}_0, \tilde{w}_1\} \in H$, there exists a control $u \in L_2(0,T;L_2(\Gamma_1))$, such that the corresponding solution of problem (10.1) satisfies $\{w(T,\cdot), w_t(T,\cdot)\} = \{\tilde{w}_0, \tilde{w}_1\}$.*

Theorem 10.1 generalizes [L-T.4, Theorem 1.1, case $\gamma = 0$] to which it reduces in the case of $\Delta_g = \Delta$ (that is, constant coefficient principal part on a Euclidean domain: $-a_{ij} = \delta_{ij}$ in (10.3a) with no energy-level terms $p_1 \equiv P \equiv 0$).

As mentioned in the introduction, this basic case $\gamma = 0$ is excluded from the results of [Z.1] and also of [Zh.1, Theorem 4.2, p. 831], as the latter falls into the former.

Remark 10.2. No compactness/uniqueness argument is involved in obtaining estimate (10.12). On the contrary, (10.12) is used (and has been used in this paper) to claim a global uniqueness result for the over-determined problem (10.11a–c) plus $\frac{\partial \phi}{\partial \nu}|_{\Sigma_1} \equiv 0$. The proof of (10.12) is based on pointwise Carleman estimates with *no* lower-order terms, in the presence of low regularity of the coefficients, as it is pursued in ([K-K.1], [L-T-Z.1], [Zh.1] in the Euclidean setting and extended further in the Riemannian setting in) [T-Y.1]. This approach is more refined than (by now) classical multipliers' approaches. The resulting control-theoretic *a-priori* inequalities obtained through this pointwise Carleman estimate approach have the definite advantage of coming with an explicit estimate of the final constants involved. This, despite the low regularity of the coefficients, in particular $L_\infty(Q)$-potentials q. This is a new feature over much of the control-theoretic literature since the early '80s [B-L-R.1], [L-T.2], [Lio.1], [Tr.2]. Thus, compactness/uniqueness arguments are dispensed with. There is, however, at present a big difference between the Dirichlet and the Neumann boundary control cases for wave equations. In the Dirichlet boundary control case of the present Section 9, this pointwise Carleman estimate approach yields estimate (10.12), which is *precisely* the required COI for problem (10.11), in particular, with potential $q_0 \in L_\infty(Q)$. By contrast, in the Neumann boundary control case of the present paper—say (3.9) = (6.3) = (A.4)—this approach does yield an estimate, whereby the $H = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$—norm of the I.C. $\{\phi_0, \phi_1\}$ is bounded above by boundary traces of ϕ and ϕ_t in $L_2(\Sigma_1)$ [L-T-Z.1]:

$$\|\{\phi_0, \phi_1\}\|^2_{\mathcal{D}(A^{\frac{1}{2}}) \times L_2(\Omega)} \leq C_T \int_{\Sigma_1} [\phi^2 + \phi_t^2] d\Sigma_1. \tag{10.14}$$

But this estimate is *not* the required COI (5.10) = (8.1). It is *equivalent* to it, *but only if* $q_t \in L_\infty(Q)$, see Remark A.1, which is not an acceptable assumption to treat the semilinear problem (1.1). That is why, in the Neumann case of the present paper, we had to resort to different approaches, described in Parts I, II, III, and IV.

Appendix A: Preliminaries on the Linearized Problem. Main Result

Controlled dynamics. As in Section 1, we let $\{\Omega, \Gamma_0, \Gamma_1\}$ be a given triple, $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Omega \in \mathbb{R}^n$ a bounded domain, $n \geq 2$; $\Gamma_0 \neq \emptyset$. Consider the following wave mixed problem

$$\begin{cases} y_{tt} = \Delta y + q(t,x)y & \text{in } Q = (0,T] \times \Omega; & \text{(A.1a)} \\ y(0,\,\cdot\,) = y_0,\ y_t(0,\,\cdot\,) = y_1 & \text{in } \Omega; & \text{(A.1b)} \\ y|_{\Sigma_0} = 0;\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = u & \text{in } \Sigma_i = (0,T] \times \Gamma_i,\ i = 0,1. & \text{(A.1c)} \end{cases}$$

We note that (A.1c) (LHS) is a technical condition made to invoke Poincaré inequality; it can be eliminated by using the equivalence in [L-T-Z.1, Eqn. (6.1)].

Notation. Throughout Appendices A, B, and C, y denotes the solution of (A.1). By contrast, in Parts I and II, to conform to [L-T.4], we defined y as $y = \{w, w_t\}$ where w is the solution of (1.1). No confusion is likely to arise, as the Appendices are essentially independent of Parts I and II in this regard.

Case of fixed 'potential' q. We assume that $q(t,x)$ is fixed and satisfies

$$q(t,x) \in L_\infty(Q). \tag{A.2}$$

Problem (A.1) coincides with the ζ-problem (2.37) of Section 2, for the choice $q = f'(\eta_1(\,\cdot\,))$, where eventually $\eta_1 = w(t,x) \in C([0,T]; H^1_{\Gamma_0}(\Omega))$, see regularity statement in Step 1, at the beginning of the proof of Section 4. It is convenient to study problem (A.1) as a prototype of the linearization (2.37) of the original semilinear problem (1.1), with a fully general $L_\infty(Q)$-potential.

Remark A.1. Assumption (A.2) is made with the aim of solving the exact controllability problem for semi-linear models. In contrast, if we assume $q_t(t,x) \in L_\infty(Q)$, the approach in [L-T.2] would apply. The approach that follows is dictated by assumption (A.2).

Lemma A.1. *Exact Controllability of problem (A.1) on the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2(\Sigma_1) = L_2(0,T;L_2(\Gamma_1))$-controls holds true if and only if the following COI (Continuous Observability Inequality) is satisfied: there exists a constant $C_{T,q} > 0$, depending only on T and q, but not on the I.C., such that*

$$\int_{\Sigma_1} \phi^2 d\Sigma_1 \geq C_{T,q} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times(H^1_{\Gamma_0}(\Omega))'} \tag{A.3}$$

(whenever the LHS is finite), where ϕ solves the following homogeneous backward problem:

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{on } Q; \quad \text{(A.4a)} \\ \phi(T,\,\cdot\,) = \phi_0 \in L_2(\Omega), \ \phi_t(T,\,\cdot\,) = \phi_1 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' & \text{on } \Omega; \quad \text{(A.4b)} \\ \phi|_{\Sigma_0} \equiv 0; \ \left.\dfrac{\partial \phi}{\partial \nu}\right|_{\Sigma_1} \equiv 0 & \text{on } \Sigma_i = (0,T]\times\Gamma_i. \quad \text{(A.4c)} \end{cases}$$

The regularity of problem (A.4) is stated in Theorem 3.1 and proved in Theorem C.1 of Appendix C.

Notation. Problem (A.4) coincides with problem (3.9) = (6.3). As in Eqn. (2.2), (2.4), we set

$$H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega) : \ f|_{\Gamma_0} = 0\} = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \text{ (equivalent norms)}; \tag{A.5}$$

$$\mathcal{A}f = -\Delta f, \ \mathcal{D}(\mathcal{A}) = \left\{ f \in H^1(\Omega) : \ f|_{\Gamma_0} = 0, \ \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = 0 \right\}; \tag{A.6}$$

$$\mathcal{A}^{-1} \in \mathcal{L}(L_2(\Omega)) \text{ (because of } \Gamma_0 \neq \emptyset);$$

$$H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega); \ Y \equiv L_2(\Omega) = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'; \tag{A.7}$$

$$(H^1_{\Gamma_0}(\Omega))' = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \text{ (duality w.r.t. } L_2(\Omega) \text{ as a pivot space)}. \tag{A.8}$$

Proof of Lemma A.1. By usual duality.

Step 1. Consistently with the notation in (3.15) or (4.33) for the abstract linearized z-problem (2.18)—corresponding to the linearized ζ-problem in (2.37) in PDE terms; that is, the y-problem (A.1) with $q = f'(\eta_1(\,\cdot\,))$—we introduce the input $\rightarrow$ solution operator $\mathcal{M}_T \equiv \mathcal{M}_T[q]$ (*unbounded*, densely defined, *closed*), see (3.3) in the case of $\mathcal{L}_T$, i.e., for $q \equiv 0$:

$$\mathcal{M}_T u = \{y(T), y_t(T)\} \in H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$$
$$\text{for } u \in \mathcal{D}(\mathcal{M}_T) = \mathcal{D}(\mathcal{L}_T) \subset L_2(\Sigma_1). \tag{A.9}$$

The required exact controllability property of (A.1) on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ at time $t = T$ within the class of $L_2(0, T; L_2(\Gamma_1))$-controls u is *reformulated* as the property that the operator $\mathcal{M}_T$ in (A.9) be surjective:

$$\mathcal{M}_T : L_2(\Sigma_1) \supset \mathcal{D}(\mathcal{M}_T) \overset{\text{onto}}{\longrightarrow} H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega). \tag{A.10}$$

This ontoness property for the closed operator $\mathcal{M}_T$ is, in turn, equivalent to the property that its (Hilbert space) adjoint $\mathcal{M}^*_T : H \supset \mathcal{D}(\mathcal{M}^*_T) \rightarrow L_2(\Sigma_1)$ has a continuous inverse [T-L.1, p. 235]:

$$\left\| \mathcal{M}^*_T \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|_{L_2(\Sigma_1)} \geq C_T \|\{h_0, h_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)}, \quad \forall \; \{h_0, h_1\} \in \mathcal{D}(\mathcal{M}^*_T) \subset H. \tag{A.11}$$

[This is the "continuous observability" inequality in this case.] We shall see now that inequality (A.3) is nothing but a PDE-reformulation of inequality (A.11). [We have $(\mathcal{M}_T u, z)_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)} = (u, \mathcal{M}^*_T z)_{L_2(\Sigma)}$, consistently with the notation $*$ in Remark 4.1.]

Step 2. We shall show that $\mathcal{M}^*_T \equiv \mathcal{M}^*_T[q]$ satisfies

$$\left(\mathcal{M}^*_T \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)(t) = \phi(t; \phi_0, \phi_1)|_{\Gamma_1}; \;\; \phi_0 = h_1 \in L_2(\Omega); \phi_1 = -\mathcal{A} h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'. \tag{A.12}$$

To prove (A.12), we set $y_0 = y_1 = 0$ in (A.1b); multiply the y-problem in (A.1) by ϕ (solution of (A.4)) and integrate by parts (Green's second theorem) to get

$$\int_\Omega \int_0^T y_{tt}\phi \, dQ = \int_0^T \int_\Omega \Delta y \phi \, dQ + \iint\limits_Q qy\phi \, dQ. \tag{A.13}$$

In the $L_2(\Omega)$-inner product we obtain, using (A.4a), (A.1c) (LHS) and (A.4c) (RHS):

$$(y_t(T), \phi(T)) - (y(T), \phi_t(T)) - (y_1, \phi_0) + (y_0, \phi_t(0)) + \int_Q \cancel{y\phi_{tt}} \, dQ$$

$$= \int_Q \cancel{y\Delta\phi} \, dQ + \int_\Sigma \frac{\partial y}{\partial \nu}\phi \, d\Sigma - \int_\Sigma \cancel{y\frac{\partial \phi}{\partial \nu}} d\Sigma + \int_Q \cancel{yq\phi} \, dQ,$$

or, as $y_0 = y_1 = 0$; $\phi(T) = \phi_0$, $\phi_t(T) = \phi_1$ via (A.1c) (RHS) and (A.4c) (LHS):

$$(y_t(T), \phi_0)_{L_2(\Omega)} - (y(T), \phi_1)_{L_2(\Omega)} = \int_{\Sigma_1} u\phi \, d\Sigma_1, \tag{A.14}$$

or

$$\left(\begin{bmatrix} y(T) \\ y_t(T) \end{bmatrix}, \begin{bmatrix} -\phi_1 \\ \phi_0 \end{bmatrix} \right)_{L_2(\Omega)\times L_2(\Omega)} = (u, \phi|_{\Gamma_1})_{L_2(\Sigma_1)}. \tag{A.15}$$

On the other hand, by definition of $\mathcal{M}_T$ and $\mathcal{M}_T^*$, we have for $\{h_0, h_1\} \in \mathcal{D}(\mathcal{M}_T^*) \subset H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$:

$$\begin{aligned} \left(\mathcal{M}_T u, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)} &= \left(\begin{bmatrix} y(T) \\ y_t(T) \end{bmatrix}, \begin{bmatrix} \mathcal{A}h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Omega)\times L_2(\Omega)} \\ &= \left(u, \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Sigma_1)}. \end{aligned} \tag{A.16}$$

By comparison between (A.15) and (A.16), we conclude that (A.12) holds true, as desired.

Step 3. Thus, by (A.12), we have

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|^2_{L_2(\Sigma_1)} = \int_{\Sigma_1} \phi^2(t) d\Sigma_1; \tag{A.17}$$

$$\begin{aligned} \|\{h_0, h_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)} &= \|\{\mathcal{A}^{\frac{1}{2}} h_0, h_1\}\|^2_{L_2(\Omega)\times L_2(\Omega)} \\ &= \|\{\mathcal{A}^{-\frac{1}{2}} \phi_1, \phi_0\}\|^2_{L_2(\Omega)\times L_2(\Omega)} \end{aligned}$$

$$= \quad \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \qquad \text{(A.18)}$$

and then the necessary and sufficient condition ["continuous observability inequality"] (A.11) reads off as

$$\int_{\Sigma_1} \phi^2(t; \phi_0, \phi_1)|_{\Gamma_1} d\Sigma_1 \geq C_T \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \qquad \text{(A.19)}$$

where $\phi_0 = h_1 \in L_2(\Omega)$, $\phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Thus, Lemma A.1 is established. □

Thus, by duality, the *crux of the matter* is to prove the validity of inequality (A.3). The latter requires, in general, geometric conditions on $\{\Omega, \Gamma_0, \Gamma_1\}$. One illustration of this is given by the setting given in Eqns. (3.6)–(3.8) of Section 3. We have

Theorem A.2. *(E.C. of (A.1) for fixed 'potential' q.) Assume (A.2) on q and, moreover, let the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ satisfy the geometrical assumptions (3.6), (3.7) of Section 3. Let $T > T_0$ with T_0 defined by (3.8). Then:*

(i) The Continuous Observability inequality (A.3) holds true. For a fixed $q \in L_\infty(Q)$, the constant $C_{T,q}$ in (A.3) depends on T as well as on the $\|q\|_{L_\infty(Q)}$-norm of q: $C_{T,\|q\|_{L_\infty(Q)}}$.

(ii) Hence, by Lemma A.1, the y-problem (A.1) is exactly controllable on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, by means of $L_2(0,T; L_2(\Gamma_1))$-controls.

This result is included *a-fortiori* in either one of Parts I, II, III, IV. Refer to Fig. 5.1.

Case of 'potential' q varying in a ball. As seen in the discussion leading to the Uniform Continuous Observability Inequality (5.10), as well as at the outset of Part III, in the study of exact controllability for the corresponding semilinear problem (1.1), we need to have "control" on the dependence of the above constant $C_{T,\|q\|_{L_\infty(Q)}}$, when q varies in a ball of $L_\infty(Q)$ of preassigned radius, as to establish (5.10). In particular, we need to ascertain that such constant $C_{T,\|q\|_\infty}$ depends only on the radius of such ball. For $r > 0$, define

$$B(0,r) = \{p \in L_\infty(Q) : \ \|p\|_{L_\infty(Q)} \le r\}. \tag{A.20}$$

Then we have

Theorem A.3. *(E.C. of (A.1) for potential q varying on a ball of radius r, as in (A.20).) Consider problem (A.1) where the potential q varies in the $L_\infty(Q)$-ball $B(0,r)$ defined in (A.20). Assume the validity of the COI (A.3) for some $q \in B(0,r)$ and for $T > T_0$ defined in (3.8). Then, the constant $C_{T,q}$ in (A.3) can be chosen as to depend only on the radius $r > 0$ in (A.20). More precisely, we have that: There exists a constant $C_{T,r} > 0$ such that inequality (A.3) can be written as*

$$\begin{aligned} &C_{T,r}\|\{\phi_0,\phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \\ &\qquad \le \int_{\Sigma_1} \phi^2 d\Sigma_1, \textit{uniformly in all } q \in B(0,r) \subset L_\infty(Q). \end{aligned} \tag{A.21}$$

This is the Uniform Continuous Observability Inequality (5.10).

This result is established in either one of Parts I, II, III, IV. Refer to Fig. 5.1.

Remark A.2. It is well known [Tr.2, Appendix], [L-T.2, Appendix] that the constant occurring in a COI is related to the minimal norm among the steering controls. Thus Theorem A.3 says that the minimal energy (norm) $L_2(0,T;L_2(\Gamma_1))$ among all steering controls is uniformly bounded with respect to potentials q in a fixed $L_\infty(Q)$-ball.

Appendix B: Lemma A.1 Revisited Via an Evolution Operator Approach

The goal of this section is to *re-prove* Lemma A.1. In the process, we shall obtain formula (B.13) for $\mathcal{M}_T^* = \mathcal{M}_T^*[q]$, which was used in Section 6, Eqn. (6.4).

Instead of a PDE-approach as in Appendix A, we follow here an evolution operator strategy (the time-dependent counterpart of the semigroup or cosine operator strategy in [L-T.2], [L-T.4], [Tr.2]). Introduce the operator

$$\mathbb{A}(t) = \begin{bmatrix} 0 & I \\ -\mathcal{A} + q(t,\,\cdot\,) & 0 \end{bmatrix} = A + F'[q]; \quad \mathcal{D}(\mathbb{A}(t)) \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \tag{B.1}$$

where $\mathcal{A}$ is defined in (A.6) and A is defined in (2.2). One needs to invoke a generation result from the general theory [Tan.1, Sections 7.3, 7.7]: There exists an evolution operator $U(t,s) \in \mathcal{L}(Y),\ Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, possessing the following properties:

$$U(t,\tau)U(\tau,s) = U(t,s),\ 0 \le s \le \tau \le t; \tag{B.2}$$

$$\frac{\partial U(t,s)}{\partial t} = \mathbb{A}(t)U(t,s); \tag{B.3a}$$

$$\frac{\partial U^*(t,s)}{\partial t} = U^*(t,s)\mathbb{A}^*(t)$$
$$\text{(adjoint in the } H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)\text{-topology)}; \tag{B.3b}$$

$$\frac{\partial U(t,s)}{\partial s} = -U(t,s)\mathbb{A}(s); \tag{B.3c}$$

$$\frac{\partial U^*(t,s)}{\partial s} = -\mathbb{A}^*(s)U^*(t,s). \tag{B.3d}$$

Moreover, as usual, problem (A.1) for y may be written abstractly as follows. Recall the ('Neumann') map N in (2.5)

$$h = Ng \Longleftrightarrow \left\{ \Delta h = 0 \text{ in } \Omega;\ h|_{\Gamma_0} = 0 \text{ in } \Gamma_0;\ \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_1} = g \text{ in } \Gamma_1 \right\}. \tag{B.4}$$

Thus, rewrite problem (A.1) as

$$\begin{cases} y_{tt} = \Delta(y - Nu) + qy & \text{in } Q; \quad \text{or recalling (A.6), (B.4)}: \\ [y - Nu]_{\Gamma_0} = 0 & \text{in } \Sigma_0; \quad y_{tt} = -\mathcal{A}(y - Nu) + qy \in L_2(\Omega) \\ & \qquad\quad \text{or} \\ \dfrac{\partial}{\partial \nu}[y - Nu]_{\Gamma_1} = 0 & \text{in } \Sigma_1; \quad y_{tt} = -\mathcal{A}y + \mathcal{A}Nu + qy \in [\mathcal{D}(\mathcal{A})]'. \end{cases} \tag{B.5}$$

Thus, the second-order/first-order abstract models of problem (A.1) are

$$y_{tt} = (-\mathcal{A} + q)y + \mathcal{A}Nu \in [\mathcal{D}(\mathcal{A})]'; \tag{B.6}$$

or via (B.1), $\tilde{y} = [y, y_t]$:

$$\begin{aligned} \tilde{y}_t &= \frac{d}{dt}\begin{bmatrix} y \\ y_t \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ -\mathcal{A}+q & 0 \end{bmatrix}\begin{bmatrix} y \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix} = \mathbb{A}(t)\tilde{y} + \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix}, \end{aligned} \tag{B.7a}$$

$$\tilde{y}_t = \mathbb{A}(t)\tilde{y} + Bu, \quad Bu = \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix}; \ \mathbb{A}(t) = A + F'[q], \text{ via (2.7)}, \tag{B.7b}$$

B as in (2.6). As usual, we have ([L-T.2, Lemma 3.2, p. 272])

$$(Bu, h)_H = (u, B^*h)_{L_2(\Gamma)} = (u, N^*\mathcal{A}h_2)_{L_2(\Gamma)}; \tag{B.8a}$$

$$B^*h = B^*\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = N^*\mathcal{A}h_2 = \begin{cases} 0 & \text{on } \Gamma_0 \\ h_2|_{\Gamma_1} & \text{on } \Gamma_1 \end{cases}; \quad h_2 \in \mathcal{D}(\mathcal{A}) \tag{B.8b}$$

[the validity of (B.8) can be extended to, say, all $f \in H^{\frac{3}{2}+\epsilon}(\Omega)$ s.t. $\frac{\partial f}{\partial \nu}|_{\Gamma_1} = 0$, $f|_{\Gamma_0} = 0$]. The map $\mathcal{M}_T$ introduced in (A.9) can now be made explicit by means of the 'variation of constant' formula:

$$\begin{bmatrix} y(T) \\ y_t(T) \end{bmatrix} = \mathcal{M}_T u = \int_0^T U(T,t)Bu(t)dt = \int_0^T U(T,t)\begin{bmatrix} 0 \\ \mathcal{A}Nu(t) \end{bmatrix} dt. \tag{B.9}$$

By using (B.9), we now pick $\{h_0, h_1\} \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ and compute $\left(\mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)(t)$, as required by (A.11) We intend to re-obtain (A.12). [For the present approach in the time-independent case $q(t,x) = q(x)$, see [L-T.2, p. 271–73]. We compute via (B.9),

$$\begin{aligned} &\left(\mathcal{M}_T u, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)_{H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)} \\ &\quad = \left(\int_0^T U(T,t)Bu(t)dt, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)_H \end{aligned}$$

$$= \int_0^T \left(u(t), B^* U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Gamma_1)} \tag{B.10a}$$

$$\text{(by (B.7b))} \quad = \left(\int_0^T U(T,t) \begin{bmatrix} 0 \\ \mathcal{A}Nu(t) \end{bmatrix} dt, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)}$$

$$= \int_0^T \left(\begin{bmatrix} 0 \\ \mathcal{A}Nu(t) \end{bmatrix}, U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)}$$

$$= \int_0^T \left(u(t), N^*\mathcal{A} \left[U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right]_2 \right)_{L_2(\Gamma_1)} dt$$

$$= \left(u, \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Sigma_1)}, \tag{B.10b}$$

where $\left[U^*(T,t) \left[\begin{smallmatrix} h_0 \\ h_1 \end{smallmatrix}\right] \right]_2$ denotes the second component of the 2-component vector $U^*(T,t) \left[\begin{smallmatrix} h_0 \\ h_1 \end{smallmatrix}\right]$. Thus, (B.10) gives

$$\left(\mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)(t)$$

$$= B^* U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = B^* V(t; V_0) \tag{B.11a}$$

(by (B.8b))

$$= N^*\mathcal{A} \left[U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right]_2 = \left[U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right]_2 \Bigg|_{\Sigma_1}, \tag{B.11b}$$

recalling that $N^*\mathcal{A}$ is the Dirichlet trace on Γ_1, by (B.8b). Here we have set

$$\begin{bmatrix} v_1(t; V_0) \\ v_2(t; V_0) \end{bmatrix} \equiv V(t; V_0) \equiv U^*(T;t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix};$$

$$\text{thus } V_0 \equiv V(T) = [h_0, h_1] \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega), \tag{B.12}$$

so that (B.11b) is rewritten as

$$\left(\mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)(t) = B^* V(t; V_0) = v_2(t; V_0)|_{\Sigma_1}. \tag{B.13}$$

Remark B.1. Notice that, consistently with Remark 4.1, the symbol $*$ for an adjoint always refer to the space $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$. Thus, $(Bu, h)_H = (u, B^*h)_{L_2(\Gamma)}$, $h \in \mathcal{D}(B^*) \subset H$, while $\mathcal{M}_T^*$, $U^*(T,t)$ are adjoints in H. Similarly, the operator $\mathbb{A}^*(t)$ in (B.3b,d).

Recalling (B.3d), we obtain by (B.12),

$$V_t(t; V_0) = U_t^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = -\mathbb{A}^*(t) U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = -\mathbb{A}^*(t) V(t; V_0). \tag{B.14}$$

From $\mathbb{A}(t)$ in (B.1) and (B.7b), we readily compute that the $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$-adjoint of $\mathbb{A}(t)$ is

$$\mathbb{A}^*(t) = \begin{bmatrix} 0 & -I + \mathcal{A}^{-1}(q\,\cdot) \\ \mathcal{A} & 0 \end{bmatrix} = A^* + (F'[q])^*;\ \mathcal{D}(\mathbb{A}^*(t)) = \mathcal{D}(\mathbb{A}(t)), \tag{B.15a}$$

or

$$\mathbb{A}^*(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 + \mathcal{A}^{-1}(qx_2) \\ \mathcal{A}x_1 \end{bmatrix}; \tag{B.15b}$$

Thus, we rewrite (B.14) for $V(t; V_0) = \begin{bmatrix} v_1(t; V_0) \\ v_2(t; V_0) \end{bmatrix}$ by virtue of (B.15) as

$$\begin{cases} v_{1t} = v_2 - \mathcal{A}^{-1}(qv_2) \\ v_{2t} = -\mathcal{A}v_1; \qquad\qquad \Rightarrow v_{2tt} = -\mathcal{A}v_2 + qv_2; \\ v_{2t}(T) = -\mathcal{A}v_1(T) = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]',\ \text{by (B.12)} \end{cases} \tag{B.16}$$

In conclusion, $v_2(t; V_0)$ solves the problem (same as (3.9) or (6.3))

$$\begin{cases} v_{2tt} = \Delta v_2 + q(t,x) v_2 & \text{on } Q; \\ v_2(T) = h_1 \in L_2(\Omega);\ v_{2t}(T) = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' & \text{on } \Omega; \\ v_2|_{\Sigma_0} \equiv 0,\ \left.\dfrac{\partial v_2}{\partial \nu}\right|_{\Sigma_1} \equiv 0 & \text{on } \Sigma_i; \end{cases} \tag{B.17}$$

whose regularity is given by Theorem 3.1. Thus, via (B.13) we obtain

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|_{L_2(\Sigma_1)}^2 \equiv \int_{\Sigma_1} v_2^2(t; V_0) d\Sigma; \tag{B.18}$$

while via (B.17) we obtain

$$\begin{aligned} \|\{h_0, h_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)} &\equiv \|\{\mathcal{A}^{\frac{1}{2}} h_0, v_2(T)\}\|^2_{L_2(\Omega) \times L_2(\Omega)} \\ &= \|\{\mathcal{A}^{-\frac{1}{2}} v_{2t}(T), v_2(T)\}\|^2_{L_2(\Omega) \times L_2(\Omega)} \\ &= \|\{v_2(T), v_{2t}(T)\}\|^2_{L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \end{aligned} \tag{B.19}$$

Then the Continuous Observability Inequality (A.11):

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|_{L_2(\Sigma_1)}^2 \geq C_T \|\{h_0, h_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)}$$

is now rewritten via (B.18), (B.19) as

$$\int_0^T \int_{\Gamma_1} v_2^2 d\Sigma \geq C_T \|\{v_2(T), v_{2t}(T)\}\|^2_{L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{B.20}$$

where v_2 solves problem (B.17). Comparing (B.17), (B.20) with (A.4), (A.3), we see that the v_2-problem (B.17) is *precisely* the ϕ-problem in (A.4) and (B.20) is precisely (A.3). Thus, we have *re-proved Lemma A.1.*

Appendix C: Proof of Theorem 3.1: Regularity of the ϕ-Problem (3.9) = (6.3) = (A.4) [Same as the v_2-Problem (B.17)] with I.C. in $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$

Actually, since the ϕ-problem is time-reversible, we may take the Initial Conditions $\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ at $t = 0$, rather than $t = T$. We shall prove Theorem 3.1, relabeled here as

Theorem C.1. *With reference to the ϕ-problem (3.9) = (6.3) = (A.4), with q as in (3.10) = (A.2) and Initial Conditions $\{\phi_0, \phi_1\}$ at $t = 0$, we have*

$$\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \Rightarrow \{\phi, \phi_t\} \in C([0,T]; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'), \tag{C.1}$$

continuously, where the constant of continuity depends explicitly on the $\|q\|_{L_\infty(Q)}$-norm as in (C.16) below.

One may provide several more or less standard proofs: By the evolution operator $U(t,s)$ as in Appendix B; by fixed point (contraction); etc. Here, for completeness, we provide a proof which has the advantage of showing the *dependence on q of certain estimates*, see (C.13) below, to be needed later in various contexts.

Proof of Theorem C.1. Step 1. (See (C.13) below) As in (6.36), we split the ϕ-problem in (3.9) = (6.3) = (A.4) (with I.C. at $t = 0$) as the sum of the less regular problem ψ and the more regular problem χ, by setting

$$\phi = \psi + \chi, \tag{C.2}$$

where ψ and χ are solutions of the following problems:

$$\begin{cases} \psi_{tt} = \Delta\psi; \\ \psi(0,\,\cdot\,) = \phi_0, \;\; \psi_t(0,\,\cdot\,) = \phi_1; \\ \psi|_{\Sigma_0} \equiv 0, \;\; \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0; \end{cases} \quad \begin{cases} \chi_{tt} = \Delta\chi + q(t,x)\phi = \Delta\chi + q(t,x)\chi \\ \qquad\qquad\qquad\qquad\qquad + q(t,x)\psi & \text{in } Q; \\ \chi(0,\,\cdot\,) = \chi_t(0,\,\cdot\,) = 0 & \text{in } \Omega; \\ \chi|_{\Sigma_0} \equiv 0; \;\; \left.\dfrac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i. \end{cases} \tag{C.3}$$

The idea of the proof is that by using the unperturbed ψ-problem, we can shift the topologies up or down. For the pure wave equation in ψ, the regularity result

$$\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \Rightarrow \{\psi, \psi_t\} \in C([0,T]; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]') \tag{C.4}$$

continuously, is standard. It can be shown in various ways; e.g., by using $\psi(t) = C(t)\psi_0 + S(t)\psi_1$, where $C(\,\cdot\,)$ is the 'cosine' operator generated by $-\mathcal{A}$, and $S(t)$ the corresponding 'sine' operator, see (2.3).

Step 2. Regarding the χ-problem in the RHS of (C.3), a *more regular* result is actually true. Namely, setting

$$E(t) \equiv \int_\Omega [|\nabla\chi(t)|^2 + \chi_t^2(t)]d\Omega \text{ norm-equivalent to } \|\{\chi(t), \chi_t(t)\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)}, \tag{C.5}$$

by the Poincaré inequality ($\Gamma_0 \neq \emptyset$), we shall first prove that

$$E(t) \equiv \int_\Omega [|\nabla\chi(t)|^2 + \chi_t^2(t)]d\Omega \leq C_T e^{M_q t}\|q\|_{L_\infty(Q)}\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \tag{C.6a}$$

$$M_q \equiv \max\{c_P\|q\|_{L_\infty(Q)}, 2\|q\|_{L_\infty(Q)}\}, \quad c_P = \text{constant in Poincaré inequality.} \tag{C.6b}$$

In fact, multiplying the χ-problem by χ_t and integrating by parts, we obtain, as usual, using the homogeneous I.C., and denoting $\|q\|_\infty = \|q\|_{L_\infty(Q)}$:

$$\begin{aligned}\int_\Omega [\chi_t^2(t) + |\nabla\chi(t)|^2]d\Omega &= 2\int_0^t\int_\Omega q(\,\cdot\,,\,\cdot\,)\chi\chi_t dQ + 2\int_0^t\int_\Omega q(\,\cdot\,,\,\cdot\,)\psi\chi_t dQ \\ &\leq \|q\|_\infty \int_0^t\int_\Omega [\chi^2(\tau) + \chi_t^2(\tau)]d\Omega\, d\tau \\ &+ \|q\|_\infty \int_0^t\int_\Omega [\psi^2(\tau) + \chi_t^2(\tau)]d\Omega\, d\tau. \end{aligned} \tag{C.7}$$

We now invoke the Poincaré inequality: $\int_\Omega \chi^2 d\Omega \leq c_P \int_\Omega |\nabla\chi|^2 d\Omega$, which is valid since we have assumed $\Gamma_0 \neq \emptyset$. Thus, (C.7) becomes

$$\begin{aligned}&\int_\Omega [\chi_t^2(t) + |\nabla\chi(t)|^2]d\Omega \\ &\qquad\leq c_P\|q\|_\infty \int_0^t\int_\Omega |\nabla\chi(\tau)|^2 d\Omega\, d\tau \\ &\qquad+ 2\|q\|_\infty\int_0^t\int_\Omega \chi_t^2(\tau)d\Omega\, d\tau + \|q\|_\infty\int_0^t\int_\Omega \psi^2(\tau)d\Omega\, d\tau. \end{aligned} \tag{C.8}$$

Setting $M_q \equiv \max\{c_P \|q\|_\infty, 2\|q\|_\infty\}$, and invoking a weaker L_2-regularity result included in conclusion (C.4) in the last term in (C.8), we obtain

$$\int_\Omega [\chi_t^2(t) + |\nabla \chi(t)|^2] d\Omega \quad \leq \quad C_T \|q\|_\infty \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} + M_q \int_0^t \int_\Omega [\chi_t^2(\tau) + |\nabla \chi(\tau)|^2] d\Omega \, d\tau; \tag{C.9}$$

or via (C.5),

$$E(t) \leq C_T \|q\|_\infty \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} + M_q \int_0^t E(\tau) d\tau. \tag{C.10}$$

Application of the Gronwall's inequality on (C.10) yields

$$E(t) \equiv \int_\Omega [|\nabla \chi(t)|^2 + \chi_t^2(t)] d\Omega \leq C_T e^{M_q t} \|q\|_\infty \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{C.11}$$

which is precisely (C.6a).

Step 3. From the pointwise estimate (C.11) = (C.6a), we then obtain

$$\sup_{0\leq t\leq T} \int_\Omega [|\nabla \chi(t)|^2 + \chi_t^2(t)] d\Omega \leq C_T e^{M_q T} \|q\|_{L_\infty(Q)} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \tag{C.12}$$

Due to the norm equivalence noted in (C.5) (via $\Gamma_0 \neq \emptyset$), we finally obtain from (C.12) (first in $L_\infty(0, T; \cdot)$, then by approximation in $C([0, T]; \cdot)$)

$$\|\{\chi, \chi_t\}\|^2_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega))} \leq C_T e^{M_q T} \|q\|_{L_\infty(Q)} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{C.13}$$

which shows the desired regularity property

$$\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \Rightarrow \{\chi, \chi_t\} \in C([0, T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)), \tag{C.14}$$

with an explicit dependence on $\|q\|_{L_\infty(Q)}$ of the constant of continuity.

Step 4. Then statements (C.4) and (C.14), combined in (C.2), yield, *a-fortiori*

$$\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \Rightarrow \{\phi, \phi_t\} \in C([0, T]; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}}]'), \tag{C.15}$$

which is the desired conclusion (C.1). Actually (C.4) and (C.13) give the more precise result

$$\begin{aligned} &\|\{\phi, \phi_t\}\|^2_{C([0,T];L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')} \\ &\quad \le \quad C_T(1+e^{M_q T}\|q\|_{L_\infty(Q)})\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \end{aligned} \tag{C.16}$$

Theorem C.1 is proved. □

That is, Theorem 3.1 (= Theorem C.1) of Section 3 is proved.

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TOPIC II:

Well-Posedness and Sharp Uniform Decay Rates at the $L_2(\Omega)$-Level of the Schrödinger Equation with Nonlinear Boundary Dissipation[2]

Abstract

We prove that the Schrödinger equation defined on a bounded open domain of $\mathbb{R}^n$ and subject to a certain attractive, nonlinear, dissipative boundary feedback is (semigroup) well-posed on $L_2(\Omega)$ for any $n = 1, 2, 3, \ldots$, and, moreover) stable on $L_2(\Omega)$ for $n = 2, 3$, with sharp (optimal) uniform rates of decay. Uniformity is with respect to all initial conditions contained in a given $L_2(\Omega)$-ball. This result generalizes the corresponding linear case which was proved recently in [L-T-Z.2]. Both results critically rely—at the outset—on a far general result of interest in its own right: an energy estimate at the $L_2(\Omega)$-level for a fully general Schrödinger equation with gradient and potential terms. The latter requires a heavy use of pseudo-differential/micro-local machinery [L-T-Z.2, Sect. 10], to shift down the more natural $H^1(\Omega)$-level energy estimate to the $L_2(\Omega)$-level. In the present nonlinear boundary dissipation case, the resulting energy estimate is then shown to fit into the general uniform stabilization strategy, first proposed in [La-Ta.1] in the case of wave equations with nonlinear (interior and) boundary dissipation.

1 Introduction and Motivation: Uniform Boundary Stabilization and Exact Controllability of the Linear Schrödinger Equation at the $L_2(\Omega)$-Energy Level

The goal of the present paper is to provide a uniform boundary stabilization result at the $L_2(\Omega)$-energy level (Theorem 2.2) for a multi-dimensional Schrödinger equation model in feedback form, with nonlinear dissipation in the Neumann-boundary conditions. The mathematical model is given in Eqns.

[2]Well-posedness and uniform decay rates at the $L_2(\Omega)$-level of Schrödinger equations with nonlinear boundary dissipation, *J. Evol. Eqns.* 6 (2006), 485–537.

(2.1a–c) in Section 2 below. The aforementioned boundary feedback energy decay result with nonlinear boundary dissipation is, in turn, motivated by, and a generalization of, the corresponding linear problem, which was recently obtained in [L-T-Z.2, Sect. 11]; see also [L-T-Z.3, Sect. 11]. Accordingly, we need to review such linear result first.

Conservative open-loop and dissipative closed-loop Schrödinger problems in $L_2(\Omega)$. Recently the following problem was considered in [L-T-Z.2, Sect. 11]. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded domain with sufficiently smooth boundary $\partial\Omega = \Gamma$ of class C^2, $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. In Ω, we consider the following two Schrödinger problems

$$\begin{cases} iy_t + \Delta y = 0; \\ y(0,\,\cdot\,) = y_0; \\ y|_{\Gamma_0} \equiv 0,\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Gamma_1} \equiv u; \end{cases} \quad \begin{cases} iv_t + \Delta v = 0 & \text{in } Q = (0,T] \times \Omega; \quad (1.1a) \\ v(0,\,\cdot\,) = v_0 & \text{in } \Omega; \quad (1.1b) \\ v|_{\Gamma_0} \equiv 0,\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Gamma_1} \equiv iv & \text{in } \Sigma_k = (0,T] \times \Gamma_k, \quad (1.1c) \end{cases}$$

$k = 0, 1$, $\nu(x) =$ outward unit (real) normal at $x \in \Gamma_1$, where the v-problem can be viewed as a closed-loop version of the y-Neumann problem, with boundary control u on Γ_1 in the feedback form $u = iv$. For $u \equiv 0$, the y-problem is conservative ('energy' preserving, see (1.3) below, with skew-adjoint generator $\mathbb{A} = -\mathbb{A}^*$, see (1.14) below). Instead, in contrast, the v-problem is dissipative, as quantitatively stated in Theorem 1.1 and Theorem 1.2 below. The solution y or v is complex-valued.

A detailed description of the literature on uniform boundary stabilization of linear Schrödinger equations (with boundary dissipation in both the Dirichlet or Neumann B.C.) is deferred to the end of Section 2, after the statement of the main results of the present paper. This concerns the boundary nonlinear version of the v-problem in (1.1) [see problem (2.1)]. For now, it suffices to note that, unlike prior literature, the v-problem in (1.1) has a local (as opposed to 'non-local') boundary feedback involving the solution v (instead of its time derivative $v_t \sim \Delta v$), and the resulting state space is $L_2(\Omega)$.

Remark 1.1. In (1.1c), with cosmetic changes, we could also allow $\Gamma_0 = \emptyset$ and $\Gamma \equiv \Gamma_1$. More importantly, in (1.1c) we could also include the case where the homogeneous Dirichlet B.C. $y|_{\Gamma_0} \equiv 0$, $v|_{\Gamma_0} \equiv 0$ on Γ_0 are replaced by the corresponding homogeneous Neumann B.C. $\frac{\partial y}{\partial \nu}|_{\Gamma_0} \equiv 0$, $\frac{\partial v}{\partial \nu}|_{\Gamma_0} \equiv 0$, respectively, in which case the condition $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ is dispensed with. However, in this latter 'purely' Neumann case, more stringent geometrical conditions are called for: there exists a smooth strictly convex function $d : \overline{\Omega} \to \mathbb{R}$ such that the vector field $\ell(x) = \nabla d(x)$ satisfies: $\ell \cdot \nu \equiv 0$ on Γ_0 (instead of the more relaxing condition $\ell \cdot \nu \leq 0$ on Γ_0 in the Dirichlet case, as assumed in Theorem 1.2 and (H.3) = (2.4) below). Various classes of $\{\Omega, \Gamma_0, \Gamma_1\}$ where such strictly convex function d may be constructed are given in [L-T-Z.1, Appx. A, pp. 287–307]. In particular, *a sufficient condition is that* Γ_0 *be convex or concave* [L-T-Z.1, Thm. A.4.1, p. 301]. This result is extended in the Riemannian setting in [T-Y.1, Appx. B]. Same considerations apply to the nonlinear problem (2.1) below.

Well-posedness and stabilization. The following two results are proved in [L-T-Z.2, Sect. 11]; see also [L-T-Z.3, Sect. 11].

Theorem 1.1. *[L-T-Z.2, Thm. 11.1.1] (Well-posedness and strong stabilization in $L_2(\Omega)$) With reference to the v-problem in (1.1), we have:*

(i) the map $v_0 \to v(t)$ defines a s.c. contraction semigroup on $L_2(\Omega)$: $v(t) = e^{L_F t} v_0 \in C([0,T]; L_2(\Omega))$, where L_F is a maximal dissipative operator, to be explicitly defined in (1.13) or (1.15) below.

(ii) Setting

$$E_v(t; v_0) = E_v(t) \equiv \|e^{L_F t} v_0\|^2_{L_2(\Omega)} \equiv \|v(t)\|^2_{L_2(\Omega)} = \int_\Omega |v(t)|^2 d\Omega, \tag{1.2}$$

we have the dissipativity relations

$$\frac{dE_v(t)}{dt} = -2 \int_{\Gamma_1} \left| \frac{\partial v}{\partial \nu} \right|^2 d\Gamma_1 \leq 0; \quad E_v(t) + 2 \int_0^t \int_{\Gamma_1} \left| \frac{\partial v}{\partial \nu} \right|^2 d\Gamma_1 dt = E_v(0); \tag{1.3}$$

$$\int_0^\infty \int_{\Gamma_1} \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma_1 dt = \int_0^\infty \int_{\Gamma_1} |v|^2 d\Gamma_1 dt \le \frac{1}{2} E_v(0). \tag{1.4}$$

(iii) The resolvent $R(\lambda, L_F)$ of L_F is well defined, at least for all λ with Re $\lambda \geq 0$, and is compact on $L_2(\Omega)$.

(iv) (Strong stabilization) For $v_0 \in L_2(\Omega)$, we have

$$e^{L_F t} v_0 \to 0 \text{ in } L_2(\Omega), \qquad \text{as } t \to \infty. \tag{1.5}$$

Notice that the strong stabilization result in (1.5) requires no geometric condition. Indeed, it is proved by "soft methods," see comments below. In contrast, the key result of [L-T-Z.2, Sect. 11] is the following uniform stabilization result, which instead requires geometric conditions.

Theorem 1.2. *[L-T-Z.2, Thm. 11.1.2] (Uniform stabilization) Assume that there exists a (i) coercive real-valued vector field $\ell(x) \in (C^2(\overline{\Omega}))^n$ [that is, with Jacobian matrix J satisfying $Re\{Jv \cdot \bar{v}\} \geq \rho |v|^2$, $\rho > 0$; in particular, $\ell(x) = \nabla d(x)$, for a real strictly convex function $d(x)$ on Ω, the radial case $\ell(x) = x - x_0$, for some $x_0 \in \mathbb{R}^n$, being the canonical case], such that (ii) $\ell \cdot \nu \leq 0$ on Γ_0. Then, there exist constants $M \geq 1$, $\delta > 0$, such that*

$$\|e^{L_F t}\|_{\mathcal{L}(L_2(\Omega))} \leq M e^{-\delta t}, \quad t \geq 0; \text{ equivalently, } E_v(t) \leq M e^{-\delta t} E_v(0),\ t \geq 0. \tag{1.6}$$

Comments about the proofs of these results—particularly the key Theorem 1.2—are deferred below. The proof of Theorem 1.1 uses also the following setting, to be invoked also in the proof of Section 2 in the nonlinear case.

Abstract models. Following an established procedure, e.g., [L-T.1]–[L-T.3], [Tr.1], we have that the abstract model of each of the two problems in (1.1) is [L-T-Z.2, Sect. 11.2],

$$iy_t = A(y - Nu); \qquad iv_t = Av - iANN^*Av = iL_F v \in [\mathcal{D}(A)]', \tag{1.7}$$

where A is the positive self-adjoint operator $L_2(\Omega) \supset \mathcal{D}(A) \to L_2(\Omega)$,

$$A\psi = -\Delta\psi, \quad \mathcal{D}(A) = \left\{ f \in H^2(\Omega) : \ f|_{\Gamma_0} = 0,\ \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = 0 \right\}, \tag{1.8}$$

N is the Neumann map [L-T.2], [Tr.1], [L-T-Z.1]

$$\chi = Ng \Longleftrightarrow \left\{ \Delta\chi = 0 \text{ in } \Omega;\ \chi|_{\Gamma_0} = 0,\ \left.\frac{\partial \chi}{\partial \nu}\right|_{\Gamma_1} = g \right\}, \quad \Gamma_0 \neq \emptyset; \tag{1.9}$$

$$N : \text{continuous } H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega),\ s \in \mathbb{R}; \tag{1.10}$$

$$N : \text{continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) \equiv \mathcal{D}(A^{\frac{3}{4}-\epsilon}),\ \forall\ \epsilon > 0; \tag{1.11}$$

$$N^*A^*\varphi = N^*A\varphi = \begin{cases} 0 & \text{on } \Gamma_0 \\ \varphi & \text{on } \Gamma_1 \end{cases}, \quad \varphi \in \mathcal{D}(A) : (Ng, u)_{L_2(\Omega)} = (g, N^*u)_{L_2(\Gamma)}; \tag{1.12}$$

where φ can be extended to all of $\mathcal{D}(A^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega)$; and L_F in (1.7) is the operator

$$L_F v = [\mathbb{A} - BB^*]v, \quad v \in \mathcal{D}(L_F) = \{v \in L_2(\Omega) : [\mathbb{A} - BB^*]v \in L_2(\Omega)\}; \tag{1.13}$$

$$\mathbb{A} = -iA = -\mathbb{A}^*; \quad B = AN; \quad \mathbb{A}^{-\frac{1}{2}}B \equiv R \in \mathcal{L}(L_2(\Gamma); L_2(\Omega));$$
$$R^* = B^*\mathbb{A}^{*-\frac{1}{2}} = -iB^*\mathbb{A}^{-\frac{1}{2}}, \tag{1.14}$$

and, for $x \in \mathcal{D}(L_F)$, we can write

$$L_F x = \mathbb{A}^{\frac{1}{2}}[I + iRR^*]\mathbb{A}^{\frac{1}{2}}x = f \in L_2(\Omega);\ iL_F x = A^{\frac{1}{2}}[I - iA^{\frac{1}{2}}NN^*A^{\frac{1}{2}}]A^{\frac{1}{2}}x; \tag{1.15}$$

$$x = L_F^{-1}f = \mathbb{A}^{-\frac{1}{2}}[I + iRR^*]^{-1}\mathbb{A}^{-\frac{1}{2}}f \in \mathcal{D}(L_F); \tag{1.16}$$

$$L_F^{-1} \in \mathcal{L}(L_2(\Omega)),\ \text{with } \mathcal{D}(L_F) \subset \mathcal{D}(\mathbb{A}^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B^*). \tag{1.17}$$

Comments on the proof of Theorem 1.1. The proof of *generation* is a rather standard application of the Lumer-Phillips Theorem, once the above setting (1.7) through (1.17) is in place. Eqn. (1.13) or (1.15), along with $\mathbb{A} = -\mathbb{A}^*$, yields that L_F is *dissipative*, while (1.17) on L_F^{-1} implies the 'range condition,' so that L_F is *maximal dissipative*: $(\lambda I - L_F)$ is surjective onto $L_2(\Omega)$, for some $\lambda > 0$.

The proof of *strong stabilization* follows, in a by-now established procedure, tested in numerous prior strong stabilization results, e.g., [L-T.1], [L-T.2], [Tr.1], etc. It uses the Stone's theorem combined with the Nagy-Foias-Foguel's theorem on contraction semigroups [Le.1].

Comments on the proof of Theorem 1.2. By contrast, the proof of Theorem 1.2 available from [L-T-Z.2, Sect. 11] is definitely non-trivial. As a matter of fact, the main goal of [L-T-Z.2] was not to establish Theorem 1.2. Rather, Theorem 1.2 was obtained in [L-T-Z.2] as a ready application of a far more general, *a-priori* energy estimate, of interest in its own right, which we now state in Theorem 1.3 below. The primary object of [L-T-Z.2] was to obtain an *a-priori* energy estimate *at the $L_2(\Omega)$-energy level* for a general linear Schrödinger equation of the form

$$iz_t + \Delta z = q_1(t,x) \cdot \nabla z + q_0(t,x) z(t,x) + f \text{ in } Q. \tag{1.18}$$

By contrast, the *natural energy level* for a Schrödinger equation such as (1.18) is the $H^1(\Omega)$-level, as is well known [L-T.2].

We provide below the key steps of the proof of the uniform stabilization Theorem 1.2, in the linear case (1.1), as these will be the basis for the proof of the uniform stabilization Theorem 2.2 in the non-linear case (2.1) as well.

Proof of Theorem 1.2. Step 1. The crux of the proof is the following *a-priori* $L_2(\Omega)$-level energy estimate for solutions z of Eqn. (1.18), a specialization of a Carleman estimate result.

Theorem 1.3. *Assume that $q_1(t,x) = ir_1(t,x)$ for a real-valued vector r_1. Let z be a solution of (1.18) satisfying, in addition, $z|_{\Sigma_0} \equiv 0$, where $\ell \cdot \nu \le 0$ on Γ_0, for the vector field $\ell(x)$ of Theorem 1.2. Let $T > 0$ be arbitrary. Then, if the forcing term $f \in L_2(Q)$, under some mild "minimal" assumptions of regularity on the coefficients $q_1(t,x)$ and $q_0(t,x)$ in (1.18)—which is not critical to recall here, as in the case of our v-problem of interest in (1.1) they are all zero—the following inequality holds true: there exists a constant $C_T > 0$ such that*

$$\int_0^T \left[\|z\|^2_{L_2(\Omega)} + \|z_t\|^2_{H^{-2}(\Omega)}\right] dt + \|z(0)\|^2_{L_2(\Omega)} + \|z_t(0)\|^2_{H^{-2}(\Omega)}$$

$$\leq \quad C_T \left\{ \|z\|^2_{L_2(\Sigma_1)} + \left\|\frac{\partial z}{\partial \nu}\right\|^2_{H_a^{-1}(\Sigma_1)} \right.$$

$$\left. + \int_0^T \int_{\Gamma_1} \left|\frac{\partial z}{\partial \nu}\right| |z| d\Gamma_1 dt + \|z\|^2_{H^{-1}(Q)} + \|f\|^2_{L_2(Q)} \right\}, \quad (1.19)$$

where $H_a^{-1}(\Sigma_1)$ is the dual space to the anisotropic space $H_a^1(\Sigma_1)$, with respect to the pivot space $L_2(\Sigma_1)$:

$$H_a^{-1}(\Sigma_1) = (H_a^1(\Sigma_1))'; \; H_a^1(\Sigma_1) \equiv H^{\frac{1}{2}}(0,T; L_2(\Gamma_1)) \cap L_2(0,T, H^1(\Gamma_1)). \quad (1.20)$$

Remark 1.2. The proof of the energy estimate at the $L_2(\Omega)$-level for a fully general Schrödinger equation with gradient and potential terms given in (1.19) requires a heavy use of pseudo-differential/micro-local analysis machinery [L-T-Z.2, Sect. 10], to shift the more natural $H^1(\Omega)$-level energy estimate to the $L_2(\Omega)$-level.

Step 2. We apply estimate (1.19) to our v-problem in (1.1), whereby then $\frac{\partial v}{\partial \nu}|_{\Gamma_1} = iv$ and $f = 0$. Moreover, the energy $E_v(t)$ in (1.2) satisfies $E_v(T) \leq E_v(t)$, $0 \leq t \leq T$ by (1.3). Thus, dropping from (1.19) the $H^{-2}(\Omega)$-terms for v_t plus the first term, we see that (1.19) specializes to

$$E_v(T) \leq E_v(0) \leq C_T \left\{ \int_0^T \int_{\Gamma_1} \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma_1 dt + \|v\|^2_{H^{-1}(Q)} \right\}, \quad (1.21)$$

since the $H_a^{-1}(\Sigma_1)$-norm is dominated by the $L_2(\Sigma_1)$-norm.

Step 3. The lower-order term on the RHS of inequality (1.21) can be absorbed: that is,

$$\|v\|^2_{H^{-1}(Q)} \leq c_T \left\|\frac{\partial v}{\partial \nu}\right\|^2_{L_2(\Sigma_1)}, \quad (1.22)$$

for a suitable constant $c_T > 0$. This follows by the by-now standard compactness/uniqueness contradiction argument [Lit.1]. To be carried out, this argument requires the following global uniqueness theorem

$$\begin{cases} i\psi_t + \Delta\psi \equiv 0 \text{ in } Q \\ \psi|_\Sigma \equiv 0, \ \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 \end{cases} \quad \text{on } [0,T] \Rightarrow \psi \equiv 0, \tag{1.23}$$

which is true by Holmgren's uniqueness theorem, in addition to [A.1] and [S.1].

Step 4. Using (1.22) into the RHS of (1.21) yields

$$E_v(T) \le E_v(0) \le C_T \int_0^T \int_{\Gamma_1} \left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma_1 dt, \tag{1.24}$$

so that the energy at $t = T$ and at $t = 0$ is dominated by the boundary dissipation on $L_2(0,T; L_2(\Gamma_1))$. A standard semigroup argument using Theorem 1.1 then allows one to establish that

$$\|e^{L_F T}\|_{\mathcal{L}(L_2(\Omega))} < 1, \tag{1.25}$$

and hence [Bal.1, p. 178] that the uniform estimate (1.6) holds true. Theorem 1.2 is proved. $\square$

Comments about the literature on uniform stabilization results for linear Schrödinger equations on bounded domains are deferred to the end of Section 2, after we present the nonlinear result of Theorem 2.2.

Exact controllability of the y-problem (1.1) on $L_2(\Omega)$.

Theorem 1.4. *Assume the same geometrical condition as in Theorem 1.2. Then, the y-problem (1.1) is exactly controllable in the following sense. Let $T > 0$ be arbitrary. Given $y_0 \in L_2(\Omega)$ [respectively, $y_1 \in L_2(\Omega)$], there exists a boundary control $u \in L_2(0,T; L_2(\Gamma_1))$ such that the corresponding solution of the y-problem (1.1) due to the data $\{y_0, u\}$ [respectively, due to the data $\{y_0 = 0, u\}$] satisfies $y(T) = 0$ [respectively, $y(T) = y_1$].*

Proof. The key ingredient is again the energy estimate (1.19) of Theorem 1.3. We shall consider only the case $y_0 = 0$, $y_1 \in L_2(\Omega)$ preassigned, as by time reversibility the other case $y_0 \in L_2(\Omega)$ preassigned and $y_1 = 0$ will follow.

Step 1. With reference to the abstract model (1.7) for the y-problem (1.1), rewritten here as

$$y_t = -iAy + iANu \in [\mathcal{D}(A)]', \qquad y(0) = 0, \tag{1.26}$$

its solution at $t = T$ is given by

$$y(T) = L_T u \equiv i \int_0^T e^{-iA(T-t)} ANu(t)dt : \ L_{@}(\Sigma_{1T}) \to L_2(\Omega), \tag{1.27}$$

where $L_2(\Sigma_{1T}) \equiv L_2(0,T;L_2(\Gamma_1))$. Exact controllability in the sense of the theorem's statement means that

$$L_T : \text{ surjective } L_2(\Sigma_{1T}) \text{ onto } L_2(\Omega); \text{ equivalently, } \|L_T^* \chi\|_{L_2(\Sigma_{1T})} \geq c_T \|\chi\|_{L_2(\Omega)} \tag{1.28}$$

via [T-L.1, p. 235]. But, in turn, via (1.12), condition (1.28) (RHS) is equivalent tot he following Continuous Observability Inequality

$$\|\varphi_0\|_{L_2(\Omega)} \leq c_T \int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1, \tag{1.29}$$

with $c_T > 0$ independent of φ_0, where φ solves the homogeneous problem

$$\begin{cases} i\varphi_t + \Delta\varphi \equiv 0 \quad \text{ in } Q; & \text{(1.30a)} \\ \varphi(0, \cdot\,) = \varphi_0 \in L_2(\Omega); & \text{(1.30b)} \\ \varphi|_{\Sigma_0} \equiv 0; \ \left.\dfrac{\partial\varphi}{\partial\nu}\right|_{\Sigma_1} \equiv 0, & \text{(1.30c)} \end{cases}$$

which is dual to the y-problem (1.1) save for the inessential detail that, in (1.30b), the I.C. is taken at $t = 0$, rather than $t = T$.

Step 2. The validity of the COI (1.29) for the φ-problem (1.30) is now readily seen as a direct application of estimate (1.19) in Theorem 1.3, modulo an interior lower-order term in $H^{-1}(Q)$, which is then absorbed, as usual, e.g., [L-T.1], by a compactness/uniqueness argument [such argument for the homogeneous φ-problem (1.30) is simpler than the corresponding argument for the dissipative v-problem (1.1) to obtain estimate (1.22)]. Theorem 1.4 is proved. $\square$

Remark 1.3. The above result, Theorem 1.4, complements [L-T.5, Sect. 8.2], which shows—at least for the Schrödinger's equation on a half-space, in dimension ≥ 2, with Neumann $L_2(\Sigma_T)$-boundary control—that the *range* of the input-solution operator L_T, applied to all of $L_2(\Sigma_T)$, is *not* contained in $H^\epsilon(\Omega)$, $\forall\, \epsilon > 0$.

2 The Nonlinear Boundary Dissipation Case. Assumptions and Main Results

Nonlinear boundary dissipation model. The core of the present paper begins with the present section. Here Ω is an open bounded domain of $\mathbb{R}^n$, $n \geq 1$, as in Section 1, with sufficiently smooth boundary $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. Prompted by the dissipative linear v-problem in (1.1a–c) reported in Section 1, we now consider the corresponding problem with nonlinear boundary dissipation:

$$\begin{cases} iw_t + \Delta w = 0 & \text{in } Q = (0,T] \times \Omega; \quad (2.1a) \\ w(0,\,\cdot\,) = w_0 & \text{in } \Omega; \quad (2.1b) \\ w|_{\Gamma_0} \equiv 0, \ \dfrac{\partial w}{\partial \nu} = ig(w) & \text{in } \Sigma_k = (0,T] \times \Gamma_k, \ k = 0,1. \quad (2.1c) \end{cases}$$

We have already noted in Remark 1.1 that we could take $\Gamma_0 = \emptyset$, and, moreover, we could take $\frac{\partial w}{\partial \nu}|_{\Gamma_0} \equiv 0$ instead of $w|_{\Gamma_0} \equiv 0$ in (2.1c), in which case the condition $\Gamma_0 \cap \Gamma_1 = \emptyset$ is dispensed with. However, stronger geometrical conditions, noted in Remark 1.1, are then called for.

Assumptions on the nonlinearity g. First, for purposes of the well-posedness result, Theorem 2.1 below, we impose on the complex-valued function g the following assumptions:

(H.1): g is a continuous complex, single-valued function of the complex variable $z \in \mathbb{C}$, $g(z)$, with $g(0) = 0$; moreover, $g(z)$ is the sub-differential, $g(z) = \partial j(z)$, of a lower semicontinuous, convex, proper function $j : \mathbb{C} \to \overline{\mathbb{R}} =]-\infty, +\infty]$, so that it satisfies

$$\operatorname{Im}\{g(z)\bar{z}\} \equiv 0; \quad \operatorname{Re}\{(g(z) - g(v))(\bar{z} - \bar{v})\} \geq 0, \quad \forall\, z, v \in \mathbb{C}. \tag{2.2a}$$

Thus, in particular, for $v = 0$ we obtain $\operatorname{Re}\{g(z)\bar{z}\} \geq 0$, so that

$$\operatorname{Re}\{g(z)\bar{z}\} = g(z)\bar{z} = |g(z)\bar{z}| \geq 0, \quad \forall\, z \in \mathbb{C}. \tag{2.2b}$$

Property (2.2a) is the present counterpart of the monotonicity assumption in the case g is real-valued function of a real variable. We may view $g : \mathbb{C} \to \mathbb{C}$ with the range space $\mathbb{C}$ being an inner-product space under the inner product $(z, v)_{\mathbb{C}} = \operatorname{Re}\{z\bar{v}\}$, $\forall z, v \in \mathbb{C}$. See more precisely Lemma 2.3 below. In the linear case of Section 1, we have $g(z) = z$. We remark that if the functions $g_j(z)$ satisfy (H.1), so does $\sum_{j=1}^{J} d_j g_j(z)$ for $d_j \geq 0$.

Typical examples for $g(z)$ are: $g(z) = |z|^r z$, $r > 0$; or $g(z) = \frac{1}{|z|^r} z$, $0 < r < 1$. See Lemma 2.3; Examples 3.1 and 3.5; Appendix A.

Second, for purposes of the main result, the uniform stabilization Theorem 2.2 below, we require—besides (H.1)—additional growth conditions:

(H.2): There exist positive constants $m > 0$, $M > 0$, such that

(a)

$$m|z|^2 \leq g(z)\bar{z} \text{ (recall (2.2b)), for } |z| \geq 1, \quad \forall\, z \in \mathbb{C}; \tag{2.3a}$$

(b)

$$\begin{cases} |g(z)| \leq M|z|^p, \text{ for } |z| \geq 1, \quad \forall\, z \in \mathbb{C}; \\ \text{where: } p = 5 \quad \text{for } n = \dim \Omega = 2, \\ \qquad\quad\;\; p = 3 \quad \text{for } n = \dim \Omega = 3. \end{cases} \tag{2.3b}$$

We remark that *no growth assumptions on $g(z)$ are made near the origin*, though the decay asserted by Theorem 2.2 does depend on such behavior. Moreover, $g(\,\cdot\,)$ is allowed to be superlinear at infinity, unlike the corresponding case of the wave equation [La-Ta.1]. Finally, in contrast with most of the literature on uniform stabilization of nonlinear dynamics, g need not be differentiable.

Classes of functions satisfying assumptions (H.1), (H.2) are discussed below in Lemma 2.3; Examples 3.1–3.5; Appendix A.

Geometrical assumption. Finally, for purposes of the uniform stabilization Theorem 2.2, we need a geometrical condition imposed on the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\Gamma_0 \neq \emptyset$, the same as in Theorems 1.2 and 1.4: there exists a real coercive vector field $\ell(x) \in (C^2(\overline{\Omega}))^n$, such that

(H.3))

$$\ell(x) \cdot \nu(x) \leq 0 \text{ on } \Gamma_0. \tag{2.4}$$

Main results. Well-posedness and regularity.

Theorem 2.1. *Let $n = 1, 2, \ldots$. Assume hypothesis (H.1) = (2.2) on g. Then, the following results hold true for the w-problem (2.1):*

(a) (well-posedness) For any initial condition $w_0 \in L_2(\Omega)$, problem (2.1) defines a unique (non-linear contraction semigroup) mild solution $w(t; w_0)$ [B.2, p. 202, 204, 230] satisfying

$$w_0 \in L_2(\Omega) \Rightarrow w(\,\cdot\,; w_0) \in C([0, \infty); L_2(\Omega)) \tag{2.5}$$

continuously. The generator A_F of the corresponding nonlinear semigroup is given explicitly in (4.2) below: it is maximal dissipative, hence closed; moreover, $\overline{\mathcal{D}(A_F)} = L_2(\Omega)$.

(b) (regularity) Let, in particular, $w_0 \in H^2(\Omega)$ subject to compatibility conditions:

$$w_0 \in H^2(\Omega) : w_0|_{\Gamma_0} = 0; \ \left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_1} = ig(w_0), \quad \textit{so that } w_0 \in \mathcal{D}(A_F), \tag{2.6a}$$

the domain of the generator A_F defined in (4.2) below. Then, the corresponding unique solution $w(t; w_0)$ guaranteed by part (a), satisfies

$$\begin{cases} w(\,\cdot\,; w_0) \in C([0, \infty); \mathcal{D}(A_F)), \quad \mathcal{D}(A_F) \subset \mathcal{D}(A^{\frac{1}{2}}) \equiv H^1_{\Gamma_0}(\Omega), \\ w_t \in C([0, \infty); L_2(\Omega)); \\ \\ w(\,\cdot\,; w_0)|_{\Gamma_1} \in C([0, \infty); H^{\frac{1}{2}}(\Gamma_1)). \end{cases} \tag{2.6b}$$

(c) (higher regularity) Assume (2.6a) on w_0 and, moreover,

(c_1) if dim $\Omega = 2$, assume that $g(z)$ is of polynomial growth:

$$|g(z)| \leq C|z|^k, \quad z \in \mathbb{C}, \text{ for some positive integer } k, \tag{2.7a}$$

or, more generally, that $g : H^{\frac{1}{2}}(\Gamma) \to L_2(\Gamma)$, in particular, $g : L_p(\Gamma) \to L_2(\Gamma)$, $p \geq 1$;

(c_2) if dim $\Omega = 3$, assume that

$$|g(z)| \leq C|z|^r, \quad z \in \mathbb{C}, \text{ for } r < 3. \tag{2.7b}$$

Then, in both cases (c_1) and (c_2), we have that

$$\mathcal{D}(A_F) \subset H^{\frac{3}{2}}(\Omega), \text{ so that } w(\,\cdot\,; w_0) \in C([0,\infty); H^{\frac{3}{2}}(\Omega)), \tag{2.8a}$$

for w_0 as in (2.6a). In particular (from (2.6b) and (2.7a–b)):

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Gamma_1} = ig(w(\,\cdot\,; w_0)|_{\Gamma_1}) \in L_2(0,T; L_2(\Gamma_1)). \tag{2.8b}$$

Additionally, higher regularity requires differentiability of $g(\,\cdot\,)$. We shall not include this result, as Theorem 2.1 is sufficient to justify the subsequent calculations: see statement preceding Eqn. (2.18a).

Of course, Theorem 2.1 is a generalization of the linear well-posedness Theorem 1.1(i) to which it reduces for $g(z) = z$. Theorem 2.1 follows from monotone operator theory. Its proof is given in Section 4. It is proved in Section 4 and Appendix C.

The main focus of the present paper is an asymptotic energy decay rates of solutions, as $t \to \infty$. To this end, assumptions (H.2) on g and (H.3) on geometrical conditions are invoked.

Uniform stabilization (or uniform decay rates). Orientation. Before stating our main result, Theorem 2.2, we need to introduce some concepts and relative notation. In our proof, to be given in Section 5, we shall need to establish some *a-priori* estimates, after which we will be able to fall into the general setting for "hyperbolic-like" linear PDE-dynamics (wave, Schrödinger, plates, shells equations) with nonlinear (interior and) boundary dissipation, first introduced in [La-Ta.1] in the specific case of wave equations with nonlinear boundary dissipation in the Neumann-boundary conditions. This approach is, however, fully general. It has been used in several other settings, including the following cases: von Karman plates [H-L.1]; full von Karman model [L.2]; Maxwell equations [E-L-N.1]; equations of shells (coupled system of two hyperbolic PDEs defined on a 2-dimensional surface) [L-T.4]; wave equations with interior localized dissipation [La-To.1] and [Bu.1]. It will be invoked and applied also in the present setting of the Schrödinger problem (2.1). We need to recall such strategy.

Step (i): The concave function $h(x)$. Following [La-Ta.1], we let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a real-valued continuous, concave, strictly increasing function, with $h(0) = 0$, satisfying

$$h(g(z)\bar{z}) \geq |z|^2 + |g(z)|^2, \text{ for } |z| \leq \delta, \text{ for some } \delta > 0, \ z \in \mathbb{C} \tag{2.9}$$

(recall that $g(z)\bar{z} \geq 0$ by (2.2b)). Such function $h(s)$, $s \geq 0$, can always be constructed on the strength of assumptions (H.1), (H.2). Thus, (2.9) is a property, not an assumption.

Having constructed the required function $h(\cdot)$, we rescale it by letting

$$\tilde{h}(x) = h\left(\frac{x}{\text{meas}(\Sigma_1)}\right), \quad x \geq 0, \tag{2.10}$$

where $\Sigma_1 = (0, T] \times \Gamma_1$, with $T > 0$ arbitrary, and "meas" is the cylinder Q's surface measure. Of course, $\tilde{h}(0) = 0$ and $\tilde{h}(x)$ is strictly increasing, so that the operator $CI + \tilde{h}$ is invertible for any constant $C \geq 0$, where I is the identity operator.

Step (ii): The convex function $p(x)$. For any positive constant C, whereby the concave function $(C + \tilde{h})$ is invertible, as noted above and for any positive constant K, we define the convex function $p(x)$ by setting

$$p(x) = (CI + \tilde{h})^{-1}(Kx) : \text{positive for } x > 0,\ p(0) = 0, \text{ continuous, strictly increasing.} \tag{2.11}$$

Thus, the function $I + p(x)$ is invertible. [The present function $p(x)$ should not be confused with the parameter p in (2.3b).]

Step (iii): The function $q(x)$. Finally, we define the function $q(x)$ by

$$q(x) = x - (I+p)^{-1}(x) = p(I+p)^{-1}(x) = (I+p)^{-1}p(x) : \text{positive for } x > 0,\ q(0) = 0, \text{ continuous, strictly increasing.} \tag{2.12}$$

[Differentiating $q(x) + p(q(x)) = p(x)$ yields $q'(x) = p'(x)/[1 + p'(q(x))] > 0$, since $p(x)$ is strictly increasing. Also, $[I+p]q(0) = p(0) = 0 \Rightarrow q(0) = 0$.] We note that the above procedure—step (i) through step (iii) is both *constructive* and *explicit*, given the data of the problem: the nonlinear function g satisfying assumption (H.1); the part Γ_1 of the boundary, and the constant $T > 0$. See Section 3 below.

We can now state our main uniform decay rate result for problem (2.1) to be proved in Section 5.

Theorem 2.2. *Let $n = \dim \Omega = 2, 3$. With reference to the w-problem (2.1), we assume hypotheses (H.1) = (2.2), (H.2) = (2.3) for g and (H.3) = (2.4) for $\{\Omega, \Gamma_0, \Gamma_1\}$. Then, the energy*

$$E(t) \equiv \|w(t, w_0)\|^2_{L_2(\Omega)}, \quad w_0 \in L_2(\Omega) \tag{2.13}$$

of the solution w of problem (2.1), guaranteed by Theorem 2.1, satisfies the following decay rate

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right)(E(0)) \searrow 0 \quad \text{for all } t \geq T_0,\ t \to \infty, \tag{2.14}$$

for some $T_0 > 0$, where the scalar function $S(t)$ (nonlinear contraction) is the solution of the following nonlinear ODE:

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0) \equiv \|w_0\|^2_{L_2(\Omega)}, \tag{2.15}$$

where the function q is defined by (2.12), via (2.11), where the positive constants C and K there are defined by

$$C = \frac{\frac{1}{m} + 1 + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}}{\text{meas}(\Sigma_1)}; \qquad K = \frac{1}{2C_T \ \text{meas}(\Sigma_1)}. \tag{2.16}$$

where $p = 5$ for dim $\Omega = 2$, $p = 3$ for dim $\Omega = 3$, see (2.3b); m is defined in (2.3a); $\tilde{C}_p$ is given explicitly in Proposition 5.2.1; and C_T is the constant in estimate (5.1.3). Thus, from (2.14) it follows (via Appendix B) that

$$E(t) \to 0 \text{ as } t \to \infty, \text{ with rates specified by } S(t). \tag{2.17}$$

Theorem 2.2 generalizes the linear Theorem 1.2. We remark that the function q (like p) depends on the constants C and K. In the stabilization result of Theorem 2.2 above, C depends, in turn, on the data as well as on $E(0)$, the initial energy. Thus, the decay provided by Theorem 2.2 is uniform with respect to all initial conditions within a same ball of $L_2(\Omega)$, centered at the origin.

Several illustrations on the application of Theorem 2.2, for various functions g and corresponding rates are given in Section 3.

Preliminary dissipation energy identity. Once the well-posedness result Theorem 2.1(b) has been established—in particular (2.6a) for w_t, $\nabla w \in L_2(\Omega)$ a.e., and $\frac{\partial w}{\partial \nu}|_{\Gamma_1} \in L_2(\Gamma_1)$ a.e. for strong solutions—the following standard energy method is justified. We multiply Eqn. (2.1a), rewritten equivalently as $w_t = i\Delta w$ in Q, by $\bar{w}$, take real parts of the resulting identity and integrate by parts in time and space over $\int_s^t \int_\Omega d\Omega\, d\tau$. We use $\frac{\partial}{\partial t}|w|^2 = 2 \ \text{Re}\{w_t \bar{w}\}$ in the integration in t on the left side, and Green first theorem in the space integration on the right side with boundary conditions given by (2.1c). In the process, we get the cancellation $\text{Re}(i|\nabla w|^2) = 0$, and finally obtain the identity

$$\int_\Omega |w(t)|^2 d\Omega + 2 \int_s^t \int_{\Gamma_1} \text{Re}\{g(w)\bar{w}\} d\Gamma_1 d\tau = \int_\Omega |w(s)|^2 d\Omega, \ t \geq s \geq 0. \tag{2.18a}$$

Invoking property (2.2b) on g, $\text{Re}\{g(w)\bar{w}\} = g(w)\bar{w} \geq 0$, as well as the energy $E(\,\cdot\,)$ in (2.13), we can rewrite identity (2.18a) as

$$E(t) + 2\int_s^t \int_{\Gamma_1} g(w)\bar{w}\, d\Gamma_1 d\tau = E(s), \text{ where then } E(t) \le E(s),\ t \ge s \ge 0, \tag{2.18b}$$

so that the energy is monotonically decreasing, as desired.

Identity (2.18b) is the present nonlinear counterpart of the energy identity (1.3) for the energy $E_v(\,\cdot\,)$ in (1.2) of the linear v-problem (1.1a–c).

Remark 2.1. The energy method leading to identity (2.18a) gives $\text{Re}\{g(w)\bar{w}\}$ in the boundary integral. On the other hand, the boundary integral in the critical estimate (1.19), when applied with $z = w$, the solution of problem (2.1), gives the integrand $|g(w)|\,|w|$: see (5.1.1) below. It is the need to match these two integrands: $\text{Re}\{g(w)\bar{w}\} = |g(w)\bar{w}|$, that forces (part of) assumption (H.1) = (2.2a-b). Indeed, as a result, it is precisely in the form (2.18b) that we shall need and invoke the dissipativity energy identity (2.18a). See e.g., (5.2.29) via (5.2.1).

We close this section by giving an *ad-hoc* construction of classes of functions g satisfying (H.1), (H.2). A general setting for assumption (H.1) is given in Appendix A.

A first direct construction of continuous functions $g(z)$ satisfying (2.2a) near the origin.

Lemma 2.3. *(a) With $s_0 > 0$, let*

$\gamma : [0, s_0] \to \mathbb{R}_+$ be a continuous function of a real variable, $\gamma(0) = \lim_{s\downarrow 0} \gamma(s) \ge 0$, $\gamma(s) > 0$ for $s > 0$, such that $s \to s\gamma(s)$ is monotone increasing, (2.19)

the case $\gamma(0) = +\infty$ being included. Define the continuous function $g(z)$ by

$$g(z) = \gamma(|z|)z, \quad z \in \mathbb{C} \text{ and assume that } g(0) = 0, \tag{2.20}$$

so that $g(s)$ is increasing on $[0, s_0]$. Then $g(z)$ in (2.20) satisfies assumption (2.2a).

(b) As a partial converse, let $g(z)$ be a continuous function of $z \in \mathbb{C}$. Then, with reference to (2.2a) (LHS), we have

$$Im\{g(z)\bar{z}\} \equiv 0, \ \forall \ z \in \mathbb{C} \quad \Rightarrow g(z) = f(x,y)z, \tag{2.21}$$

where $f(x,y)$ is a continuous, real-valued function of the real variables $x = Re\ z$, $y = Im\ z$, $z = x + iy$. Moreover, with reference to (2.2b) we have

$$Re\{g(z)\bar{z}\} \geq 0, \ \forall \ z \in \mathbb{C} \quad \Rightarrow f(x,y) \geq 0, \ \forall \ x,y \in \mathbb{R}. \tag{2.22}$$

Proof of Lemma 2.3. (a) Property (2.2b) is obvious: $g(z)\bar{z} = \gamma(|z|)|z|^2 \geq 0$, $z \in \mathbb{C}$, while $g(0) = 0$.

As to property (2.2a), we compute from (2.2b), for $z, v \in \mathbb{C}$, via (2.17), (2.18):

$$\begin{aligned}
\mathrm{Re}\{(g(z) - g(v))(\bar{z} - \bar{v})\} &= \mathrm{Re}\{[\gamma(|z|)z - \gamma(|v|)v](\bar{z} - \bar{v})\} \\
&= \gamma(|z|)|z|^2 + \gamma(|v|)|v|^2 - \gamma(|z|)\ \mathrm{Re}\{z\bar{v}\} - \gamma(|v|)\ \mathrm{Re}\{v\bar{z}\} \\
&= \gamma(|z|)|z|^2 + \gamma(|v|)|v|^2 - [\gamma(|z|) + \gamma(|v|)]\ \mathrm{Re}\{z\bar{v}\} \qquad (2.23) \\
&\geq \gamma(|z|)|z|^2 + \gamma(|v|)|v|^2 - [\gamma(|z|) + \gamma(|v|)]|z|\,|v| \\
&= \gamma(|z|)|z|^2 + \gamma(|v|)|v|^2 - \gamma(|z|)|z|\,|v| - \gamma(|v|)|z|\,|v| \\
&= \gamma(|z|)|z|[|z| - |v|] + \gamma(|v|)|v|[|v| - |z|] \\
&= (\gamma(|z|)|z| - \gamma(|v|)|v|)(|z| - |v|) \\
&= (\gamma(r)r - \gamma(\rho)\rho)(r - \rho) = (g(r) - g(\rho))(r - \rho) \geq 0, \quad \forall\ r, \rho > 0,
\end{aligned} \tag{2.24}$$

where we have set $|z| = r$, $|v| = \rho$, and where non-negativity follows since the function $r \to r\gamma(r)$ is increasing. Thus, (2.24) proves (2.2a), as desired.

(b) Let $z = x + iy$, $x, y \in \mathbb{R}$ and write $g(z) = u(x,y) + i\mu(x,y)$, $u, \mu \in \mathbb{R}$. We have $g(z)\bar{z} = (ux + \mu y) + i(x\mu - uy)$. The condition $\mathrm{Im}\{g(z)\bar{z}\} \equiv 0$ in (2.21) (LHS) implies: $x\mu = uy$, or $u = f(x,y)x$, $\mu = f(x,y)y$, for some continuous, real-valued function $f(x,y)$. Hence, $g(z)$ must be of the form $g(z) = f(x,y)(x+iy) = f(x,y)z$, and property (2.21) (RHS) is established. Property (2.22) is then obvious. $\square$

The direct proof of (2.2a) in Lemma 2.3(a) is then integrated in the general abstract setting given in Appendix A, point (5), to obtain further that $g(z) = \partial_j(z)$, as required by the full assumption (H.1).

A canonical distinctive class of functions $\gamma(s)$ covered by Lemma 2.3 is given by

$$\gamma(s) = s^r, \text{ hence } g(z) = |z|^r z, \text{ for all real } r > 0, \tag{2.25}$$

since $s\gamma(s) = s^{r+1}$ is monotone increasing, $s > 0$. This class is also noted in [Li.1, p. 133].

The *canonical* cases to distinguish are three:

Case 1 (fast decay to zero: superlinear):

$$\gamma(s) = s;\ g(s) = s^2,\, 0 \le s \le s_0;\ g(z) = |z|z,\, 0 \le |z| \le s_0. \tag{2.26}$$

Case 2 (linear case):

$$\gamma(s) \equiv 1,\, g(s) = s,\, 0 \le s \le s_0;\ g(z) = z, \quad 0 < |z| \le s_0. \tag{2.27}$$

Case 3 (slow decay to zero: sublinear):

$$\gamma(s) = \frac{1}{\sqrt{s}},\, g(s) = \sqrt{s},\, 0 < s \le s_0;\ g(z) = \frac{1}{\sqrt{|z|}}\, z,\, 0 < |z| \le s_0. \tag{2.28}$$

The two corollaries in Section 3 below deal with general classes of which Case 1 and Case 3 are, respectively, canonical representatives. Further illustrations are given in Section 3.

Case 1 and Case 3 correspond to slow energy decay rates (polynomial type or worse). Case 2 corresponds to exponential energy decay rate. See Examples 3.1 and 3.5 at the end of Section 3.

Literature on boundary feedback stabilization of Schrödinger equations. Linear case. The problem of uniform stabilization of Schrödinger equations defined on an open bounded domain $\Omega \subset \mathbb{R}^n$ by means of a boundary feedback (dissipation) was taken up in the early 90's. It focused on *linear* dissipative boundary feedbacks exercised in either the Dirichlet or Neumann B.C.

Neumann B.C. In [M.1], the uniform stabilization of the v-problem (1.1a–b) is considered under the boundary dissipation $v|_{\Sigma_0} \equiv 0$, $\frac{\partial v}{\partial \nu}|_{\Sigma_1} = -((x - x_0) \cdot \nu) v_t$ in Σ_k, $k = 0, 1$. Uniform (exponential) decay is established in the $H^1(\Omega)$-norm. Mathematically, this is the most amenable case, with mathematical technology inspired by prior successes with energy methods for the wave equation [L-L-T.1], [L-T.1], etc. This case appears to have two main mathematical drawbacks: (i) the boundary dissipation on v_t ($\sim \Delta v$) is too strong topologically (and not physically appealing); (ii) with dissipation $\frac{\partial v}{\partial \nu}|_{\Sigma_1} \in L_2(0, \infty; L_2(\Gamma_1))$ as is usually the case in dissipative problems, the topology $H^1(\Omega)$ where decay occurs is likewise two strong (compare with our Theorem 1.4, whereby an $L_2(\Sigma_1)$-boundary control u in (1.1c) for the y-problem guarantees the solution $y(T)$ to be (at least) in $L_2(\Omega)$, one unit less smooth). In addition, [M.1] has to impose the (usual) requirement of non-overlapping portions Γ_0 and Γ_1, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, as well as the star-shaped requirement (2.4) on Γ_0. The nonlinear version is in [C-M-S.1].

Dirichlet B.C. In contrast, the independent and contemporaneous reference [L-T.2] considers (optimal regularity, exact controllability, as well as) uniform stabilization of the y-problem and the v-problem (1.1a–b), this time under Dirichlet boundary control. For the v-problem, the boundary dissipation is now $v|_{\Sigma_0} \equiv 0$ $v|_{\Sigma_1} \equiv i \, \frac{\partial (A_D^{-1} v)}{\partial \nu}|_{\Sigma_1}$, where $A_D = -\Delta$, $\mathcal{D}(A_D) = H^2(\Omega) \cap H_0^1(\Omega)$. Uniform decay occurs in the space $H^{-1}(\Omega)$, which is the topologically correct space: it is the space of optimal regularity and exact controllability of a $L_2(\Sigma_1)$-Dirichlet boundary control such as the one above. This result is more challenging as the natural energy level for the Schrödinger equation is $H^1(\Omega)$, while here decay is required and asserted in $H^{-1}(\Omega)$. It thus requires an operator-theoretic shift of topologies, from $H^1(\Omega)$ down to $H^{-1}(\Omega)$ (of the type needed in the prior work on dissipative wave equation [L-T.1] which provided the inspiration). Though within the mathematical optimal setting, this result also suffers from two drawbacks: (i) the dissipation operator is non-local; (ii) the mathematically optimal space $H^{-1}(\Omega)$, where decay occurs, may not be physically appealing. In this Dirichlet boundary feedback case, the two portions of the boundary, Γ_0 and Γ_1, may touch $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$, and Γ_0 is (as usual) star-shaped.

The Neumann case revisited. The two aforementioned *linear* results [M.1], [L-T.2] characterized the boundary feedback stabilization of Schrödinger equations for approximately 15 years. Between them, they created a gap of results and, accordingly, they raised the consequent question: is it possible to obtain boundary feedback stabilization of the Schrödinger equation in the mathematically and physically more desirable $L_2(\Omega)$-topology (hence, by necessity, in the Neumann case) by using a mathematically more appealing and physically more implementable boundary feedback? The answer was given in [L-T-Z.2, Sect. 11] in the *linear* case reproduced in Theorem 1.2, of which the present paper is a generalization to the case of nonlinear boundary feedback.

Nonlinear case. Contributions of the present paper. In short, the main novel features of our paper include the following points:

Well-posedness. (i) A new abstract well-posedness result—Theorem C.1.1 in Appendix C, specialized next to the case of the Schrödinger equation with nonlinear monotone interior and boundary damping in Theorem C.2.1 and, further, in Theorem 2.1. Such result requires a special proof within the theory of maximal monotone operators, which is definitely more challenging than in the corresponding case of the wave equation (see Remark C.1.1), as to necessitate the approximation argument of Secion C.1.

Uniform stabilization. (ii) sharp (optimal) energy decay rates, under a nonlinear, attractive boundary dissipation in the Neumann B.C., are obtained in the desirable $L_2(\Omega)$-norm in Theorem 2.2 with five illustrative computed examples.

(iii) The non-dissipative (homogeneous or unobserved) part Γ_0 of the boundary is allowed also to be of Neumann-type (case where the Lopatinski Condition is not satisfied), as noted in Remark 1.1: in this case, the price to pay is a stronger geometrical condition (after [L-T-Z.1, Appx. A] in the wave equation case, e.g., Γ_0 convex or concave); at any rate, $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ need not be disjoint.

(iv) Superlinear growth of the boundary dissipation is allowed at infinity: up to polynomial growth of order 5, for dim $\Omega = 2$; and of order 3, for dim $\Omega = 3$. By contrast, in the case of the wave equation, [La-Ta.1] allowed only linear growth at infinity for the boundary monotone damping. Superlinear growth in the Schrödinger's case is the result of Carleman's estimates penalizing normal derivatives on the boundary in negative anisotropic norms (Theorem 1.3).

(v) No growth assumption on the (monotone) nonlinearity is required at the origin, which therefore may be arbitrary. However, the decay rates that correspondingly are obtained via a constructive algorithm (a refinement of [La-Ta.1]) are entirely determined by the behavior of the (monotone) nonlinearity at the origin (see the numerous examples in Section 3). In particular, in contrast with most of the literature on uniform stabilization of nonlinear dynamics, no differentiability of the dissipation is assumed.

Regarding point (iii), we remark that a recent analysis of the wave equation with nonlinear localized interior damping and source terms and no growth restrictions at infinity is carried out in [La-To.1]. In both the linear and nonlinear cases, the underlying supporting pillar for obtaining decay rates in the $L_2(\Omega)$-topology is a Carleman-type energy estimate at the $L_2(\Omega)$-level (Theorem 1.3). It is obtained by use of a pseudo-differential shift of topology, and related micro-local analysis [L-T-Z.2, Sect. 10] (see also [L-T-Z.2]) to go from the natural $H^1(\Omega)$-energy level to the desired $L_2(\Omega)$-energy level.

We also refer to [Ta.1] in the Dirichlet case.

3 Corollaries and Illustrations: Computation of Optimal Decay Rates

In this section, in order to illustrate Theorem 2.2, we provide two general corollaries. They refer to the 'end cases' where the function $g(s)$, in the real positive variable s, has either a *fast decay* as in Case 1, (2.26), (Corollary 3.1) or else a *slow decay* as in Case 3, (2.28), (Corollary 3.2) *to zero* near $s = 0$, where we recall that $g(0) = 0$. Next, we present a few significant illustrations of these two corollaries. In each of them, we compute explicitly the rates of decay through the function q in (2.15)) which are, in fact, optimal. These are obtained through the explicit sequential algorithmic procedure that was introduced in [La-Ta.1] and is reproduced here—as adapted to the present class of complex-valued boundary terms $g(z)$ in (2.20). It consists in the successive construction of the following array of functions: $g \to h \to p \to q$, starting from the given boundary term $g(z) = \gamma(|z|)z$, as in (2.20).

Corollary 3.1. *(fast decay to zero of $g(s)$ as $s \downarrow 0$) Assume (H.1) = (2.2), (H.2) = (2.3), (H.3) = (2.4).*

(a) Let $\gamma(s)$ be the function: $[0, s_0] \to \mathbb{R}_+$ defined by (2.19), and let then $g(s) = s\gamma(s)$ monotone increasing, $g(0) = 0$, $g(z) = \gamma(|z|)z$ as in (2.20). Assume further that

(a_1) $0 \le \gamma(0) < 1$;

(a_2) the function $s\gamma(\sqrt{s}) = \sqrt{s}\, g(\sqrt{s})$ is convex near $s = 0$, say for $0 < s < s_0^2$, for some s_0.

Then, the procedure described in Step (i), Eqn. (2.9), to construct the required continuous, concave, strictly increasing function $h(x) : \mathbb{R}_+ \to \mathbb{R}_+$, with $h(0) = 0$, can be arranged as to yield the following choice

$$h^{-1}(x) = \frac{x}{2}\gamma\left(\sqrt{\frac{x}{2}}\right) = \sqrt{\frac{x}{2}}\, g\left(\sqrt{\frac{x}{2}}\right), \quad \text{near } x = 0, \ \text{say, } 0 \le x \le x_0 = 2s_0^2. \tag{3.1}$$

(b) Let $C > 0$ be the constant in the definition of the sought-after continuous, strictly increasing function $p(x)$, $p(0) = 0$, of Step (ii) in (2.11). For illustration purposes, assume that meas$(\Sigma_1) = 1$, so that $\tilde{h} \equiv h$, see (2.10). Assume further that

$$\gamma(s) \text{ is } C^1 \text{ for } s \downarrow 0, \text{ and that } \quad \frac{1}{4}[\gamma(0) + g'(0)] < \frac{1}{C}, \tag{3.2}$$

where $g'(0)$ must then be $g'(0) \geq 0$. Then, we can arrange to take the function $p(x)$ as to satisfy

$$\begin{aligned} h^{-1}\left(\frac{x}{2}\right) &\leq p(x) \leq h^{-1}(x) = \frac{x}{2}\,\gamma\left(\sqrt{\frac{x}{2}}\right) \\ &= \sqrt{\frac{x}{2}}\,g\left(\sqrt{\frac{x}{2}}\right) < \frac{x}{2}, \ 0 < x \leq x_1, \ x_1 \text{ small.} \end{aligned} \tag{3.3}$$

Moreover, under assumptions (a), (b), we conclude that we can always arrange to have the function $q(x)$ in (2.12) satisfy for x near $x = 0$:

$$\begin{aligned} \tilde{q}(x) \equiv \frac{2}{3}\,h^{-1}\left(\frac{x}{2}\right) \leq \frac{2}{3}\,p(x) \leq q(x) \leq p(x) \leq h^{-1}(x) < \frac{x}{2}; \\ 0 \leq x \leq x_2; \end{aligned} \tag{3.4a}$$

$$h^{-1}\left(\frac{x}{2}\right) = \frac{x}{4}\,\gamma\left(\sqrt{\frac{x}{4}}\right) = \sqrt{\frac{x}{4}}\,g\left(\sqrt{\frac{x}{4}}\right), \ 0 < x \leq x_3 = \text{small.} \tag{3.4b}$$

The function $\tilde{q}(x)$ defined in (3.4) satisfies: $\tilde{q}(0) = 0$, $\tilde{q}(x) > 0$ for $0 < x \leq x_3$, and $\tilde{q}(x)$ is strictly increasing and convex. Consider the new ODE

$$\tilde{S}_t(t) + \tilde{q}(\tilde{S}(t)) \equiv 0, \tag{3.5a}$$

or

$$\begin{aligned} &\tilde{S}_t(t) + \frac{2}{3}\,\frac{\tilde{S}(t)}{4}\,\gamma\left(\sqrt{\frac{\tilde{S}(t)}{4}}\right) \\ &\quad = \tilde{S}_t(t) + \frac{2}{3}\sqrt{\frac{\tilde{S}(t)}{4}}\,g\left(\sqrt{\frac{\tilde{S}(t)}{4}}\right) = 0, \tilde{S}(T) = S(T), \end{aligned} \tag{3.5b}$$

where T is sufficiently large, so that the solution $S(t)$ of the ODE (2.15) evaluated at $t = T$ satisfies $S(T) < x_3$ (this is possible by (2.14), (B.3)). Then, the solutions of the corresponding w-problem (2.1) with such $g(z) = \gamma(|z|)z$, for $|z|$ small [and otherwise subject to (H.1) = (2.2) and (H.2) = (2.3)] satisfy the energy decay rate

$$E(t) \leq C(E(0))S(t) \leq C(E(0))\tilde{S}(t), \; t > T \text{ and } \tilde{S}(t) \searrow 0 \text{ as } t \to \infty, \quad (3.6)$$

((B.4)) where $\tilde{S}(t)$ is obtained from integrating

$$\int_{\sqrt{\frac{\tilde{S}(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g(u)} = \frac{2}{3}(T - t), \; u = \sqrt{\frac{\tilde{S}}{4}}. \quad (3.7)$$

Proof. **(a) Construction of** $h(x)$ (see (2.9)). By the assumptions on γ, we have that $0 \leq \gamma(s) < 1$ for $0 \leq s \leq s_0$ for some s_0. Let $|z| \leq s_0$. Then, the dominant term d.t.,

$$\text{d.t. of } \left\{|z|^2 + |g(z)|^2 = [1 + \gamma^2(|z|)]|z|^2\right\} = |z|^2, \; 0 \leq |z| \leq s_0. \quad (3.8)$$

Thus, forcing (2.9) to be satisfied as $h(g(z)\bar{z}) = 2|z|^2 \geq \{|z|^2 + |g(z)|^2\}$, for small $|z|$, we may define the sought-after function $h(\,\cdot\,)$, via (3.8) as

$$h(g(z)\bar{z}) \equiv 2|z|^2; \; \text{ or } \; h(\gamma(|z|)|z|^2) \equiv 2|z|^2, \quad 0 \leq |z| \leq s_0. \quad (3.9)$$

It suffices to restrict to $z = s =$ real positive, so that (3.9) yields for the sought-after, strictly increasing continuous function $h(\,\cdot\,)$:

$$h(\gamma(s)s^2) = 2s^2; \text{ or } \gamma(s)s^2 = h^{-1}(2s^2);$$

or

$$\sqrt{\frac{x}{2}}\, g\left(\sqrt{\frac{x}{2}}\right) = \frac{x}{2}\,\gamma\left(\sqrt{\frac{x}{2}}\right) = h^{-1}(x) < \frac{x}{2}, \quad 0 \leq x \leq x_0 = 2s_0^2, \quad (3.10)$$

setting $x = 2s^2$, since $\gamma(s) < 1$ for $0 \leq s \leq s_0$. By assumption (a_2), the function $h^{-1}(\,\cdot\,)$ in (3.10) is convex near $x = 0$; by (a_1), $h^{-1}(0) = 0$, and so $h(t)$ is then concave near $t = 0$, and $h(0) = 0$, as required. Thus, $h(t)$ is well-defined as to satisfy the properties of Step (i), Eqn. (2.9). Thus, (3.1) is proved.

(b) Construction of $p(x)$**.** We use (2.11), with $C > 0$ given, $\tilde{h} = h$ as obtained in part (a) (since meas$(\Sigma_1) = 1$), and $K = 1$. We define the convex function $p(x)$ by

$$p(x) = (CI + h)^{-1}(x), \text{ or } \quad Cp(x) + h(p(x)) = x, \text{ near } x = 0, \tag{3.11}$$

so that we have $p(0) = 0$; $p(x) > 0$, $x > 0$, $p(x)$ strictly increasing, continuous, as required. We need to show (3.3). We always have via (3.11):

$$h(p(x)) \leq Cp(x) + h(p(x)) \equiv x, \text{ or } p(x) \leq h^{-1}(x), \text{ near } x = 0. \tag{3.12}$$

Moreover, starting from $h(g(s)s) = 2s^2$, see (3.10), with $g(s) = \gamma(s)s$, one shows that, under assumption (3.2) on γ, we have $h'(sg(s))[g(s) + sg'(s)] = 4s$, or

$$h'(0) = h'(sg(s))|_{s=0} = 4\left[\frac{1}{g'(s) + g(s)/s}\right]_{s=0} = \frac{4}{g'(0) + \gamma(0)} > C, \tag{3.13}$$

since $g'(0) \geq 0$, $\gamma(0) \geq 0$ in (3.2). Thus, for the dominant term d.t., we have

$$\begin{aligned} &\text{d.t. of } \{Cs + h(s)\} = h(s), \text{ or } Cs + h(s) \leq 2h(s) \\ &\qquad\qquad \text{for } s \text{ near } 0, \text{ say } 0 \leq s \leq s_1, \end{aligned} \tag{3.14}$$

for some s_1; and hence, by (3.11), (3.12), (3.14), that

$$\begin{aligned} &x = Cp(x) + h(p(x)) \leq 2h(p(x)), \text{ or } h^{-1}\left(\frac{x}{2}\right) \leq p(x) \leq h^{-1}(x), \\ &\qquad\qquad \text{near } x = 0, \quad \text{say, } 0 \leq x \leq x_1, \end{aligned} \tag{3.15}$$

for some $x_1 > 0$ small, as desired, and (3.3) is proved.

Construction of $q(x)$ **and** $\tilde{q}(x)$**.** From $q(x) \equiv (I + p)^{-1}p(x)$ in (2.12), since $p(x) \geq 0$, we have via (3.12), for x near $x = 0$:

$$q(x) \leq q(x) + p(q(x)) = p(x) \leq h^{-1}(x) < \frac{x}{2}, \ 0 \leq x \leq x_0, \tag{3.16}$$

hence, by (3.12) and (3.16),

$$\begin{aligned} p(x) &= q(x) + p(q(x)) \leq q(x) + h^{-1}(q(x)) \\ &\leq q(x) + \frac{q(x)}{2} = \frac{3}{2}\, q(x), \quad x \text{ near } x = 0. \end{aligned} \tag{3.17}$$

Moreover, differentiating (3.11) and using $p(0) = 0$ yields

$$Cp'(0) + h'(p(0))p'(0) = 1,$$

or

$$0 \leq p'(0) = \frac{1}{C + h'(0)} = \frac{g'(0) + \gamma(0)}{C(g'(0) + \gamma(0)) + 4} < \frac{1}{2C}, \tag{3.18}$$

recalling (3.13). [We may well have $p'(0) = 0$, see examples below.] At any rate, conclusion (3.4) follows via (3.17). The LHS of (3.4a) defines the function $\tilde{q}(x) \leq q(x)$ near $x = 0$, which is used to define the ODE problem (3.5). We then invoke Proposition B.1 in Appendix B on the comparison between Eqn. (2.15) on $S(t)$ and the (computable) Eqn. (3.5) on $\tilde{S}(t)$ to deduce that $S(t) \leq \tilde{S}(t)$, $\forall\ t \geq T$, with $\tilde{S}(t) \searrow 0$ as $t \to \infty$. Then conclusion (2.14) of Theorem 2.2 yields (3.6), as desired. Finally, (3.7) is obtained by separation of variables on (3.5). Corollary 3.1 is fully proved. $\square$

Corollary 3.2. *(slow decay to zero of $g(s)$ as $s \downarrow 0$) Assume (H.1) = (2.2), (H.2) = (2.3), (H.3) = (2.4).*

(a) Let $\gamma(s)$ be the function: $(0, s_0] \to \mathbb{R}_+$ defined by (2.19), and let then $g(s) = s\gamma(s)$ monotone increasing, $g(0) = 0$, $g(z) = \gamma(|z|)z$ as in (2.20). Assume further that

(a_1) $\gamma(0) > 1$; thus, $\lim_{s\searrow 0} \frac{s}{g(s)} = \lim_{s\searrow 0} \frac{1}{\gamma(s)} = \frac{1}{\gamma(0)} \leq 1$; where the case $\gamma(0) = \lim_{s\downarrow 0} \gamma(s) = +\infty$ is included (and is typical of Case 3, (2.28)).

(a_2) the function $g(s)$, in the real positive variable s, is (not only increasing, as contained in (2.2a) via Lemma 2.3 for z and v restricted to real positive variables, but also) strictly increasing near $s = 0$, with inverse $g^{-1}(\,\cdot\,)$. Moreover, the function $\sqrt{s}\, g^{-1}(\sqrt{s})$ is convex near $s = 0$.

Then, the procedure described in Step (i), Eqn. (2.9), to construct the required continuous, concave, strictly increasing function $h(x):\mathbb{R}_+\to\mathbb{R}_+$, with $h(0)=0$, can be arranged as to yield the following choice

$$h^{-1}(x)=\sqrt{\frac{x}{2}}\,g^{-1}\left((\sqrt{\frac{x}{2}}\right)<\frac{x}{2},\ \textit{near } x=0,\ \textit{say, } 0\le x\le x_0 \textit{ small.} \tag{3.19}$$

(b) Let $C>0$ be the constant in the definition of the sought-after continuous, strictly increasing function $p(x)$, $p(0)=0$, of Step (ii) in (2.11). For illustration purposes, assume that meas$(\Sigma_1)=1$, so that $\tilde{h}=h$, see (2.10). Assume further that

$$\gamma(s) \textit{ is } C^1 \textit{ for } s\downarrow 0 \textit{ and } \left[\frac{1}{g'(0)}+\frac{1}{\gamma(0)}\right]^{-1}>\frac{C}{4} \tag{3.20}$$

[where the cases $g'(0)=\gamma(0)=+\infty$ are included and indeed typical of Case 2, (2.28)]. Then, we can arrange to take the function $p(x)$ as to satisfy

$$h^{-1}\left(\frac{x}{2}\right)\le p(x)\le h^{-1}(x)=\sqrt{\frac{x}{2}}\,g^{-1}\left(\sqrt{\frac{x}{2}}\right)<\frac{x}{2},$$
$$\textit{near } x=0,\ \textit{say } 0<x\le x_1,\ x_1 \textit{ small.} \tag{3.21}$$

Moreover, under assumptions (a), (b), we conclude that we can always arrange to have the function $q(x)$ in (2.12) satisfy

$$\begin{aligned}\tilde{q}(x)\ &\equiv\ \frac{2}{3}h^{-1}\left(\frac{x}{2}\right)\le\frac{2}{3}p(x)\le q(x)\le p(x)\le h^{-1}(x)\\ &=\ \sqrt{\frac{x}{2}}\,g^{-1}\left(\sqrt{\frac{x}{2}}\right)<\frac{x}{2},\quad 0\le x\le x_3=\textit{small},\end{aligned} \tag{3.22a}$$

$$h^{-1}\left(\frac{x}{2}\right)=\sqrt{\frac{x}{4}}\ g^{-1}\left(\sqrt{\frac{x}{4}}\right),\quad x \textit{ small}, \tag{3.22b}$$

so that the ODE (3.5a): $\tilde{S}_t(t)+\tilde{q}(\tilde{S}(t))\equiv 0$ now takes the form via (3.2)

$$\tilde{S}_t(t)+\frac{2}{3}\sqrt{\frac{\tilde{S}(t)}{4}}\,g^{-1}\left(\sqrt{\frac{\tilde{S}(t)}{4}}\right)=0,\quad \tilde{S}(T)=S(T), \tag{3.23}$$

where T is sufficiently large, so that the solution $S(t)$ of the ODE (2.15) evaluated at $t=T$ satisfies $S(T)<x_3$ (this is possible by (A.3) or (2.14)). Thus,

the solutions of the corresponding w*-problem (2.1) with such* $g(z) = \gamma(|z|)z$ *for* $|z|$ *small [and otherwise subject to (H.1) = (2.2) and (H.2) = (2.3)], satisfy the energy decay rate*

$$E(t) \le C(E(0))S(t) \le C(E(0))\tilde{S}(t), \quad \forall\, t \ge T; \text{ and } \tilde{S}(t) \searrow 0 \text{ as } t \to \infty, \tag{3.24}$$

((A.4)) where $\tilde{S}(t)$ *is obtained from integrating*

$$\int_{\tilde{S}(T)}^{\tilde{S}(t)} \frac{d\tilde{S}}{\sqrt{\frac{\tilde{S}}{4}}\, g^{-1}\left(\sqrt{\frac{\tilde{S}}{4}}\right)} = 8 \int_{\sqrt{\frac{\tilde{S}(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g^{-1}(u)} = \frac{2}{3}(T - t); \quad u = \sqrt{\frac{\tilde{S}}{4}}. \tag{3.25}$$

Proof. **(a) Construction of** $h(x)$ (see (2.9)). Since, now, $\gamma(0) > 1$, we have $\gamma(s) > 1$ for $s > 0$ small, hence $g(s) = s\gamma(s) > s$ and $s > g^{-1}(s)$ for s near $s = 0$. Thus we now choose $h(\,\cdot\,)$ as to satisfy

$$h(g(z)\bar{z}) = 2|g(z)|^2 \ge |z|^2 + |g(z)|^2; \tag{3.26a}$$

since

$$\text{d.t. } \{|z|^2 + |g(z)|^2 = [1 + \gamma^2(|z|)|z|^2]\} = \gamma^2(|z|)|z|^2 = |g(z)|^2, \tag{3.26b}$$

in place of (3.8), (3.9). Thus, restricting to a real positive variable s, we have

$$h(g(s)s) = 2g^2(s) \text{ or } \; g(s)s = h^{-1}(2g^2(s)), \text{ or}$$

$$h^{-1}(x) = \sqrt{\frac{x}{2}}\, g^{-1}\left(\sqrt{\frac{x}{2}}\right) < \frac{x}{2}, \text{ for } x \text{ small}, \tag{3.27}$$

for the sought-after, strictly increasing function $h(\,\cdot\,)$. In the penultimate step of (3.27), we have set: $2g^2(s) = x$, hence $g(s) = \sqrt{\frac{x}{2}}$, hence $s = g^{-1}\left(\sqrt{\frac{x}{2}}\right)$, where $g^{-1}(\,\cdot\,)$ is the inverse of the strictly increasing function $g(\,\cdot\,)$; hence $g(s)s = \sqrt{\frac{x}{2}}\, g^{-1}\left(\sqrt{\frac{x}{2}}\right)$, as desired, where we have seen above that $g^{-1}(s) < s$, for s small. By assumption (a_2), the function $h^{-1}(\,\cdot\,)$ in (3.24) is convex in x near $x = 0$, and $h^{-1}(0) = 0$, since $g^{-1}(0) = 0$ or $g(0) = 0$, as assumed; and so $h(t)$ is then concave near $t = 0$, and $h(0) = 0$, as required. Thus, $h(t)$ is well-defined as to satisfy the properties of Step (i), Eqn. (2.9). Thus, (3.19) is proved via (3.27).

(b) Construction of $p(x)$. The argument leading from (3.11) to (3.12): $p(x) \leq h^{-1}(x)$, near $x = 0$, still holds unchanged. Next, we still seek to impose $h'(0) > C$. To this end, we must now differentiate $h(g(s)s) = 2g^2 s$, see (3.27) to get $h'(g(s)s)[g(s) + sg'(s)] = 4g(s)g'(s)$, and upon dividing

$$h'(0) = \lim_{s \to 0} h'(g(s)s) = 4 \lim_{s \to 0} \left[\frac{1}{g'(s)} + \frac{s}{g(s)} \right]^{-1} = 4 \left[\frac{1}{g'(0)} + \frac{1}{\gamma(0)} \right]^{-1} > C, \tag{3.28}$$

invoking assumption (3.20). Then, the same argument as given in (3.14), (3.15) holds true and leads to (3.21).

Construction of $\tilde{q}(x)$. Inequalities (3.16) and (3.17) still hold, the first also in view of (3.19). Hence (3.22) follows likewise via also (3.15):

$$\frac{2}{3} h^{-1}\left(\frac{x}{2}\right) \leq \frac{2}{3} p(x) \leq q(x) \leq p(x) \leq h^{-1}(x) < \frac{x}{2}, \quad x \text{ small.} \tag{3.29}$$

The rest of the proof is the same as for Corollary 3.1: the LHS of (3.22) defines the function $\tilde{q}(x) \leq q(x)$ near $x = 0$, whereby Proposition B.1 of Appendix B completes the proof along with Theorem 2.2. Corollary 3.2 is likewise established. $\square$

We now provide a few examples.

Example 3.1. In this illustration, we take the superlinear case for $g(s)$ as in (2.25):

$$\begin{aligned} &\gamma(s) = s^r \text{ or } g(s) = s^{r+1}, r > 0; \text{ near the origin,} \\ &\qquad 0 \leq s \leq s_0; \ g(z) = \gamma(|z|)z = |z|^r z. \end{aligned} \tag{3.30}$$

We shall first apply Corollary 3.1 and next analyze this case directly and explicitly, along the lines of the proof of Corollary 3.1.

Application of Corollary 3.1. We have

$$\gamma(0) = 0; \ s\gamma(\sqrt{s}) = s^{\frac{r+2}{2}} \text{ convex near } s = 0, \ s > 0; \ g'(0) = 0, \tag{3.31}$$

and thus assumptions (a_1), (a_2), (b) = (3.2) of Corollary 3.1 are fulfilled for *any* constant $C > 0$. By (3.13) and (3.31), we have $h'(0) = \infty$, and by (3.18) we have $p'(0) = 0$. Recalling the definition of $\tilde{q}(x)$ in (3.4a)

$$\tilde{q}(x) = \frac{2}{3}\, h^{-1}\left(\frac{x}{2}\right) = \frac{2}{3}\sqrt{\frac{x}{4}}\, g\left(\sqrt{\frac{x}{4}}\right) = \frac{2}{3}\left(\sqrt{\frac{x}{4}}\right)^{r+1}, \text{ near } x = 0, \tag{3.32}$$

via (3.30). Then we need to integrate Eqn. (3.7) with $\tilde{S}(T) = S(T)$ to get via (3.30),

$$\begin{aligned} \frac{2}{3}\,(T-t) &= \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g(u)} = \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} u^{-(r+1)} du \\ &= \frac{1}{r}\left[\left(\frac{S(T)}{4}\right)^{-\frac{r}{2}} - \left(\frac{\tilde{S}(t)}{4}\right)^{-\frac{r}{2}}\right]; \end{aligned} \tag{3.33}$$

$$\left(\frac{\tilde{S}(t)}{4}\right)^{-\frac{r}{2}} = \left(\frac{S(T)}{4}\right)^{-\frac{r}{2}} + \frac{2}{3}\, r(t-T), \text{ or}$$

$$\frac{\tilde{S}(t)}{4} = \left[\frac{1}{\left(\frac{S(T)}{4}\right)^{-\frac{r}{2}} + \frac{2}{3}\, r(t-T)}\right]^{\frac{2}{r}}, \quad \forall\, t \geq T. \tag{3.34}$$

By (3.6) of Corollary 3.1, since $S(0) = E(0) > S(T) = \tilde{S}(T)$, as $S(t) \searrow 0$ by (2.14) or (B.3), we conclude via (3.34) that: with $g(z) = \gamma(|z|)z = |z|^r z,\ r > 0$ by (3.30), for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (2.1) satisfy the following energy decay rates

$$E(t) \leq C(E(0)) \left[\left(\frac{E(0)}{4}\right)^{-\frac{r}{2}} + \frac{2}{3}\, r(t-T)\right]^{-\frac{2}{r}}, \ t \geq T. \tag{3.35}$$

Direct analysis. Construction of function $h(s)$. The analysis in the proof of Corollary 3.1, once specialized to the present case (3.30) yields that $h(\cdot)$ is chosen as to satisfy, by $\gamma(0) = 0$:

$$h(g(z)\bar{z}) = h(\gamma(|z|)|z|^2) = h(|z|^{r+2}) = 2|z|^2, \tag{3.36}$$

since d.t. $\{|z|^2 + |g(z)|^2\} = |z|^2, \quad |z|$ small. This is a specialization of (3.9). Thus, setting via (3.30) and (2.20),

$$x = g(z)\bar{z} = \gamma(|z|)|z|^2 = |z|^{r+2}, \quad \text{or } |z| = x^{\frac{1}{r+2}}, \tag{3.37}$$

we obtain from (3.36), (3.37),

$$\begin{aligned} \tilde{h}(x) \equiv h(x) = 2x^{\frac{2}{r+2}} \equiv s, \quad 0 \leq x \leq x_0 = \text{small;} \\ h^{-1}(s) = \left(\frac{s}{2}\right)^{\frac{1}{m}}, \ m = \frac{2}{r+2}, \end{aligned} \tag{3.38}$$

taking, for illustrative purposes, $\text{meas}(\Sigma_1) = 1$, so that $\tilde{h} = h$.

Construction of $p(x)$. This is given by (2.11). Taking, again the constant $K = 1$, we have that $p(x)$ is defined by

$$\begin{aligned} p(x) = (CI + h)^{-1}(x); \text{ or } Cp(x) + h(p(x)) = x, \text{ or} \\ Cp(x) + 2p^m(x) = x, \ m = \frac{2}{r+2}, \end{aligned} \tag{3.39}$$

invoking (3.38). The function x^m with $m < 1$ has infinite (positive) slope at the origin. Thus, for any constant $C > 0$, we have that x^m dominates Cx near $x = 0$. Then, for $x > 0$ small, since $p(0) = 0$ and $m < 1$, the dominant term d.t. of $\{Cp(x) + 2p^m(x)\} = 2p^m(x)$. Thus, by (3.38), (3.39), we have

$$\begin{cases} 4p^m(x) \geq Cp(x) + 2p^m(x) = x \geq 2p^m(x), \text{ or} \\ h^{-1}\left(\frac{x}{2}\right) = \left(\frac{x}{4}\right)^{\frac{1}{m}} \leq p(x) \leq \left(\frac{x}{2}\right)^{\frac{1}{m}} = \left(\frac{x}{2}\right)^{\frac{r+2}{2}} = h^{-1}(x), \\ \qquad 0 < x \leq x_0 = \text{small.} \end{cases} \tag{3.40}$$

Construction of $\tilde{q}(x)$. By (2.12), we define $q(x)$ by $q(x) = (I + p)^{-1}p(x)$, $x > 0$. For $x > 0$ small and $p(x)$ small (where $p(0) = 0$), and $q(x)$ small (where $q(0) = 0$), we have $\left(\frac{q(x)}{2}\right)^{\frac{1}{m}} < \frac{q(x)}{2}$ since $m < 1$. Thus, in view of this and

(3.40), we obtain

$$\begin{cases} q(x) \leq p(x) = q(x) + p(q(x)) \leq q(x) + \left(\dfrac{q(x)}{2}\right)^{\frac{1}{m}} \leq q(x) + \frac{q(x)}{2} = \frac{3q(x)}{2}, \\ \text{or} \\ \tilde{q}(x) \equiv \dfrac{2}{3}\, h^{-1}\left(\dfrac{x}{2}\right) = \dfrac{2}{3}\left(\dfrac{x}{4}\right)^{\frac{1}{m}} \leq 2\dfrac{p(x)}{3} \leq q(x) \leq p(x) \leq h^{-1}(x) = \left(\dfrac{x}{2}\right)^{\frac{1}{m}}, \end{cases} \tag{3.41}$$

thus re-obtaining the function $\tilde{q}(x)$ in (3.32).

Example 3.2. In this illustration, we take

$$\gamma(s) = s^2 e^{-\frac{1}{s^2}}, \text{ or } g(s) = s^3 e^{-\frac{1}{s^2}} \text{ near } s = 0,\ 0 \leq s \leq s_0;$$

$$g(z) = \gamma(|z|)z = |z|^2 e^{-\frac{1}{|z|^2}} z. \tag{3.42}$$

Application of Corollary 3.1. We have

$$\gamma(0) = 0;\ s\gamma(\sqrt{s}) = s^2 e^{-\frac{1}{s}} \text{ convex near } s = 0,\ g'(0) = 0, \tag{3.43}$$

and thus assumptions (a_1), (a_2), (b) = (3.2) of Corollary 3.1 are fulfilled, again with *any* constant $C > 0$. By (3.13) and (3.43), we have $h'(0) = \infty$ and by (3.18), we have $p'(0) = 0$. The definition of $\tilde{q}(x)$ in (3.4) near $x = 0$ is, via (3.42):

$$\tilde{q}(x) = \frac{2}{3}\, h^{-1}\left(\frac{x}{2}\right) = \frac{2}{3}\sqrt{\frac{x}{4}}\, g\left(\sqrt{\frac{x}{4}}\right) = \frac{2}{3}\left(\frac{x}{4}\right)^2 \exp\left(-\frac{1}{x/4}\right). \tag{3.44}$$

Then, we need to integrate Eqn. (3.7) with $\tilde{S}(T) = S(T)$ to get

$$\begin{aligned} \frac{2(T-t)}{3} &= \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g(u)} = \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{e^{\frac{1}{u^2}}}{u^3} du \\ &= -\frac{1}{2}\int_{\frac{4}{S(T)}}^{\frac{4}{\tilde{S}(t)}} e^{\tau} d\tau = \frac{1}{2}\left(e^{\frac{4}{S(T)}} - e^{\frac{4}{\tilde{S}(t)}}\right) \end{aligned} \tag{3.45}$$

[setting $\tau = u^{-2}, -2u^{-3}du = d\tau$]. We obtain

$$e^{\frac{4}{\tilde{S}(t)}} = e^{\frac{4}{S(T)}} + \frac{4}{3}(t-T), \text{ or } \tilde{S}(t) = \frac{4}{\ln\left(e^{\frac{4}{S(T)}} + \frac{4}{3}(t-T)\right)},\ \forall\, t \geq T. \tag{3.46}$$

By (3.6) of Corollary 3.1, we conclude that, with $g(z) = \gamma(|z|)z = |z|^2 e^{-\frac{1}{|z|^2}} z$, for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (2.1) satisfy the following energy decay rates

$$\begin{aligned} E(t) &\leq C(E(0))S(t) \leq C(E(0))\tilde{S}(t) \\ &\leq C(E(0)) \left[\ln \left(e^{\frac{4}{E(0)}} + \frac{4}{3}(t-T) \right) \right]^{-1}, \; t \geq T. \end{aligned} \tag{3.47}$$

since $S(0) = E(0) > S(T) = \tilde{S}(T)$, as $S(t) \searrow 0$ by (2.14) or (B.3).

Example 3.3. In this illustration, we take

$$\begin{aligned} &\gamma(s) = se^{-\frac{1}{s}}, \text{ or } g(s) = s^2 e^{-\frac{1}{s}} \text{ near } s = 0, \;\; 0 \leq s \leq s_0; \\ &\qquad g(z) = \gamma(|z|)z = |z|e^{-\frac{1}{|z|}} z. \end{aligned} \tag{3.48}$$

Application of Corollary 3.1. We have

$$\gamma(0) = 0; \;\; s\gamma(\sqrt{s}) = s^{\frac{3}{2}} e^{-\frac{1}{\sqrt{s}}} \text{ convex near } s = 0, \;\; g'(0) = 0, \tag{3.49}$$

and thus assumptions (a_1), (a_2), (b) = (3.2) of Corollary 3.1 are fulfilled, again with *any* constant $C > 0$. By (3.13) and (3.49), we have $h'(0) = \infty$, and hence by (3.18) we have $p'(0) = 0$. The definitions of $\tilde{q}(x)$ in (3.4) near $x = 0$ is, via (3.48):

$$\tilde{q}(x) = \frac{2}{3} h^{-1}\left(\frac{x}{2}\right) = \frac{2}{3}\sqrt{\frac{x}{2}}\, g\left(\sqrt{\frac{x}{2}}\right) = \frac{2}{3}\left(\frac{x}{2}\right)^{\frac{3}{2}} \exp\left(-\frac{1}{\sqrt{\frac{x}{2}}}\right). \tag{3.50}$$

Then we need to integrate Eqn. (3.7) with $\tilde{S}(T) = S(T)$ to get

$$\begin{aligned} \frac{2(T-t)}{3} &= \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g(u)} = \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{e^{\frac{1}{u}}}{u^2} du \\ &= -\int_{\sqrt{\frac{4}{S(T)}}}^{\sqrt{\frac{4}{\tilde{S}(t)}}} e^{\tau} d\tau = e^{\frac{2}{\sqrt{S(T)}}} - e^{\frac{2}{\sqrt{\tilde{S}(t)}}}, \quad t \geq T \end{aligned} \tag{3.51}$$

[setting $\tau = u^{-1}, d\tau = -u^{-2}du$]. We obtain from (3.51)

$$e^{\frac{2}{\sqrt{\tilde{S}(t)}}} = e^{\sqrt{\frac{4}{S(T)}}} + \frac{2(t-T)}{3},$$

or

$$\tilde{S}(t) = \frac{4}{\ln^2\left[e^{\sqrt{\frac{4}{S(T)}}} + \frac{2(t-T)}{3}\right]} \leq \frac{4}{\ln^2\left[e^{\sqrt{\frac{4}{E(0)}}} + \frac{2(t-T)}{3}\right]}, \ t \geq T, \qquad (3.52)$$

since $S(0) = E(0) > S(T) = \tilde{S}(T)$, as $S(t) \searrow 0$ by (B.3) or (2.14).

By (3.6) of Corollary 3.1, we conclude that: with $g(z) = \gamma(|z|)z = z|z|e^{-\frac{1}{|z|}}$, for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (2.1) satisfy the following energy decay rates

$$\begin{aligned} E(t) &\leq C(E(0))S(t) \leq C(E(0)\tilde{S}(t) \\ &= C(E(0))\left\{\ln^2\left[e^{\sqrt{\frac{4}{E(0)}}} + \frac{2(t-T)}{3}\right]\right\}^{-1}, t \geq T. \end{aligned} \qquad (3.53)$$

Example 3.4. In this illustration, we take the *linear* case near the origin:

$$\gamma(s) \equiv 1, \text{ or } g(s) = s \text{ near } s = 0, \text{ say } 0 \leq s \leq s_0; \ g(z) = z; \ x = g^{-1}(x). \qquad (3.54)$$

In this case, all the relevant quantities in the constructive algorithm: $h(\cdot)$, $p(\cdot)$, $q(\cdot)$ are directly computable.

Computation of $h(\cdot)$. By (2.9) we define $h(\cdot)$ by imposing for $|z|$ small, or s small

$$h(g(z)\bar{z}) = |z|^2 + |g(z)|^2 = 2|z|^2; \ h(g(s)s) = h(s^2) = 2s^2;$$

$$h(y) = 2y, h^{-1}(x) = \frac{x}{2}, \qquad (3.55)$$

near $x = 0$.

Computation of $p(x)$. By (2.11) with $\tilde{h} = h$ and $K = 1$, we have via (3.55):

$$Cp(x) + h(p(x)) = x, \text{ or } Cp(x) + 2p(x) = x, \text{ or } p(x) = \frac{x}{2+C} \text{ near } x = 0. \qquad (3.56)$$

Computation of $q(x)$. By (2.12), we have by (3.56),

$$q(x) + p(q(x)) = p(x), \text{ or } q(x) + \frac{q(x)}{2+C} = \frac{x}{2+C},$$

$$\text{or } q(x) = \frac{x}{3+C} \text{ near } x = 0. \tag{3.57}$$

ODE (2.15). With $q(\cdot)$ given by (3.57), the ODE (2.15) is

$$S_t(t) + \frac{1}{3+C} S(t) \equiv 0, \; S(0) = E(0), \text{ or } S(t) = E(0)e^{-\frac{t}{3+C}}, \; t > 0. \tag{3.58}$$

We apply Theorem 2.2, Eqn. (2.14) and we conclude that: with $g(z) = \gamma(|z|)z = z$, for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (2.1) satisfy the following energy decay rates

$$E(t) \le C(E(0))S(t) = C(E(0))e^{-\frac{t}{3+C}}, \; t > 0. \tag{3.59}$$

Example 3.5. In this illustration we take near the origin:

$$\gamma(s) = \frac{1}{s^r} \text{ or } g(s) = s^{1-r}, \; 0 < r < 1, \; 0 < s \le s_0;$$

$$g(z) = \frac{1}{|z|^r} z, \; |z| \text{ small}, \; g^{-1}(y) = y^{\frac{1}{1-r}}. \tag{3.60}$$

Application of Corollary 3.2. By (3.60), (3.28), (3.18), we have

$$\gamma(0) = \infty; \; g'(0) = \infty; \; h'(0) = \infty; \; p'(0) = \frac{1}{C + h'(0)} = 0. \tag{3.61}$$

Thus, the definition of $\tilde{q}(x)$ in (3.22) is, by (3.60), near $x = 0$:

$$\tilde{q}(x) = \frac{2}{3} h^{-1}\left(\frac{x}{2}\right) = \frac{2}{3}\sqrt{\frac{x}{4}}\, g^{-1}\left(\sqrt{\frac{x}{4}}\right) = \frac{2}{3}\left(\frac{x}{4}\right)^m,$$

$$1 < m = \frac{1}{2}\left(1 + \frac{1}{1-r}\right) < \infty. \tag{3.62}$$

Then, we need to integrate Eqn. (3.23),

$$\tilde{S}_t(t) + \tilde{q}(\tilde{S}(t)) \equiv 0 \text{ or } \tilde{S}_t(t) + \frac{2}{3}\left(\frac{\tilde{S}(t)}{4}\right)^m = 0, \; \tilde{S}(T) = S(T), \tag{3.63}$$

for T sufficiently large. We obtain by separation ((3.25))

$$\begin{aligned}\frac{2}{3}(t-T) &= -4\int_{S(T)}^{\tilde{S}(t)}\left(\frac{\tilde{S}}{4}\right)^{-m} d\left(\frac{\tilde{S}}{4}\right) \\ &= \frac{4}{m-1}\left[\left(\frac{\tilde{S}(t)}{4}\right)^{1-m} - \left(\frac{S(T)}{4}\right)^{1-m}\right]; \qquad (3.64)\end{aligned}$$

$$\begin{aligned}\tilde{S}(t) &= \frac{4}{\left[\left(\frac{4}{S(T)}\right)^{m-1} + \frac{m-1}{6}(t-T)\right]^{\frac{1}{m-1}}}, \quad \forall\, t \geq T \qquad (3.65) \\ &\leq \frac{4}{\left[\left(\frac{4}{E(0)}\right)^{m-1} + \frac{m-1}{6}(t-T)\right]^{\frac{1}{m-1}}}, \quad \forall\, t \geq T, \qquad (3.66)\end{aligned}$$

since $S(0) = E(0) > S(T) = \tilde{S}(T)$ as $S(t) \searrow 0$ as $t \to \infty$ by (2.14) or (B.3). By (3.24) of Corollary 3.2 we conclude that: with $g(z) = \frac{1}{|z|^r}\, z,\ 0 < r < 1$, $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (2.1) satisfy the following energy decay rates

$$\begin{aligned}E(t) &\leq C(E(0))S(t) \leq C(E(0))\tilde{S}(t) \\ &\leq C(E(0))\left[\left(\frac{4}{E(0)}\right)^{m-1} + \frac{m-1}{4}(t-T)\right]^{1-m}, \ \forall\, t \geq T. \qquad (3.67)\end{aligned}$$

4 Proof of Well-Posedness Theorem 2.1

Preliminaries. Throughout the paper, all functions spaces $L_2(\Omega)$, $L_2(\Gamma)$, $\ldots$, are complex valued. We return to the abstract model (1.7) for the open-loop, mixed y-problem in (1.1a–c) (LHS). Accordingly, in the present nonlinear model (2.1), the abstract model of the dissipative nonlinear w-problem (2.1a–c) is given by

$$w_t = A_F w = -iAw - ANg(N^*Aw) \in [\mathcal{D}(A)]' \qquad (4.1)$$

(the subscript F stands for 'feedback') to be interpreted as follows:

$$\begin{aligned} A_F : L_2(\Omega) \supset \mathcal{D}(A_F) \to L_2(\Omega);\ \mathcal{D}(A_F) & \\ = \quad \{x \in L_2(\Omega) : A[-ix - Ng(N^*Ax)] \in L_2(\Omega)\}\,. & \qquad (4.2a) \end{aligned}$$

$$\begin{cases} x \in \mathcal{D}(A_F) \iff x \in H^2(\Omega) : \quad -ix - Ng(N^*Ax) \in \mathcal{D}(A) \\ \qquad\qquad\qquad\qquad\qquad \iff -i\left.\dfrac{\partial x}{\partial \nu}\right|_{\Gamma_1} = g(x|_{\Gamma_1}) \text{ on } \Gamma_1 \text{ and } \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x = 0 \text{ on } \Gamma_0, \end{cases} \qquad (4.2b)$$

recalling (1.8). For $g(z) \equiv z$, we re-obtain the linear case (1.7), (1.13): $A_F \equiv L_F$. Since $A^{\frac{1}{2}}N \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$ *a-fortiori* from (1.11), we can also write

$$A_F = A^{\frac{1}{2}}[-iA^{\frac{1}{2}}(\,\cdot\,) - A^{\frac{1}{2}}Ng(N^*A^{\frac{1}{2}}A^{\frac{1}{2}}(\,\cdot\,))] : \mathcal{D}(A^{\frac{1}{2}}) \to [\mathcal{D}(A^{\frac{1}{2}})]'. \qquad (4.3)$$

Claim. Under the assumption in (2.2a) on g: $\operatorname{Im}\{g(z)\bar{z}\} \equiv 0$, we have (as in the linear case (1.17)) via (1.12):

$$\begin{cases} \mathcal{D}(A_F) \subset \mathcal{D}(A^{\frac{1}{2}}) \equiv H^1_{\Gamma_0}(\Omega) \equiv \{x \in H^1(\Omega) : x|_{\Gamma_0} = 0\} & (4.4a) \\ \text{so that } x \in \mathcal{D}(A_F) \to x \in H^1_{\Gamma_0}(\Omega) \to x|_\Gamma = N^*Ax \in H^{\frac{1}{2}}(\Gamma). & (4.4b) \end{cases}$$

Moreover, A_F is injective: $\mathcal{D}(A_F) \to L_2(\Omega)$. Indeed, if $x \in \mathcal{D}(A_F)$, we obtain from (4.2):

$$\begin{aligned} A_F x &= A[-ix - Ng(N^*Ax)] = f \in L_2(\Omega) \Rightarrow (A_F x, x) \\ &= -i(Ax, x) - (g(N^*Ax), N^*Ax) = (f, x) \qquad (4.5) \end{aligned}$$

in the complex $L_2(\Omega)$-inner product $(\,\cdot\,,\,\cdot\,)$. Taking the imaginary part of the identity in (4.5), where A is positive self-adjoint, and recalling that

$$\operatorname{Im}[(g(N^*Ax), N^*Ax)] = 0$$

by assumption (2.2a), we obtain

$$\mathrm{Im}(A_F x, x) = (Ax, x) = -\mathrm{Im}(f, x) \Rightarrow$$

$$\|A^{\frac{1}{2}}x\|^2_{L_2(\Omega)} \le \|A^{-\frac{1}{2}}f\|_{L_2(\Omega)}\|A^{\frac{1}{2}}x\|_{L_2(\Omega)} \le \frac{\epsilon}{2}\|A^{\frac{1}{2}}x\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon}\|A^{-\frac{1}{2}}f\|^2_{L_2(\Omega)}, \tag{4.6}$$

and $\|A^{\frac{1}{2}}x\|_{L_2(\Omega)} \le C\|f\|_{L_2(\Omega)}$, as desired. Thus, (4.4a) is established. Then, trace theory and (1.12) yields (4.4b). The same argument shows that $f = 0 \Rightarrow x = 0$ in (4.5) and A_F is injective.

Proof of Theorem 2.1(a). The main goal of the present section is to establish that the operator A_F in (4.2) is maximal dissipative [B.1, p. 71], [B.2, p. 99], [Sh.1, p. 155], on $L_2(\Omega)$; that is, $(-A_F)$ is maximal monotone on $L_2(\Omega)$. Then, the well-posedness result of Theorem 2.1a–b follows from the theory of m-monotone or m-dissipative operators [B.1, p. 33; p. 71], [B.2, p. 35], [Sh.1]. This task is accomplished by the following result:

Proposition 4.1. *Assume hypothesis (H.1) = (2.2) on g. Then, the operator A_F resp. $(-A_F)$) in (4.2) is m-dissipative (resp. m-monotone) on $L_2(\Omega)$. By Claim (4.4), $\mathcal{D}(A_F) \subset \mathcal{D}(A^{\frac{1}{2}})$.*

Proof of Proposition 4.1. Step 1: Dissipativity of A_F: $\mathcal{D}(A_F) \to L_2(\Omega)$; or else $V \to V'$, $V = \mathcal{D}(A^{\frac{1}{2}})$. Let $x, y \in \mathcal{D}(A_F)$ so that $x, y \in \mathcal{D}(A^{\frac{1}{2}})$ by (4.4). Then, the following expressions are well-defined, via (4.2), (4.4a), and $A^{\frac{1}{2}}N \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$ by (1.11):

$$\begin{aligned}
&\mathrm{Re}\{(A_F x - A_F y, x - y)_{L_2(\Omega)}\} \\
&\quad = \mathrm{Re}\{-i(A^{\frac{1}{2}}(x-y), A^{\frac{1}{2}}(x-y))_{L_2(\Omega)}\} \\
&\quad\quad - \mathrm{Re}\{(g(N^*Ax) - g(N^*Ay), N^*Ax - N^*Ay)_{L_2(\Gamma)}\} && (4.7) \\
&\quad = -\mathrm{Re}\left\{\int_\Gamma (g(N^*Ax) - g(N^*Ay))(\overline{N^*Ax} - \overline{N^*Ay})d\Gamma\}\right\} \\
&\quad \le 0, \ x, y \in \mathcal{D}(A_F), && (4.8)
\end{aligned}$$

where negativity in (4.8) follows from assumption (H.1) = (2.2a) on g. The inequality in (4.8) says that A_F in (4.2) is a dissipative operator $\mathcal{D}(A_F) \to L_2(\Omega)$, or $(-A_F)$ is monotone $\mathcal{D}(A_F) \to L_2(\Omega)$, as desired. The same computations in (4.7), (4.8), this time with $x, y \in V = \mathcal{D}(A^{\frac{1}{2}})$ and duality pairing $V \to V'$ in place of the $L_2(\Omega)$-inner product, show that $(-A_F)$ is monotone $V \to V'$.

Step 2: Maximal property. To this end, (Minty's Theorem [B.1, Thm. 1.2, p. 39], [B.2, Thm. 1.2, p. 43], it suffices to prove surjectivity of the operator $[I - A_F]$ from $\mathcal{D}(A_F)$ onto $L_2(\Omega)$. Thus, let $F \in L_2(\Omega)$: by (4.1), (4.2), we seek to solve

$$(I - A_F)x = x + A[ix + Ng(N^*Ax)] = F \in L_2(\Omega), \tag{4.9}$$

for $x \in \mathcal{D}(A_F)$.

(a) To this end, we shall first show that the operator $[I - A_F]$ is *surjective from* $V \equiv \mathcal{D}(A^{\frac{1}{2}})$ onto $V' \equiv [\mathcal{D}(A^{\frac{1}{2}})]'$. Thus, let now $f \in V'$: by (4.1), we seek to solve

$$[I - A_F]x = x + iAx + ANg(N^*Ax) = f \in V', \tag{4.10}$$

for $x \in V = \mathcal{D}(A^{\frac{1}{2}})$. Technical details are given in Appendix C, as a specialization of an abstract result—Theorem C.1.1—which includes the present damped Schrödinger problem (2.1) as a special case. Key ingredients of the proof of (the abstract version of) (4.10) in Appendix C are: (a_1) the operator $ANg(N^*A): V \to V'$ is maximal monotone, as a consequence of Appendix A and of [L.1, Lem. 2.1, p. 345] $g = \partial\varphi$ and $AN\partial\varphi(N^*A\,\cdot\,) = \partial(\varphi N^*A)$, where φ and (φN^*A) are lower semi-continuous, convex, proper functions $L_2(\Gamma) \to \overline{R} =]-\infty, \infty]$: ($a_2$) the operator $[I - A_F] : V \to V'$ is maximal monotone, being the sum of the maximal monotone, linear, continuous operator $(I+iA) : V \to V'$, and the maximal monotone operator $ANg(N^*A\,\cdot\,) : V \to V'$, so that any of the results [B.1, Cor. 1.1, p. 39; Thm. 1.7, p. 46; Cor. 1.3, p. 48], [B.2, Cor. 1.1, p. 44; Thm. 1.5, p. 54] applies. A final step to prove (4.10) is an approximation argument using Minty's Theorem [B.1, Thm. 1.2, p. 39], [B.2, Thm. 2.2, p. 43]. Statement (4.10), this time with $f \in V'$ replaced by $F \in L_2(\Omega)$, readily then establishes (4.9), as desired.

Proposition 4.2. *(further regularity of $\mathcal{D}(A_F)$)) Assume $Im\{g(z)\bar{z}\} = 0$ as in hypothesis (H.1) = (2.2a) for g. Moreover,*

(a) if dim $\Omega = 2$, dim $\Gamma_1 = 1$, assume further that

$$g : L_p(\Gamma_1) \to L_2(\Gamma_1), \text{ for some } 1 \le p < \infty; \tag{4.11}$$

a sufficient condition thereof is that g has a polynomial bound as assumed in (2.7a).

(b) if dim $\Omega = 3$, dim $\Gamma_1 = 2$, assume further that

$$|g(z)| < C_r |z|^r, \text{ for all } r < 3, \ \forall\, z \in \mathbb{C} \tag{4.12}$$

[say, $r = 3 - \epsilon$, $\epsilon > 0$ arbitrary]. Then, with reference to (4.2), we have in both cases (a) and (b) that

$$\mathcal{D}(A_F) \subset H^{\frac{3}{2}}(\Omega), \text{ so that } x \in \mathcal{D}(A_F) \to x|_{\Gamma_1} \in H^1(\Gamma_1). \tag{4.13}$$

Proof. We already know from Claim (4.4) that, under $\mathrm{Im}\{g(z)\bar{z}\} \equiv 0$, for any dimension, we have: $x \in \mathcal{D}(A_F) \to x|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$.

Case (a): dim $\Gamma_1 = 1$**.** In this case, we have

$$H^{\frac{1}{2}}(\Gamma_1) \subset L_p(\Gamma_1), \ \forall\, 1 \le p < \infty, \tag{4.14}$$

as it follows from the usual embedding $W^{s,p'} \subset W^{t,q}$, $0 \le t \le s < \infty$; $1 < p' \le q < \infty$ [T.1, p. 206, p. 328], with $s = \frac{1}{2}$, $p' = 2$; $t = 0$, $q = p$, so that $s - \frac{n}{p'} \ge t - \frac{n}{q}$, $n = 1$, as required. Thus, we have by combining the last two inclusions (4.4b) and (4.14):

$$x \in \mathcal{D}(A_F) \to x|_{\Gamma_1} \in L_p(\Gamma_1) \to g(x|_{\Gamma_1}) \in L_2(\Gamma_1) \to Ng(x|_{\Gamma_1}) \in H^{\frac{3}{2}}(\Omega), \tag{4.15}$$

after using assumption (4.11) on g, as well as the regularity of N in (1.11). Thus, returning to $A_F x = f \in L_2(\Omega)$ in (4.5) with $N^* A x = x|_{\Gamma_1}$ by (4.4b), we obtain via (4.15) and $A^{-1} f \in \mathcal{D}(A) \subset H^2(\Omega)$:

$$-ix = A^{-1} f + Ng(x|_{\Gamma_1}) \in H^{\frac{3}{2}}(\Omega), \tag{4.16}$$

as desired. The sought-after conclusion $\mathcal{D}(A_F) \subset H^{\frac{3}{2}}(\Omega)$ in (4.13) then holds true in this case (a), and trace theory gives then $x|_{\Gamma_1} \in H^1(\Gamma_1)$.

Case (b): dim $\Gamma_1 = 2$. We start again with $x \in \mathcal{D}(A_F) \to x|_{\Gamma_1} \in H^{\frac{1}{2}}(\Gamma_1)$ by (4.4b). In the present case, however, we have

$$H^{\frac{1}{2}}(\Gamma_1) \subset L_4(\Gamma_1), \text{ so that } x \in \mathcal{D}(A_F) \Rightarrow x|_{\Gamma_1} \in L_4(\Gamma_1). \tag{4.17}$$

This follows again from the usual embedding $W^{s,p} \subset W^{t,q}$, $0 \le t \le s < \infty$; $1 < p \le q < \infty$ [T.1, p. 206; p. 328], this time with $s = \frac{1}{2}$, $p = 2$; $t = 0$, $q = 4$, so that $s - \frac{n}{p} \ge t - \frac{n}{q}$, with $n = 2$, as required. Then, invoking assumption (4.12), we obtain by (4.17),

$$\int_{\Gamma_1} |g(x|_{\Gamma_1})|^{\frac{4}{r}} d\Gamma_1 \le C_r \int_{\Gamma_1} |x|_{\Gamma_1}|^4 d\Gamma_1 < \infty, \text{ i.e., } g(x|_{\Gamma_1}) \in L_{\frac{4}{r}}(\Gamma_1). \tag{4.18}$$

We next invoke the L_p-elliptic theory of Appendix D, Eqn. (D.20). With $g(s|_{\Gamma_1}) \in L_{\frac{4}{r}}(\Gamma_1) \equiv W^{0,\frac{4}{r}}(\Gamma_1)$, Eqn. (D.20) yields via (4.18) and (1.9) for N:

$$[Ng(x|_{\Gamma_1})]_{\Gamma_1} \in W^{1,\frac{4}{r}}(\Gamma_1) \tag{4.19}$$

for the Neumann-Dirichlet map. [We note that the interior regularity $Ng(x|_{\Gamma_1}) \in W^{1+\frac{1}{4/r},\frac{4}{r}}(\Omega)$ does *not* hold true. What is true, instead, is the related interior regularity result in (D.25) in Appendix D below.] Moreover, we further have

$$W^{1,\frac{4}{r}}(\Gamma_1) \subset L_{\frac{4}{r-2}}(\Gamma_1), \text{ hence by (4.19): } [Ng(x|_{\Gamma_1})]_{\Gamma_1} \in L_{\frac{4}{r-2}}(\Gamma_1), \tag{4.20}$$

again invoking the embedding $W^{s,p} \subset W^{t,q}$; $0 \le t \le s < \infty$; $1 < p \le q < \infty$ [[T.1, p. 206, p. 328], this time with $s = 1$, $p = \frac{4}{r}$; $t = 0$, $q = \frac{4}{r-2}$, $r = 3 - \epsilon$, so that $s - \frac{n}{p} \ge t - \frac{n}{q}$ with $n = 2$, as required. Moreover, with reference to (1.8) and (4.5), we have $A^{-1}f \in H^2(\Omega)$ for $f \in L_2(\Omega)$, hence by trace theory

$$[A^{-1}f]_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1) \subset L_a(\Gamma_1) \text{ for any } 2 \le a < \infty; \text{ in particular,}$$
$$[A^{-1}f]_{\Gamma_1} \subset L_{\frac{4}{r-2}}(\Gamma_1), \tag{4.21}$$

again by invoking the embedding $W^{s,p} \subset W^{t,q}$, $0 \le t \le s < \infty$; $1 < p \le q < \infty$ [T.1, p. 206; p. 328], this time with $s = \frac{3}{2}$, $p = 2$; $t = 0$, $q = a$, so that

$s - \frac{n}{p} \geq t - \frac{n}{q}$ with $n = 2$, which is true. Applying the Dirichlet trace to identity (4.16) and using (4.20) and (4.21) in the resulting identity yields

$$-ix|_{\Gamma_1} = [A^{-1}f]_{\Gamma_1} + [Ng(x|_{\Gamma_1})]_{\Gamma_1} \in L_{\frac{4}{r-2}}(\Gamma_1), \quad r = 3 - \epsilon. \tag{4.22}$$

Next, with $x|_{\Gamma_1}$ in (4.22), we invoke assumptions (4.12) on g and obtain

$$\begin{cases} \int_{\Gamma_1} |g(x|_{\Gamma_1})|^s d\Gamma_1 \leq C_r \int_{\Gamma_1} |x|_{\Gamma_1}|^{sr} d\Gamma_1 < \infty, \text{ with } sr = \dfrac{4}{r-2}, \\ \text{thus, } g(x|_{\Gamma_1}) \in L_{\frac{4}{r(r-2)}}(\Gamma_1) \underset{\neq}{\subset} L_{\frac{4}{r}}(\Gamma_1). \end{cases} \tag{4.23}$$

In the last proper inclusion of (4.23), we have observed that, for $r = 3 - \epsilon$, $\epsilon > 0$ arbitrary, as in assumption (4.12), we obtain $\frac{4}{r(r-2)} > \frac{4}{r}$. Thus, the original regularity $g(x|_{\Gamma_1}) \in L_{\frac{4}{r}}(\Gamma_1)$ in (4.18) has been boosted to the regularity $g(x|_{\Gamma_1}) \in L_{\frac{4}{r(r-2)}}(\Gamma_1)$ in (4.23), an improvement of $\left[\frac{4}{r(r-2)} - \frac{4}{r}\right] = \frac{4(3-r)}{r(r-2)}$ in the integrability of the boundary datum $g(x|_{\Gamma_1})$.

Then, a bootstrap argument will lead to the desired integrability

$$g(x|_{\Gamma_1}) \in L_2(\Gamma_1). \tag{4.24}$$

In fact, with $g(x|_{\Gamma_1}) \in L_\rho(\Gamma_1) \equiv W^{0,\rho}(\Gamma_1)$ by (4.23), L_p-elliptic theory in Appendix D, Eqn. (D.20), with $\rho = \frac{4}{r(r-2)}$ now yields via the definition (1.9) for N:

$$[Ng(x|_{\Gamma_1})]_{\Gamma_1} \in W^{1,\rho}(\Gamma_1) \subset L_q(\Gamma_1) \subset L_2(\Gamma_1), \quad 2 < q = \frac{4}{r(r-2)-2}, \tag{4.25}$$

for the Neumann-Dirichlet map, with $q = \frac{4}{r(r-2)-2} > 2$ for $r = 3-\epsilon$. To obtain (4.25), we have used the usual embedding: $W^{s,p} \subset W^{t,q'}$, $0 \leq t \leq s < \infty$; $1 < p \leq q' < \infty$ [T.1, p. 206; p. 328], this time with $s = 1,\ p = \rho;\ t = 0$, so that $s - \frac{n}{p} \geq t - \frac{n}{q}$, $n = 2$, which is true, as required, for $q' \leq \frac{4}{r(r-2)-2}$, in particular for $q' = q$. Hence, from the identity in (4.22), as well as from (4.21) with $a = 2$, we obtain

$$-ix|_{\Gamma_1} = [A^{-1}f]_{\Gamma_1} + [Ng(x|_{\Gamma_1})]_{\Gamma_1} \in L_2(\Gamma_1), \tag{4.26}$$

Then, $x|_{\Gamma_1} \in L_2(\Gamma_1)$ in (4.26) yields conservatively $g(x|_{\Gamma_1}) \in L_2(\Gamma_1)$ in (4.24), via assumption (4.12) on g. Finally, (4.24) then yields via the L_2-elliptic theory (1.11) that $x \in \mathcal{D}(A_F)$ implies

$$Ng(x|_\Gamma) \in H^{\frac{3}{2}}(\Omega), \text{ hence } -ix = A^{-1}f + Ng(x|_\Gamma) \in H^{\frac{3}{2}}(\Omega) \tag{4.27}$$

from (4.16). Thus (4.27) proves (4.13) also in case (b). Proposition 4.2 is established. □

5 Proof of Uniform Decay Rates. Theorem 2.2

5.1 Step 1. First A-priori $L_2(\Omega)$-Energy Estimate in Terms of Boundary Terms

The key crux of the proof is the following $L_2(\Omega)$-energy estimate expressed by means of boundary terms of the w-problem (2.1) and of a lower-order (interior) term. It is obtained as a specialization of the basic estimate (1.19) of Theorem 1.3, as applied now to the present w-problem (2.1a-b-c). Thus, the energy $E(t) = \|w(t)\|^2_{L_2(\Omega)}$ in (2.13) satisfies $E(t) \leq E(0)$, $t \geq 0$ via the dissipation energy identity (2.18b). We then readily obtain with $H_a^{-1}(\Sigma_1)$, defined in (1.20).

Theorem 5.1.1. *Let $n = 1, 2, \ldots$. Assume hypothesis (H.1) = (2.2) on g and (H.3) = (2.4) on $\{\Omega, \Gamma_0, \Gamma_1\}$. Then, the solution of problem (2.1), guaranteed by Theorem 2.1, satisfies the following inequality: there exists a constant $C_T > 0$ such that*

$$\begin{aligned} E(t) \leq E(0) \quad &\leq \quad C_T \Big\{ \|w\|^2_{L_2(\Sigma_1)} + \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} \\ &\qquad + \int_0^T \int_{\Gamma_1} |g(w)|\,|w| d\Sigma_1 + \|w\|^2_{H^{-1}(Q)} \Big\}. \end{aligned} \tag{5.1.1}$$

Estimate (5.1.1) is the counterpart to the present nonlinear w-problem (2.1) of estimate (1.21) for the linear v-problem (1.1), both being a direct application of the basic estimate (1.19) of Theorem 1.3, for the respective dissipative boundary conditions. Reference to Remark 2.1 is useful here.

The next step is to further clean estimate (5.1.1) by absorbing the interior l.o.t. $w \in H^{-1}(Q)$, via a nonlinear, compactness/uniqueness argument.

Lemma 5.1.2. *With reference to Theorem 5.1.1, we have: there exists a constant C_T (independent of $w_0 \in L_2(\Omega)$), such that the solution of problem (2.1) satisfies:*

$$\|w\|^2_{H^{-1}(Q)} \leq C_T \left\{ \|w\|^2_{L_2(\Sigma_1)} + \int_{\Sigma_1} g(w)\bar{w}\, d\Sigma_1 + \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} \right\}. \quad (5.1.2)$$

Proof. It is based, as usual, on a compactness/uniqueness argument, though in the nonlinear context of the w-problem (2.1), along the lines of [La-Ta.1, Lem. 3.1, p. 526] in the case of wave equations.

Step 1. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of solutions to problem (2.1) (hence, with $w_n|_{t=0} \equiv w_0 \in L_2(\Omega), n = 1, 2, \ldots$), thus satisfying the well-posedness Theorem 2.1, Eqn. (2.5), in particular,

$$E(w_n(t)) \equiv \|w_n(t)\|^2_{L_2(\Omega)} \leq M_{w_0}, \quad 0 \leq t \leq T, \quad (5.1.3)$$

and such that, by contradiction, (5.1.2) is violated; that is, such that

$$\lim_{n\to\infty} \frac{\|w_n\|^2_{H^{-1}(Q)}}{\|w_n\|^2_{L_2(\Sigma_1)} + \int_{\Sigma_1} g(w_n)\bar{w}_n\, d\Sigma_1 + \|g(w_n)\|^2_{H_a^{-1}(\Sigma_1)}} = \infty. \quad (5.1.4)$$

Then, it follows from (5.1.3) that

$$w_n \quad \to \quad \text{some } w, \text{ weak}^* \text{ in } L_\infty(0,T; L_2(\Omega)), \quad (5.1.5)$$

$$w_n \quad \to \quad w, \text{ weakly in } L_2(0,T; L_2(\Omega)), \quad (5.1.6)$$

$$w_n \quad \to \quad w, \text{ strongly in } H^{-1}(Q). \quad (5.1.7)$$

Statements (5.1.5) (Al Theorem) and (5.1.6) are standard consequences of (5.1.3). The passage from (5.1.6) (or (5.1.5)) to (5.1.7) invokes a well-known compactness (in time and space) result [A.1], [S.1]. Because of (5.1.7), the numerator in (5.1.4) is uniformly bounded. This implies that each (positive) term in the denominator in (5.1.4) must tend to zero, as $n \to \infty$:

$$\|w_n\|_{L_2(\Sigma_1)} \to 0; \ \int_{\Sigma_1} g(w_n)\bar{w}_n d\Sigma_1 \to 0; \ \|g(w_n)\|_{H_a^{-1}(\Sigma_1)} \to 0. \tag{5.1.8}$$

Next, the main difference over [La-Ta.1] is that, in the present case and unlike [La-Ta.1], g is *not* dominated by a linear function at infinity. Thus, one needs to provide an alternative proof of the following implication:

Claim: Let $\{w_n\}$ be a sequence of solutions of problems (2.1) such that, as $n \to \infty$,

$$w_n \to 0 \text{ in } L_2(\Sigma_1); \qquad \int_{\Sigma_1} g(w_n)\bar{w}_n d\Sigma_1 \to 0, \tag{5.1.9}$$

as asserted in the first two statements of (5.1.8). Then, in fact,

$$\int_{\Sigma_1} g(w_n)\varphi d\Sigma_1 \to 0, \qquad \forall\ \varphi \in C^\infty(\Sigma_1). \tag{5.1.10}$$

[Compare with [La-Ta.1, Eqns. (3.16) and (3.17)]). $\square$

Proof of (5.1.10). To prove (5.1.10) as a consequence of (5.1.9), given $w_n(t,x)$ and $\delta > 0$, we divide $\Sigma_1 = (0,T] \times \Gamma_1$ accordingly as follows:

$$\begin{aligned} \Sigma_A(\delta,n) &\equiv \{(t,x) \in \Sigma_1 : \ |w_n(t,x)| \le \delta\}; \\ \Sigma_B(\delta,n) &\equiv \{(t,x) \in \Sigma_1 : \ |w_n(t,x)| \ge \delta\}, \end{aligned} \tag{5.1.11}$$

with $\delta > 0$ to be selected below. Then, for $\varphi \in C^\infty(\Sigma_1)$:

$$\int_{\Sigma_1} g(w_n)\varphi\, d\Sigma_1 = \int_{\Sigma_A(\delta,n)} g(w_n)\varphi\, d\Sigma_A + \int_{\Sigma_B(\delta,n)} g(w_n)\varphi\, d\Sigma_B. \tag{5.1.12}$$

Since $g(z)$ is continuous and $g(0) = 0$ by (H.1), then given $\epsilon > 0$, there is $\delta_0 = \delta_0(\epsilon) > 0$ such that $|g(w_n(t,x))| < \epsilon$, for $(t,x) \in \Sigma_A(\delta_0, n)$. Thus,

$$\left| \int_{\Sigma_A(\delta_0,n)} g(w_n)\varphi\, d\Sigma_A \right| \le \epsilon \|\varphi\|_{L_\infty(\Sigma_1)}. \tag{5.1.13}$$

Moreover, by definition of $\Sigma_B(\delta_0, n)$ and $g(w_n)\bar{w}_n \ge 0$ by (H.1), (2.2b):

$$\left|\int_{\Sigma_B(\delta_0,n)} g(w_n)\varphi\, d\Sigma_B\right|$$

$$= \left|\int_{\Sigma_B(\delta_0,n)} g(w_n)\frac{\bar{w}_n}{\bar{w}_n}\ \varphi\, d\Sigma_B\right| \tag{5.1.14}$$

$$\leq \frac{\|\varphi\|_{L_\infty(\Sigma_1)}}{\delta_0}\int_{\Sigma_B(\delta_0,n)} g(w_n)\bar{w}_n\, d\Sigma_B \tag{5.1.15}$$

$$\leq \frac{\|\varphi\|_{L_\infty(\Sigma_1)}}{\delta_0}\int_{\Sigma_1} g(w_n)\bar{w}_n\, d\Sigma_1 \to 0, \quad \text{as } n \to \infty. \tag{5.1.16}$$

Thus, given $\epsilon > 0$, $\delta_0 > 0$, there exists $N = N_{\epsilon,\delta_0}$, such that $\forall\ n > N$ we have that the integral in (5.1.16) is less than $(\delta_0\epsilon)$, and hence

$$\left|\int_{\Sigma_B(\delta_0,n)} g(w_n)\varphi\, d\Sigma_B\right| \leq \epsilon\|\varphi\|_{L_\infty(\Sigma_1)}, \quad \forall\ n > N_{\epsilon,\delta_0}. \tag{5.1.17}$$

Combining (5.1.13) and (5.1.17) in (5.1.12) yields the Claim in (5.1.10), as desired.

Next, we specialize to $\varphi \in C^\infty(\Sigma_1)$ such that $\varphi|_{t=0} = \varphi|_{t=T} = 0$ and $\varphi|_{\Sigma_0} \equiv 0$. Integrating by parts (second green theorem) and using (2.1a-b-c) for w_n yields:

$$\begin{aligned} 0 &= (i(w_n)_t + \Delta w_n, \varphi)_{L_2(Q)} \\ &= -i\int_0^T\int_\Omega w_n\varphi d\Omega\, dt + i\int_{\Sigma_1} g(w_n)\varphi\, d\Sigma_1 \\ &\quad -\int_0^T\int_{\Gamma_1} w_m\frac{\partial\varphi}{\partial\nu}\ d\Sigma_1 + \int_0^T\int_\Omega w_n\Delta\varphi\, d\Omega\, dt. \end{aligned} \tag{5.1.18}$$

On the RHS of (5.1.18), we invoke the weak convergence (5.1.6): $w_n \to w$ in $L_2(0,T;L_2(\Omega))$ on its first and last integral terms; as well as the convergence to zero of its second integral term by (5.1.10); and the convergence $w_n \to 0$ in $L_2(\Sigma_1)$, by the firsts statement of (5.1.8), on its penultimate integral term.

The final result is

$$\int_0^T \int_\Omega w\varphi_t d\Omega\, dt + \int_0^T \int_\Omega w\Delta\varphi\, d\Omega\, dt = 0, \tag{5.1.19}$$

for $w|_\Sigma \equiv 0$ and $\frac{\partial w}{\partial\nu}|_{\Sigma_1} \equiv 0$. (The first claim follows from $w_n|_{\Sigma_0} \equiv 0$ and $w_n \to 0$ on $L_2(\Sigma_1)$; the second by $\frac{\partial w_n}{\partial \nu} = ig(w_n) \to 0$ in $H_a^{-1}(\Sigma_1)$ by (5.1.8)). Thus, specializing further $\varphi \in D(Q) \equiv C_0^\infty(Q)$, we see that (5.1.19) is the weak formulation of the following problem

$$\begin{cases} iw_t + \Delta w = 0 \quad \text{in } Q; & (5.1.20a) \\ w|_{\Sigma_0} \equiv 0; \ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv 0. & (5.1.20b) \end{cases}$$

By Holmgren's theorem, we conclude that (5.1.20) implies then that $w \equiv 0$ in Q. Thus, all the convergences in (5.1.5), (5.1.6), (5.1.7) are to the limit $w \equiv 0$.

Step 2. (Scaling) Setting for notational convenience,

$$c_n \equiv \|w_n\|_{H^{-1}(Q)}, \text{ normalize: } \hat{w}_n \equiv \frac{w_n}{c_n}, \text{ so that } \|\hat{w}_n\|_{H^{-1}(Q)} \equiv 1.$$

Divide numerator and denominator of the fraction in (5.1.4) by c_n^2, use (5.1.21) and obtain

$$\lim_{n\to\infty} \frac{1}{\|\hat{w}_n\|^2_{L_2(\Sigma_1)} + \frac{\int_{\Sigma_1} g(w_n)\bar{w}_n d\Sigma_1}{c_n^2} + \frac{\|g(w_n)\|^2_{H_a^{-1}(\Sigma_1)}}{c_n^2}} = \infty. \tag{5.1.21}$$

Thus, (5.1.22) implies

$$\|\hat{w}_n\|^2_{L_2(\Sigma_1)} \to 0; \quad \frac{\int_{\Sigma_1} g(w_n)\bar{w}_n d\Sigma_1}{c_n^2} \to 0; \quad \frac{\|g(w_n)\|^2_{H_a^{-1}(\Sigma_1)}}{c_n^2} \to 0. \tag{5.1.22}$$

On the other hand, each solution w_n satisfies the energy estimate (5.1.1) of Theorem 5.1.1. Divide such estimate by c_n^2 and obtain via (5.1.21):

$$\|\hat{w}_n\|^2_{L_2(\Omega)} \le C_T \left\{ \|\hat{w}_n\|^2_{L_2(\Sigma_1)} + \frac{\|g(w_n)\|^2_{H_a^{-1}(\Sigma_1)}}{c_n^2} + \frac{\int_0^T \int_{\Gamma_1} g(w_n)\bar{w}_n d\Sigma_1}{c_n^2} + 1 \right\}, \tag{5.1.23}$$

as $g(w_n)\bar{w}_n = |g(w_n)|\,|w_n|$ by (H.1), (2.2b). Invoking on the RHS of (5.1.24) the convergence statements in (5.1.23), we then arrive at

$$\|\hat{w}_n(t)\|_{L_2(\Omega)} \leq \text{Const}_{w_0}, \quad 0 \leq t \leq T, \; n = 1, 2, \ldots \tag{5.1.24}$$

Hence, as in (5.1.5), (5.1.6), (5.1.7), we deduce from (5.1.25) that

$$\hat{w}_n \quad \to \quad \text{some } \hat{w}, \text{ weak}^* \text{ in } L_\infty(0, T; L_2(\Omega)), \tag{5.1.25}$$

$$\hat{w}_n \quad \to \quad \hat{w}, \text{ weakly in } L_2(0, T; L_2(\Omega)), \tag{5.1.26}$$

$$\hat{w}_n \quad \to \quad \hat{w}, \text{ strongly in } H^{-1}(Q) \text{ so that } \|\hat{w}\|_{H^{-1}(Q)} = 1, \tag{5.1.27}$$

by (5.1.21). We next divide problem (2.1a–c) for w_n by c_n, thus obtaining

$$\begin{cases} i(\hat{w}_n)_t + \Delta \hat{w}_n = 0 \quad \text{ in } Q; & (5.1.28\text{a}) \\ \hat{w}_n|_{\Sigma_0} \equiv 0; \; \left.\dfrac{\partial \hat{w}_n}{\partial \nu}\right|_{\Sigma_1} = i\dfrac{g(w_n)}{c_n}. & (5.1.28\text{b}) \end{cases}$$

Proceeding as in Step 1—from (5.1.8) to (5.1.20a–b) we then arrive at the following conclusion that the limit function $\hat{w}$ satisfies the following problem:

$$\begin{cases} i\hat{w}_t + \Delta \hat{w} = 0 \quad \text{ in } Q; & (5.1.29\text{a}) \\ \hat{w}|_{\Sigma_0} \equiv 0; \; \left.\dfrac{\partial \hat{w}}{\partial \nu}\right|_{\Sigma_1} \equiv 0. & (5.1.29\text{b}) \end{cases}$$

Again, via Holmgren's theorem, problem (5.1.30) implies then that $\hat{w} \equiv 0$ in Q and this is in contradiction with $\|\hat{w}\|_{H^{-1}(Q)} = 1$ in (5.1.28). Thus, the hypothesis of contradiction (5.1.4) is false and thus conclusion (5.1.2) is true, as desired. □

As a corollary, using (5.1.2) in the RHS of estimate (5.1.1), as well as $g(w)\bar{w} \geq 0$ by (2.2b), we obtain the basic starting point of our analysis.

Theorem 5.1.3. *Under the assumptions of Theorem 5.1.1, the following estimate holds true for the solutions of the w-problem (2.1): there exists a constant $C_T > 0$ such that*

$$E(t) \leq E(0) \leq C_T \left\{ \|w\|^2_{L_2(\Sigma_1)} + \int_0^T \int_{\Gamma_1} g(w)\bar{w}\, d\Sigma_1 + \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} \right\}. \tag{5.1.30}$$

The above estimate bounds ("recovers") the energy in terms of the "observed" boundary data.

The remaining part of the proof is PDE-free, and it relies instead on arguments of nonlinear analysis.

5.2 Estimate of the Damping Term $g(w)$ in $H_a^{-1}(\Sigma_1)$

Henceforth, let $w \in C([0,\infty); L_2(\Omega))$ be the solution of the w-problem (2.1), guaranteed by Theorem 2.1(i) for $w_0 \in L_2(\Omega)$.

On the strength of assumption (H.1)= (2.2), whereby $g(z)\bar{z} = |g(z)\bar{z}|$, we set for notational convenience

$$BT_{\Sigma_1}(w) = \int_0^T \int_{\Gamma_1} g(w)\bar{w}\, d\Sigma_1 = \int_0^T \int_{\Gamma_1} |g(w)\bar{w}| d\Sigma_1 = |g(w)\bar{w}|_{L_1(\Sigma_1)}, \tag{5.2.1}$$

where BT stands for "boundary term," and $L_1(\Sigma_1) = L_1(0,T; L_1(\Gamma_1))$. Next, we split $\Sigma_1 = \Sigma_{1A} \cup \Sigma_{1B}$ in two disjoint regions, $\Sigma_{1A} \cap \Sigma_{1B} = \emptyset$ (depending on the solution w), defined by

$$\Sigma_{1A} \equiv \{(t,x) \in \Sigma_1 : |w(t,x)| \geq 1\}; \quad \Sigma_{1B} \equiv \{(t,x) \in \Sigma_1 : |w(t,x)| < 1\}. \tag{5.2.2}$$

The goal of the present subsection is to prove the following result.

Proposition 5.2.1. *Let $n = dim\ \Omega = 2, 3$. Assume hypotheses (H.1) = (2.2) and (H.2)(b) = (2.3b). Let w be the solution of (2.1) guaranteed by Theorem 2.1. Then there exists a positive constant C_p (depending on p in (2.3b)) such that the following estimate holds true for problem (2.1):*

$$\|g(w)\|^2_{H_a^{-1}(\Sigma_1)} \leq \tilde{C}_p (E(0))^{\frac{p-1}{p+1}} (BT_{\Sigma_1}(w)) + 2\|g(w)\|^2_{L_2(\Sigma_{1B})}, \tag{5.2.3}$$

where: $p = 5$ for dim $\Omega = 2$; $p = 3$ for dim $\Omega = 3$, as in Assumption (H.2b) = (2.3b); $E(0)$ and $BT_{\Sigma_1}(w)$ are defined by (2.13) and (5.2.1), respectively. Moreover, $\tilde{C}_p = 2^{-p} c_p^2 M^{\frac{2}{p+1}}$, with M defined by (2.3b), and c_p defined by (5.2.18).

Proof of Proposition 5.2.1. Our first step is the following lemma.

Step 1: Lemma 5.2.2 *Let dim* $\Omega = 2, 3$. *Under assumptions (H.1) and (H.2)(b) = (2.3b), the following estimate holds true, where* M *is defined in (2.3b):*

$$\|g(w)\|_{H_a^{-1}(\Sigma_1)} \leq \|g(w)\|_{L_2(\Sigma_{1B})} + c_p M^{\frac{1}{p+1}} \|g(w)\bar{w}\|_{L_1(\Sigma_1)}^{\frac{p}{p+1}}, \tag{5.2.4}$$

with p *as specified below (5.2.3),* c_p *the constant in (5.2.18) below.*

Proof of Lemma 5.2.2. We proceed by duality. Take $\phi \in H_a^1(\Sigma_1)$, the space defined in (1.20). Inequality (5.2.4) holds true as soon as we establish the following estimates:

(a)

$$\left| \int_{\Sigma_{1A}} g(w)\phi \, dx \, dt \right| \leq c_p M^{\frac{1}{p+1}} \|\phi\|_{H_a^1(\Sigma_1)} \|g(w)\bar{w}\|_{L_1(\Sigma_1)}^{\frac{p}{p+1}}; \tag{5.2.5}$$

(b)

$$\left| \int_{\Sigma_{1B}} g(w)\phi \, dx \, dt \right| \leq \|\phi\|_{L_2(\Sigma_1)} \|g(w)\|_{L_2(\Sigma_{1B})} \leq \|\phi\|_{H_a^1(\Sigma_1)} \|g(w)\|_{L_2(\Sigma_{1B})}. \tag{5.2.6}$$

Part (b), estimate (5.2.6), is just an application of the Schwarz inequality, followed by the fact that the $H_a^1(\Sigma_1)$-norm in (1.20) is stronger than the $L_2(\Sigma_1)$-norm.

We now proceed to establish Part (a), estimate (5.2.5).

Step (i). To this end, for each $t \in [0, T]$, recalling (5.2.2), define

$$\Gamma_{1A}(t) \equiv \{x \in \Gamma_1 : |w(t, x)| \geq 1\}, \text{ so that } \Sigma_{1A} = \bigcup_{0 \leq t \leq T} t \times \Gamma_{1A}(t). \tag{5.2.7}$$

Next, we invoke the Hölder inequality, in the space integration below, with $\frac{1}{r} + \frac{1}{r^*} = 1$, where r and r^* will be selected in (5.2.10) below, and we estimate, via (5.2.7) and (5.2.2) on Σ_{1A}:

$$\begin{aligned}
&\left|\int_{\Sigma_{1A}} g(w)\phi\,dx\,dt\right| \\
&\quad = \left|\int_0^T \int_{\Gamma_{1A}(t)} g(w)\phi\,dx\,dt\right| \\
&\quad \le \int_0^T \left\{\left[\int_{\Gamma_{1A}(t)} |g(w)|^r dx\right]^{\frac{1}{r}} \left[\int_{\Gamma_{1A}(t)} |\phi|^{r^*} dx\right]^{\frac{1}{r^*}}\right\} dt \qquad (5.2.8) \\
&\quad = \int_0^T \left\{\left[\int_{\Gamma_{1A}(t)} |g(w)|\,|g(w)|^{r-1} dx\right]^{\frac{1}{r}} \|\phi(t)\|_{L_{r^*}(\Gamma_{1A}(t))}\right\} dt
\end{aligned}$$

(by (2.3b))

$$\le \int_0^T \left\{\left[\int_{\Gamma_1} |g(w)| M^{r-1} |w|^{p(r-1)} d\Gamma_1\right]^{\frac{1}{r}} \|\phi(t)\|_{L_{r^*}(\Gamma_1)}\right\} dt, \quad (5.2.9)$$

where, in the last step, we have majorized $\Gamma_{1A}(t)$ by Γ_1 and we have recalled assumption (H.2b) = (2.3b) for $|z| \ge 1$, p given explicitly there, and (5.2.7). We now impose the condition that

$$p(r-1) = 1, \text{ so that } r = \frac{p+1}{p},\ r^* = p+1,\ \frac{r-1}{r} = \frac{1}{p+1}. \qquad (5.2.10)$$

Thus, with the choice for r made in (5.2.10), we rewrite inequality (5.2.9) as

$$\left|\int_{\Sigma_{1A}} g(w)\,\phi\,dx\,dt\right| \le M^{\frac{1}{p+1}} \int_0^T \left\{\left[\int_{\Gamma_1} |g(w)|\,|w| d\Gamma_1\right]^{\frac{p}{p+1}} \|\phi(t)\|_{L_{p+1}(\Gamma_1)}\right\} dt. \qquad (5.2.11)$$

Step (ii). Next, we invoke again the Hölder inequality, this time in the time integration, with $\frac{1}{q} + \frac{1}{q^*} = 1$, where q and q^* will be selected in (5.2.17a-b) below, according to dim $\Omega = 2, 3$. We thus estimate from (5.2.11):

$$\begin{aligned}
&\left|\int_{\Sigma_{1A}} g(w)\,\phi\,dx\,dt\right| \\
&\quad \le M^{\frac{1}{p+1}} \left\{\int_0^T \left[\int_{\Gamma_1} |g(w)|\,|w| d\Gamma_1\right]^{\frac{pq}{p+1}} dt\right\}^{\frac{1}{q}} \left\{\int_0^T \|\phi(t)\|^{q^*}_{L_{p+1}(\Gamma_1)} dt\right\}^{\frac{1}{q^*}}. \qquad (5.2.12)
\end{aligned}$$

Recalling $g(w)\bar{w} = |g(w)|\,|w| \geq 0$ by (2.2a) (assumption (H.1)), we rewrite (5.2.12) as

$$\left| \int_{\Sigma_{1A}} g(w)\,\phi\,dx\,dt \right| \leq M^{\frac{1}{p+1}} \left\{ \int_0^T \left[\int_{\Gamma_1} g(w)\bar{w} d\Gamma_1 \right]^{\frac{pq}{p+1}} dt \right\}^{\frac{1}{q}} \|\phi\|_{L_{q^*}(0,T;L_{p+1}(\Gamma_1))}. \tag{5.2.13}$$

Step (iii). The following embedding holds true:

$$\begin{aligned} H_a^1(\Sigma_1) \equiv H^{\frac{1}{2}}(0,T;L_2(\Gamma_1)) &\cap L_2(0,T;H^1(\Gamma_1)) \\ &\subset H^{\frac{\theta}{2}}(0,T;H^{1-\theta}(\Gamma_1)),\ 0 \leq \theta \leq 1. \end{aligned} \tag{5.2.14}$$

Indeed, $f \in L_2(0,T;H^1(\Gamma_1))$, $D_t^{\frac{1}{2}} f \in L_2(0,T;L_2(\Gamma_1))$ implies that $D_t^{\frac{\theta}{2}} f \in L_2(0,T;[H^1(\Gamma_1),L_2(\Gamma_1)]_\theta)$, where $[H^1(\Gamma_1),L_2(\Gamma_1)]_\theta = H^{1-\theta}(\Gamma_1)$, by [L-M.1, p. 36], and the embedding (5.2.14) is justified.

Moreover, we shall justify below in Step (v) the following embeddings, according to as dim $\Omega = 2$, or dim $\Omega = 3$:

Case: dim $\Omega = 2$, hence dim $\Gamma_1 = 1$. In this case we have

$$H^{\frac{\theta}{2}}(0,T;H^{1-\theta}(\Gamma_1)) \subset L_{\frac{2}{1-\theta}}\left(0,T;L_{\frac{2}{2\theta-1}}(\Gamma_1)\right),\ \frac{1}{2} < \theta \leq 1. \tag{5.2.15a}$$

Case: dim $\Omega = 3$, hence dim $\Gamma_1 = 2$. In this case

$$H^{\frac{\theta}{2}}(0,T;H^{1-\theta}(\Gamma_1)) \subset L_{\frac{2}{1-\theta}}\left(0,T;L_{\frac{2}{\theta}}(\Gamma_1)\right),\ 0 < \theta \leq 1. \tag{5.2.15b}$$

Thus, combining the two embeddings in (5.2.14) and in (5.2.15), we obtain the new embedding

$$H_a^1(\Sigma_1) \subset X_\theta(\Sigma_1)$$

$$\equiv \begin{cases} L_{\frac{2}{1-\theta}}\left(0,T;L_{\frac{2}{2\theta-1}}(\Gamma_1)\right),\ \frac{1}{2} < \theta \leq 1;\ \dim(\Gamma_1)\ \equiv\ 1; & (5.2.16a) \\ L_{\frac{2}{1-\theta}}\left(0,T;L_{\frac{2}{\theta}}(\Gamma_1)\right),\ 0 < \theta \leq 1;\ \dim(\Gamma_2)\ \equiv\ 2; & (5.2.16b) \end{cases}$$

$$\|f\|_{X_\theta(\Sigma_1)} \leq c_\theta \|f\|_{H_a^1(\Sigma_1)}. \tag{5.2.16c}$$

Step (iv). We next return to estimate (5.2.13), where the function ϕ taken in $H^1_a(\Sigma_1)$ is evaluated in its RHS as a function $\phi \in L_{q^*}(0,T;L_{p+1}(\Gamma_1))$. This will be possible in (5.2.18) below. Thus, by comparing this norm requirement with (5.2.16a-b), we see that we must select q^* and q as follows, where $\frac{1}{q^*}+\frac{1}{q} = 1$:

for dim $\Omega = 2$, dim $\Gamma_1 = 1$:

$$\begin{cases} q^* = \dfrac{2}{1-\theta};\ p+1 = \dfrac{2}{2\theta-1},\ \dfrac{1}{2} < \theta \le 1;\ \theta = \dfrac{3+p}{2(p+1)}; \\[2ex] \text{or } q = \dfrac{2}{1+\theta} = \dfrac{4(p+1)}{3p+5} = \dfrac{6}{5} \text{ for } p=5;\ q^* = \dfrac{4(p+1)}{p-1} = 6 \text{ for } p=5; \end{cases} \tag{5.2.17a}$$

for dim $\Omega = 3$, dim $\Gamma_1 = 2$:

$$\begin{cases} q^* = \dfrac{2}{1-\theta};\ p+1 = \dfrac{2}{\theta},\ 0 < \theta \le 1;\ \theta = \dfrac{2}{p+1}; \\[2ex] \text{or } q = \dfrac{2}{1+\theta} = \dfrac{2(p+1)}{p+3} = \dfrac{4}{3} \text{ for } p=3;\ q^* = \dfrac{2(p+1)}{p-1} = 4 \text{ for } p=3. \end{cases} \tag{5.2.17b}$$

We remark that in both cases—$\{\dim\Omega = 2; p = 5\}$ and $\{\dim\Omega = 3, p = 3\}$—we have

$$\frac{pq}{p+1} = 1;\ \text{indeed } \frac{5\cdot\frac{6}{5}}{5+1} = 1 \text{ and } \frac{3\cdot\frac{4}{3}}{3+1} = 1,\ \text{respectively.} \tag{5.2.17c}$$

With such choice of q, q^* in terms of p as given by (5.2.17a-b), according to $\dim(\Gamma_1) = 1,2$, we then obtain by (5.2.16c),

$$\|\phi\|_{L_{q^*}(0,T;L_{p+1}(\Gamma_1))} \le c_p\|\phi\|_{H^1_a(\Sigma_1)}. \tag{5.2.18}$$

Hence, inequality (5.2.13) becomes via (5.2.18), and (5.2.17c) with $\frac{pq}{p+q} = 1$, and $\frac{1}{q} = \frac{p}{p+1}$.

$$\begin{aligned} &\left| \int_{\Sigma_{1A}} g(w)\phi\, dx\, dt \right| \\ &\quad \le \quad c_p M^{\frac{1}{p+1}} \left\{ \int_0^T \int_{\Gamma_1} g(w)\bar{w}\, d\Gamma_1 dt \right\}^{\frac{p}{p+1}} \|\phi\|_{H^1_a(\Sigma_1)} \end{aligned} \tag{5.2.19}$$

$$= \quad c_p M^{\frac{1}{p+1}} \|g(w)\bar{w}\|^{\frac{p}{p+1}}_{L_1(\Sigma_1)} \|\phi\|_{H^1_a(\Sigma_1)}, \tag{5.2.20}$$

and estimate (5.2.20) proves (5.2.5) in Part (a), as desired.

Step (v). It remains to justify the embeddings (5.2.15a-b). In order to fit into established notation, we rewrite the H^s spaces as $W^{s,2}$ spaces:

$$H^{\frac{\theta}{2}}(0,T;H^{1-\theta}(\Gamma_1)) \equiv W^{\frac{\theta}{2},2}(0,T;W^{1-\theta,2}(\Gamma_1)). \tag{5.2.21}$$

We then invoke the standard embedding result [T.1, p. 206, p. 328]. We have

$$\begin{cases} W^{s,p} \subset W^{t,q}, \ 0 \le t \le s < \infty; \ 1 < p \le q < \infty, \\ \text{provided that } s - \dfrac{n}{p} \ge t - \dfrac{n}{q}. \end{cases} \tag{5.2.22}$$

Embedding in the space variable. We obtain from (5.2.22) that the embedding

$$W^{1-\theta,2}(\Gamma_1) \subset W^{0,q}(\Gamma_1) \equiv L_q(\Gamma_1) \tag{5.2.23}$$

holds true, provided that, by (5.2.22) with $s = 1-\theta$, $p = 2$, $t = 0$, we have

$$\begin{cases} 1 - \theta - \dfrac{2}{2} \ge 0 - \dfrac{2}{q}, \text{ for } n = \text{ dim } \Gamma_1 = 2, \quad \text{dim } \Omega = 3, \\ \text{or } \theta \le \dfrac{2}{q}, \text{ or } q \le \dfrac{2}{\theta}, \ 0 < \theta \le 1; \end{cases} \tag{5.2.24}$$

$$\begin{cases} 1 - \theta - \dfrac{1}{2} \ge 0 - \dfrac{1}{q}, \text{ for } n = \text{ dim } \Gamma_1 = 1, \quad \text{dim } \Omega = 2, \\ \text{or } \theta - \dfrac{1}{2} \le \dfrac{1}{q}, \text{ or } q \le \dfrac{2}{2\theta - 1}, \ \dfrac{1}{2} < \theta \le 1. \end{cases} \tag{5.2.25}$$

Embedding in the time variable. We obtain from (5.2.22) that the embedding

$$W^{\frac{\theta}{2},2}(0,T) \subset W^{0,q}(0,T) = L_q(0,T) \tag{5.2.26a}$$

holds true, provided that, by (5.2.22) with $s = \frac{\theta}{2}$, $p = 2$, $t = 0$, $n = 1$, we have

$$\frac{\theta}{2} - \frac{1}{2} \ge 0 - \frac{1}{q} \quad \text{ or } \quad q \le \frac{2}{1-\theta}, \ 0 \le \theta < 1. \tag{5.2.26b}$$

Combining (5.2.21), (5.2.23)–(5.2.26), we obtain the validity of the embeddings (5.2.15a), (5.2.15b), as desired.

With Part (a) = (5.2.5) and Part (b) = (5.2.6) fully established, then estimate (5.2.4) follows at once by duality, as already noted. Lemma 5.2.2 is proved.

The second and last step in the proof of Proposition 5.2.1 follows next.

Step 2. Recalling the boundary term $BT_{\Sigma_1}(w)$ from (5.2.1), we rewrite inequality (5.2.4) of Lemma 5.2.2 as follows, after squaring both sides:

$$\begin{aligned} \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} &\leq 2\|g(w)\|^2_{L_2(\Sigma_{1B})} + C_p[BT_{\Sigma_1}(w)]^{\frac{2p}{p+1}} && (5.2.27) \\ &= 2\|g(w)\|^2_{L_2(\Sigma_{1B})} + C_p[BT_{\Sigma_1}(w)]^{\frac{p-1}{p+1}}[BT_{\Sigma_1}(w)], && (5.2.28) \end{aligned}$$

$C_p = 2(c_p M^{\frac{1}{p+1}})^2$. Next, we rewrite the dissipation energy identity (2.18b) for $s = 0$, $t = T$, in the notation of (5.2.1) for $BT_{\Sigma_1}(w)$ as

$$E(T) + 2BT_{\Sigma_1}(w) = E(0), \text{ thus } BT_{\Sigma_1}(w) \leq \frac{1}{2}E(0). \tag{5.2.29}$$

[At this point, it is critical that the dissipativity energy identity (2.18a) can be rewritten as in (2.18b) by virtue of assumption (H.1) = (2.2b).] Using the inequality of (5.2.29) in the RHS of (5.2.28) yields:

$$\begin{aligned} \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} &\leq 2\|g(w)\|^2_{L_2(\Sigma_{1B})} \\ &+ \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}[BT_{\Sigma_1}(w)], \tilde{C}_p = C_p\left(\frac{1}{2}\right)^{\frac{p-1}{p+1}}, && (5.2.30) \end{aligned}$$

and (5.2.30) proves (5.2.3), as desired. Proposition 5.2.1 is established. □

5.3 Estimates of the Boundary Terms $w \in L_2(\Sigma_1)$ and $g(w)\bar{w} \in L_2(\Sigma_1)$

Lemma 5.3.1. *Assume (H.1) = (2.2) and (H.2)(a) = (2.3a) and recall (5.2.1). Then, the solution of problem (2.1) guaranteed by Theorem 2.1 satisfies:*

(a)

$$\int_{\Sigma_{1A}} |w|^2 dx\, dt \leq \frac{1}{m} \int_{\Sigma_{1A}} g(w)\bar{w}\, dx\, dt \leq \frac{1}{m} BT_{\Sigma_1}(w), \tag{5.3.1}$$

(b)

$$\begin{aligned} &\int_{\Sigma_{1B}} |w|^2 dx\, dt + \int_{\Sigma_{1B}} |g(w)|^2 dx\, dt \\ &\leq \ (meas\ \Sigma_1)\tilde{h}\left(\int_0^T \int_{\Gamma_1} g(w)\bar{w}\, dx\, dt\right) = (meas\ \Sigma_1)\tilde{h}(BT_{\Sigma_1}(w)), \end{aligned} \tag{5.3.2}$$

where $\tilde{h}$ is the concave strictly increasing function defined in (2.10).

Proof. (a) Inequality (5.3.1) is an immediate application of assumption (H.2a) = (2.3a) in the notation of (5.2.1).

(b) We invoke the distinctive property (2.9) for h—where the constant δ can be taken to be $\delta = 1$ without loss of generality by renormalization, as to fit the requirement of the definition of the set Σ_{1B} in (5.2.2). We also recall, from (2.2b), that $(g(w)\bar{w}) \geq 0$, and that $h(s) \geq 0$ by definition. Then, we estimate by virtue of the Jensen's inequality [L-L.1, p. 38]

$$\begin{aligned} &\int_{\Sigma_{1B}} |w|^2 dx\, dt + \int_{\Sigma_{1B}} |g(w)|^2 dx\, dt \\ &\leq \ \int_{\Sigma_{1B}} h(g(w)\bar{w}) dx\, dt \leq \int_{\Sigma_1} h(g(w)\bar{w}) d\Sigma_1 \\ &\leq \ (\text{meas } \Sigma_1) h\left(\frac{1}{\text{meas } \Sigma_1} \int_{\Sigma_1} g(w)\bar{w}\, d\Sigma_1\right) \qquad (5.3.3) \\ &\leq \ (\text{meas } \Sigma_1)\tilde{h}\left(\int_{\Sigma_1} g(w)\bar{w}\, d\Sigma_1\right) \equiv (\text{meas } \Sigma_1)\tilde{h}(BT_{\Sigma_1}(w)), \qquad (5.3.4) \end{aligned}$$

where $\tilde{h}$ is defined in (2.10) and $BT_{\Sigma_1}(w)$ is defined in (5.2.1). Then, (5.3.4) establishes (5.3.2), as desired. □

5.4 Completion of the Proof of Theorem 2.2

Lemma 5.4.1. *Assume (H.1) = (2.2), (H.2) = (2.3). Then problem (2.1) satisfies:*

$$\int_{\Sigma_1} |w|^2 d\Sigma_1 + \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} \leq \left(\frac{1}{m} + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}\right) BT_{\Sigma_1}(w) + 2(meas\ \Sigma_1)\tilde{h}(BT_{\Sigma_1}(w)). \quad (5.4.1)$$

Proof. We collect the results of Proposition 5.2.1, Eqn. (5.2.3), and of Lemma 5.3.1, Eqns. (5.3.1), (5.3.2), and we obtain

$$\int_{\Sigma_1} |w|^2 d\Sigma_1 + \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} = \int_{\Sigma_{1A}} |w|^2 dx\,dt + \int_{\Sigma_{1B}} |w|^2 dx\,dt + \|g(w)\|^2_{H_a^{-1}(\Sigma_1)} \quad (5.4.2)$$

(by (5.2.3))

$$\leq \int_{\Sigma_{1A}} |w|^2 dx\,dt + \left\{\int_{\Sigma_{1B}} |w|^2 dx\,dt + 2\int_{\Sigma_{1B}} |g(w)|^2 dx\,dt\right\} + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}(BT_{\Sigma_1}(w)) \quad (5.4.3)$$

(by (5.3.1), (5.3.2))

$$\leq \frac{1}{m} BT_{\Sigma_1}(w) + 2(\text{meas } \Sigma_1)\tilde{h}(BT_{\Sigma_1}(w)) + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}(BT_{\Sigma_1}(w)), \quad (5.4.4)$$

and (5.4.4) leads to (5.4.1), as desired. □

Lemma 5.4.2. *Assume (H.1) = (2.2) and (H.2) = (2.3). Then, the energy $E(T)$ of problem (2.1) satisfies the equation*

$$E(T) + p(E(T)) = E(0), \quad (5.4.5)$$

where $p(\,\cdot\,)$ is defined by (2.11), with C and K specified in (2.16).

Proof. Lemma 5.4.1 gives the key desired estimate of the boundary terms, which are present on the RHS of the basic energy estimate (5.1.12) of Theorem

5.1.3. Thus, returning to (5.1.12), recalling (5.2.1), and using (5.4.1) on its RHS yields

$$
\begin{aligned}
E(t) \leq E(0) \quad &\leq \quad C_T \left[\frac{1}{m} + 1 + \tilde{C}_p (E(0))^{\frac{p-1}{p+1}}\right] BT_{\Sigma_1}(w) \\
&\quad + 2C_T (\text{meas } \Sigma_1) \tilde{h}(BT_{\Sigma_1}(w)) \qquad (5.4.6) \\
&\leq \quad C_1 (2BT_{\Sigma_1}(w)) + C_2 \tilde{h}(2BT_{\Sigma_1}(w)), \qquad (5.4.7)
\end{aligned}
$$

since $h(\cdot)$, hence $\tilde{h}(\cdot)$, is strictly increasing, where we have set

$$C_1 \equiv \frac{C_T}{2} \left[\frac{1}{m} + 1 + C_p (E(0))^{\frac{p-1}{p+1}}\right]; \quad C_2 = 2C_T \text{ meas } \Sigma_1. \tag{5.4.8}$$

Dividing (5.4.7) across by C_2 and specializing to $t = T$ yields

$$KE(T) \leq [CI + \tilde{h}](2BT_{\Sigma_1}(w)); \tag{5.4.9}$$

$$K = \frac{1}{C_2} = \frac{1}{2C_T \text{meas } \Sigma_1}; \quad C = \frac{C_1}{C_2} = \frac{\frac{1}{m} + 1 + \tilde{C}_p (E(0))^{\frac{p-1}{p+1}}}{2 \text{ meas } \Sigma_1}. \tag{5.4.10}$$

Then, recalling $p(\cdot)$ in (2.11), with constants C and K defined by (5.4.10), as well as the energy identity (2.18b) for $s = 0$, $t = T$, we obtain

$$[CI + \tilde{h}]^{-1}(KE(T)) = p(E(T)) = 2BT_{\Sigma_1}(\Sigma_1) = E(0) - E(T). \tag{5.4.11}$$

Finally,

$$E(T) + p(E(T)) = E(0), \tag{5.4.12}$$

and Lemma 5.4.2 is established, since K and C in (5.4.10) coincide with (2.16).

Since $p(\cdot)$ depends only on $E(0)$, see (2.16) = (5.4.10) and the problem is dissipative, we can apply [La-Ta.1, Lem. 3.3, p. 531] to yield the desired conclusion (2.14). The proof of Theorem 2.2 is complete. □

Appendix A: Assumption (H.1) on $g(z)$

Lemma 2.3 gives an *ad hoc*, direct construction of classes of functions $g(z)$ satisfying both conditions in (2.2a). Here, we cast the monotonicity condition in (2.2a) (RHS) into the *general setting* of assumption (H.1).

(1) A single-valued function $g : \mathbb{C} \to \mathbb{C}$ satisfying the monotonicity condition

$$\mathrm{Re}\{(g(z) - g(v))(\bar{z} - \bar{v})\} \geq 0, \quad \forall\ z, v \in \mathbb{C} \tag{A.1}$$

in (2.2a) which is the sub-differential $\partial j :\ \mathbb{C} \to \mathbb{C}$:

$$g(z) \equiv \partial j(z) \equiv \{z^* \in \mathbb{C} :\ j(z) - j(v) \leq (z^*, z - v)_{\mathbb{C}} \equiv \mathrm{Re}\{z^*(\bar{z} - \bar{v})\}\} \tag{A.2}$$

of a function $j : \mathbb{C} \to \overline{\mathbb{R}} =]-\infty, \infty]$, which is lower semi-continuous, convex, proper. Here, the range space $\mathbb{C}$ is viewed as an inner-product space, under the inner product $(z, v)_{\mathbb{C}} \equiv \mathrm{Re}(z\bar{v})$. Indeed, in these circumstances, ∂j is a maximal monotone operator on $\mathbb{C}$ [B.1, Thm. 2.1, p. 54], [B.2, Thm. 2.1, p. 62]. Then, monotonicity of $\partial j :\ (z_1 - z_2, j(z_1) - j(z_2))_{\mathbb{C}} \geq 0$ reproduces (A.1) under (A.2).

(2) The full assumption (H.1) = (2.2a) then requires (A.2) as well as $\mathrm{Im}\{g(z)\bar{z}\} \equiv 0$.

(3) Classes of explicit examples were constructed *ad hoc* in Lemma 2.3. Here, we employ the general setting (A.2), by recalling the general result that:

If—moreover—j is Gateaux differentiable at $z \in \mathbb{C}$, then

$$\partial j(z) = \nabla j(z) = \text{Gateaux differential} \tag{A.3}$$

defined by $j'(z, v) = (v, \nabla j(z))_{\mathbb{C}}$ [B.1, p. 53], [B.2, pp. 59–61]. Using these techniques, we obtain:

(4) *Proposition* **A**.1. Let $\mu(r)$ be a C^1-function of the real variable $r \geq 0$, such that $r \to \mu(r^2)$ is convex. Define the convex function $j(z)$ by

$$j(z) \equiv \mu(|z|^2), \quad z \in \mathbb{C}. \tag{A.4}$$

Then, its Gateaux differential (subdifferential) is

$$\partial j(z) = \nabla j(z) = 2\mu'(|z|^2)z, \qquad z \in \mathbb{C}, \tag{A.5}$$

and the function

$$g(z) = \partial j(z) = 2\mu'(|z|^2)z, \qquad z \in \mathbb{C}, \tag{A.6}$$

satisfies assumption (H.1) = (2.2a).

Proof. For $z, v \in \mathbb{C}$, $\lambda > 0$, we compute via (A.4)

$$\begin{aligned} j(z+\lambda v) - j(z) &= \mu([z+\lambda v][\bar{z}+\lambda\bar{v}]) - \mu(|z|^2) \\ &= \mu(|z|^2 + \lambda[z\bar{v} + v\bar{z}] + \lambda^2|v|^2) - \mu(|z|^2) \\ &= \mu'(|z|^2)(\lambda[z\bar{v}+v\bar{z}] + \lambda^2|v|^2) + h(\lambda), \end{aligned} \tag{A.7}$$

by the Taylor's formula, where $h(\lambda)$ denotes higher-order terms in λ. Hence (A.7) yields

$$\begin{aligned} \lim_{\lambda\downarrow 0} \frac{j(z+\lambda v) - j(z)}{\lambda} &= \mu'(|z|^2)\, 2\, \mathrm{Re}\{z\bar{v}\} \\ &= j'(z,v) = (v, \nabla j(z))_{\mathbb{C}} = \mathrm{Re}\{\nabla j(z)\bar{v}\}, \end{aligned} \tag{A.8}$$

and then (A.8) yields (A.5), as desired, by definition of $\nabla j(z)$. The statement on $g(z)$ is now obvious. □

(5) *Lemma 2.3 revisited.* As a corollary of Proposition A.1, we re-obtain Lemma 2.3, via a second abstract proof. Thus: let $\gamma(s)$ be, as in (2.19), a positive continuous function on $[0, s_0]$ with $\gamma(0) = \lim_{s\downarrow 0} \gamma(x)$ (possibly $+\infty$), such that $s \to s\gamma(s)$ is monotone increasing. Define the function

$$F(\sqrt{r}) \equiv \int_0^{\sqrt{r}} s\gamma(s)ds = \frac{1}{2}\int_0^r \gamma(\sqrt{x})dx = \mu(r) \quad (\text{set } s = \sqrt{x}). \tag{A.9}$$

By the assumption on $s\gamma(s)$ being increasing, we have that the map $\sqrt{r} \to F(\sqrt{r}) \equiv \mu(r)$ is convex; that is, the map $r \to \mu(r^2)$ is convex, as required in Proposition A.1. We have from (A.9),

$$\mu'(r) = \frac{1}{2}\gamma(\sqrt{r}), \qquad r \geq 0. \tag{A.10}$$

According to Proposition A.1, we define the convex function $j: \mathbb{C} \to \mathbb{R}$ by

$$j(z) = \mu(|z|^2), \qquad z \in \mathbb{C}, \tag{A.11}$$

and obtain via (A.6) and (A.10) that the function

$$g(z) \equiv \partial j(z) = 2\mu'(|z|^2)z = \gamma(|z|)z \tag{A.12}$$

satisfies assumption (H.1), as desired.

Lemma 2.3 is thus re-proved as a corollary of Proposition A.1.

Remark A.1. A stronger assumption for the conclusion in (A.12) of Lemma 2.3 is that the map $x \to \gamma(\sqrt{x})$ is increasing, so that $r \to \mu(r)$ is convex, as seen from (A.9).

Appendix B: An Asymptotic Comparison Result

In this appendix we consider two nonlinear ODE problems:

$$S_t(t) + q(S(t)) \equiv 0 \qquad S(0) = E(0) \tag{B.1}$$

$$\tilde{S}_t(t) + \tilde{q}(\tilde{S}(t)) \equiv 0 \qquad \tilde{S}(T) = S(T) \tag{B.2}$$

in the following *setting*:

(i) $q(\cdot)$ is the continuous function: $\mathbb{R}_+ \to \mathbb{R}_+$ defined in (2.12). Thus, $q(0) = 0$, $q(x) > 0$ for $x > 0$ and $q(x)$ is strictly increasing. Then (B.1) is precisely the same as Eqn. (2.15). By monotonicity, there exists a unique solution $S(t)$ that exists for all times $t > 0$ [Har.1, Cor. 6.3, p. 34]. Moreover, $S_t(t)$ is negative for $t > 0$ and $S(t)$ is strictly decreasing and satisfies

$$S(t) \searrow 0 \qquad \text{as } t \to \infty, \tag{B.3}$$

as is seen by Lyapunov's second method [B-N.1, Thm. 5.6, p. 224]: indeed, we take the Lyapunov function $V(y) = y^2$, so that its Lyapunov derivative is $V^*(y) = -2y\,q(y) \leq 0$, while the origin is the only invariant subset of the set $E \equiv \{y : V^*(y) = 0\}$.

(ii) $\tilde{q}(\cdot)$ is a continuous function defined on some interval $[0, x_0]$ and having otherwise the same properties as $q(\cdot)$. Thus, $\tilde{q}(0) = 0$, $\tilde{q}(x) > 0$ for $x > 0$ and $\tilde{q}(x)$ is strictly increasing on $[0, x_0]$. In view of (B.3), T is taken sufficiently large so that $0 < S(T) < x_0$. Then, problem (B.2) admits a unique solution $\tilde{S}(t)$ that exists for all $t > T$ [Har.1, Cor. 6.3, p. 34]. Moreover, $\tilde{S}_t(t)$ is negative for $t > T$ and $\tilde{S}(t)$ is strictly decreasing, $\tilde{S}(t) < x_0$, for $t > T$, and likewise satisfies

$$\tilde{S}(t) \searrow 0 \qquad \text{as } t \to \infty \tag{B.4}$$

again by Lyapunov's Theorem [B-N.1, Thm. 5.6, p. 224].

(iii) We think of $q(\,\cdot\,)$ as defined through the algorithm described in (2.9) through (2.12), though *not necessarily explicitly known.* Hence, Eqn. (2.15) = (B.1) may not necessarily be explicitly solvable. In contrast, we think of the function $\tilde{q}(\,\cdot\,)$ as *being explicitly known* and obtained through the arguments of Corollary 3.1, Corollary 3.2, so that $\tilde{q}(\,\cdot\,)$ satisfies, moreover, the following critical property:

$$0 < \tilde{q}(x) \leq q(x) \text{ near } x = 0, \text{ say } 0 < x \leq x_0 = \text{small}, \tag{B.5}$$

as in (3.4a) of Corollary 3.1; and (3.22a) of Corollary 3.2.

(iv) According to Theorem 2.2, in particular Eqn. (2.14), the solution $S(t)$ of (B.1) = (2.15) is responsible for controlling the decay rate of the energy $E(t)$ of the w-problem (2.1). However, as noted, $S(t)$ may not be explicitly computable. In contrast, the solution $\tilde{S}(t)$ of (B.2) is thought of as being explicitly computable, via separation of variables, since $\tilde{q}(x)$ is considered as being explicitly known: either by (3.4a–b) of Corollary 3.1; or by (3.22a–b) of Corollary 3.2.

Our task in this Appendix B is to compare the decay of $S(t)$ with the decay of $\tilde{S}(t)$ for $t > T$. More precisely, we have the following result:

Proposition B.1. *Consider the above setting for (B.1) and (B.2), in particular, properties (B.3), (B.4), (B.5). Then, we have*

$$S(t) \leq \tilde{S}(t), \qquad \forall\ t \geq T. \tag{B.6}$$

Proof. At the initial time T for the $\tilde{S}$-problem (B.2), we have $\tilde{S}(T) = S(T)$, hence via (B.2), (B.5), (B.1):

$$0 < -\tilde{S}_t(T) = \tilde{q}(\tilde{S}(T)) = \tilde{q}(S(T)) \leq q(S(T)) = -S_t(T), \tag{B.7}$$

or

$$S_t(T) \leq \tilde{S}_t(T) < 0, \text{ hence } S(t) \leq \tilde{S}(t) \text{ for all } t > T \text{ near } t = T, \text{ say } T \leq t \leq T_1. \tag{B.8}$$

Indeed, we have $T_1 = \infty$. In fact, assume by contradiction that there exists a time $t_1 > T_1$ in (B.8) such that

$$\tilde{S}(t_1) < S(t_1), \text{ hence } \tilde{S}(t) < S(t) \text{ for } T_1 < \tau < t \leq t_1, \text{ where } \tilde{S}(\tau) = S(\tau) \tag{B.9}$$

by continuity of the trajectories. We then repeat the argument in (B.7), this time at the time t, $\tau < t \leq t_1$, invoking, in addition to (B.2), (B.5), (B.1), also (B.9) and the property that $q(\cdot)$ is (strictly) increasing. We obtain for $\tau < t \leq t_1$:

$$-\tilde{S}_t(t) = \tilde{q}(\tilde{S}(t)) \leq q(\tilde{S}(t)) \leq q(S(t)) = -S_t(t), \tag{B.10}$$

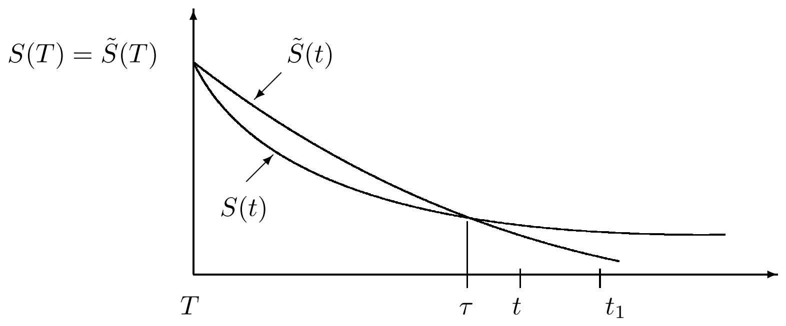
or

$$S_t(t) \leq \tilde{S}_t(t), \ \tau < t \leq t_1; \text{ hence } S(t) - S(\tau) \leq \tilde{S}(t) - \tilde{S}(\tau), \ \tau < t \leq t_1 \tag{B.11}$$

by integration. But $S(\tau) = \tilde{S}(\tau)$ by (B.9); hence (B.11) yields

$$S(t) \leq \tilde{S}(t), \quad \tau < t \leq t_1, \tag{B.12}$$

and (B.12) contradicts (B.9). Thus, we can take $T_1 = \infty$ in (B.8) and (B.6) is proved. $\square$



Appendix C: A Well-Posedness Result for an Abstract Schrödinger Feedback Problem

C.1 Abstract Result

Let $\mathcal{H}$ and $\mathcal{U}$ be two Hilbert spaces over the complex scalar field. We also consider the duality pairing $\mathcal{U}_0 \subset \mathcal{U} \subset \mathcal{U}_0'$, where $\mathcal{U}_0$ is another Hilbert space, with dense and continuous injections, where $'$ denotes duality with respect to $\mathcal{U}$ as a pivot space. The object of this Appendix C is the abstract differential inclusion of the form

$$y_t \in -i\mathcal{A}y - \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}y), \quad y(0) \in \mathcal{H}, \tag{C.1.0}$$

under the following *assumptions*:

(a$_1$) $\mathcal{A} : \mathcal{H} \supset \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is a linear, positive, self-adjoint operator;

(a$_2$) $\phi : \mathcal{U} \to \overline{R} =]-\infty, +\infty]$ is a convex, lower semi-continuous, proper function;

(a$_3$) $\mathcal{N} : \mathcal{U}_0' \to \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ is a linear bounded operator so that $\mathcal{A}^{\frac{1}{2}}\mathcal{N}$ is bounded $\mathcal{U}_0' \to \mathcal{H}$;

(a$_4$) the bounded linear operator $\mathcal{N}^*\mathcal{A}^{\frac{1}{2}} : \mathcal{H} \to \mathcal{U}_0$ is surjective, where $(\mathcal{N}u, h)_{\mathcal{H}} = (u, \mathcal{N}^*h)_{\mathcal{U}}$;

(a$_5$) $\partial\phi \subset \mathcal{U} \times \mathcal{U}$ denotes the sub-differential of the function ϕ: it is defined as follows for $u \in \mathcal{D}(\phi)$:

$$\partial\phi(u) = \{u^* \in \mathcal{U} : \phi(u) - \phi(v) \le \mathrm{Re}(u^*, u - v)_{\mathcal{U}}, \quad \forall\, v \in \mathcal{D}(\phi)\}; \tag{C.1.1}$$

(a$_6$) $\partial\phi$ satisfies the following relations:

$$\mathrm{Im}(\partial\phi(u), u)_{\mathcal{U}} \equiv 0; \quad \mathrm{Re}(\partial\phi(u), u)_{\mathcal{U}} \equiv (\partial\phi(u), u)_{\mathcal{U}}, \quad u \in \mathcal{D}(\partial\phi). \tag{C.1.2}$$

Since $\mathcal{U}_0$ is dense in $\mathcal{U}$, the sub-differential $\partial\phi$ may be viewed also as a subset of $\mathcal{U}_0 \times \mathcal{U}_0'$.

It follows from (a_2) via [B.1, Thm. 2.1, p. 54] that

$$\begin{aligned}&\partial\phi \text{ is a maximal monotone operator from either } \mathcal{U} \text{ to } \mathcal{U};\\ &\text{or else from } \mathcal{U}_0 \text{ to } \mathcal{U}_0'.\end{aligned} \tag{C.1.3}$$

Here monotonicity means

$$\mathrm{Re}(u_1 - u_2, \partial\phi(u_1) - \partial\phi(u_2))_{\mathcal{U}} \geq 0, \quad u_1, u_2 \in \mathcal{D}(\partial\phi), \tag{C.1.4}$$

where $(\quad,\quad)_{\mathcal{U}}$ denotes either the inner product on $\mathcal{U}$, or else the duality pairing from $\mathcal{U}_0$ to $\mathcal{U}_0'$.

Next, we define the operator (here $\mathcal{A}_F x$ means any $\xi \in$ set $\mathcal{A}_F x$):

$$(-\mathcal{A}_F)x = i\mathcal{A}x + \eta, \quad \eta \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]', \quad \eta \in \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}x), \tag{C.1.5a}$$

$$\begin{aligned}\mathcal{D}(-\mathcal{A}_F) = \{x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) :\ \exists\, \eta \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'\\ \text{s.t. } \eta \in \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}x),\ i\mathcal{A}x + \eta \in \mathcal{H}\},\end{aligned} \tag{C.1.5b}$$

where in (C.1.5a), $'$ denotes duality w.r.t. $\mathcal{H}$ as a pivot space. The following abstract theorem is the main result of the present subsection.

Theorem C.1.1. *Assume hypotheses (a_1)–(a_6) above. Then, the operator $(-\mathcal{A}_F) :\ \mathcal{H} \supset \mathcal{D}(\mathcal{A}_F) \to \mathcal{H}$ in (C.1.5) is maximal monotone, where monotonicity refers to $Re(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$.*

Proof. Step 1: Monotonicity of $(-\mathcal{A}_F) : \mathcal{D}(\mathcal{A}_F) \to \mathcal{H}$. Let $x, y \in \mathcal{D}(-\mathcal{A}_F) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. Let $u \in \partial\phi(\mathcal{N}^*\mathcal{A}x),\ v \in \partial\phi(\mathcal{N}^*\mathcal{A}y)$. Then, by (C.1.5a),

$$((-\mathcal{A}_F)x - (-\mathcal{A}_F)y, x - y)_{\mathcal{H}}$$

$$= (\mathcal{A}(ix + \mathcal{N}u) - \mathcal{A}(iy + \mathcal{N}v), x - y)_{\mathcal{H}} \tag{C.1.6}$$

$$= (\mathcal{A}^{\frac{1}{2}}(ix - iy) + \mathcal{A}^{\frac{1}{2}}\mathcal{N}(u - v), \mathcal{A}^{\frac{1}{2}}(x - y))_{\mathcal{H}} \tag{C.1.7}$$

$$= i\|\mathcal{A}^{\frac{1}{2}}(x - y)\|^2_{\mathcal{H}} + ((u - v), \mathcal{N}^*\mathcal{A}(x - y))_{\mathcal{U}}. \tag{C.1.8}$$

Taking real part in (C.1.8), we obtain

$$\begin{aligned}
&\mathrm{Re}((-\mathcal{A}_F)x-(-\mathcal{A}_F)y, x-y)_{\mathcal{H}} \\
&\quad = \mathrm{Re}(u-v, \mathcal{N}^*\mathcal{A}(x-y))_{\mathcal{U}} && \text{(C.1.9)} \\
&\quad = (\partial\phi(\mathcal{N}^*\mathcal{A}x)-\partial\phi(\mathcal{N}^*\mathcal{A}y), \mathcal{N}^*\mathcal{A}(x-y))_{\mathcal{U}} \geq 0, && \text{(C.1.10)}
\end{aligned}$$

where "$\geq$ 0" is obtained by monotonicity (C.1.4) of $\partial\phi$. Thus, $(-\mathcal{A}_F)$: $\mathcal{D}(\mathcal{A}_F) \to \mathcal{H}$ is monotone.

Step 2: Maximal monotonicity of $(-\mathcal{A}_F) : \mathcal{D}(\mathcal{A}_F) \to \mathcal{H}$. According to Minty's theorem [B.1, Thm. 1.2, p. 39], [B.2, Thm. 1.2, p. 43], in order to establish maximality, we need to prove the range (surjectivity) condition: range $(I-\mathcal{A}_F) = (I-\mathcal{A}_F)\mathcal{D}(\mathcal{A}_F) = \mathcal{H}$, I = identity on $\mathcal{H}$. In other words, given any $F \in \mathcal{H}$, we need to establish the existence of an element $x \in \mathcal{D}(\mathcal{A}_F)$, such that

$$Ix - \mathcal{A}_F x \ni F \in \mathcal{H}, \quad x \in \mathcal{D}(\mathcal{A}_F). \tag{C.1.11}$$

Step 2(i): Surjectivity of $(I-\mathcal{A}_F) : \mathcal{V}$ *onto* $\mathcal{V}'$. To this end, we shall first establish the related property that the operator $(I-\mathcal{A}_F)$ is surjective from $\mathcal{V} = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ onto $\mathcal{V}' = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ (duality w.r.t. $\mathcal{H}$): that is, that given any $f \in \mathcal{V}'$, there exists an element $x \in \mathcal{V}$ such that

$$Ix - \mathcal{A}_F x \ni f \in \mathcal{V}', \quad x \in \mathcal{V}. \tag{C.1.12}$$

From here, in Step 2(ii) below, we shall then deduce that, in fact, $(I-\mathcal{A}_F)$ is surjective from $\mathcal{D}(\mathcal{A}_F)$ onto $\mathcal{H}$, as required in (C.1.11). To establish the surjectivity $(I-\mathcal{A}_F) : V$ onto $\mathcal{V}'$ in (C.1.12), we rewrite the LHS operator via (C.1.5a) as

$$I - \mathcal{A}_F = (I + i\mathcal{A}) + \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}) : \ \mathcal{V} \to \mathcal{V}'. \tag{C.1.13}$$

This displays the following structure: (i) the operator $(I+i\mathcal{A}) : \ \mathcal{V} \to \mathcal{V}'$ is linear, continuous, maximal monotone by (a_1); (ii) in view of the surjectivity assumption (a_4): $\mathcal{N}^*\mathcal{A}^{\frac{1}{2}} : \mathcal{H} \to \mathcal{U}_0$, and of ($a_2$), we have that [L.1, Lem. 2.1, p. 345] (see also [Br.1, p. 33]) holds true and yields

$$\mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}) = \partial(\phi(\mathcal{N}^*\mathcal{A})) : \mathcal{V} \to \mathcal{V}', \tag{C.1.14}$$

where, by (a_2), $\phi(\mathcal{N}^*\mathcal{A}) : \mathcal{V} \to \overline{R}$ is convex, proper, lower semi-continuous. Then, accordingly, [B.1, Thm. 2.1, p. 54], the operator $\partial(\phi(\mathcal{N}^*\mathcal{A}))$ in (C.1.14) is maximal monotone $\mathcal{V} \to \mathcal{V}'$. It then follows from properties (i) and (ii), via any of the results in [B.1, Cor. 1.3, p. 48; Thm. 1.7, p. 46; Cor. 1.1, p. 39], [B.2, Cor. 1.1, p. 44; Thm. 1.5, p. 54], that the sum in (C.1.13)

$$I - \mathcal{A}_F : \mathcal{V} \to \mathcal{V}' \text{ is maximal monotone.} \tag{C.1.15}$$

Consequently, according to Minty's Theorem [B.1, Thm. 1.2, p. 39], for any $\lambda > 0$ the operator

$$\lambda J + (I - \mathcal{A}_F) = \lambda J + (I + i\mathcal{A}) + \partial(\phi(\mathcal{N}^*\mathcal{A})) : \text{ surjective } \mathcal{V} \text{ onto } \mathcal{V}' : \lambda > 0, \tag{C.1.16}$$

where J is the canonical injection $\mathcal{V}$ onto $\mathcal{V}'$, and hence can be taken to be $J = \mathcal{A}$.

Remark C.1.1. We remark that in addition to the maximal monotonicity property of $(I - \mathcal{A}_F)$: $\mathcal{V} \to \mathcal{V}'$ in (C.1.15), we also have that $-i(I - \mathcal{A}_F) = -iI + \mathcal{A} - i\partial(\phi(\mathcal{N}^*\mathcal{A}))$ is coercive $\mathcal{V} \to \mathcal{V}'$. If maximal monotonicity and coercivity were valid for the *same operator* (as is the case of the wave equation), then the desired conclusion of surjectivity $(I - \mathcal{A}_F)$: $\mathcal{V}$ onto $\mathcal{V}'$ would follow from Minty's Theorem [B.1, Thm. 1.3, p. 40], [B.2, Cor. 1.2, p. 45].

In light of Remark C.1.1, in the sequel, to establish the surjectivity result (C.1.12), we shall employ an approximating argument using (C.1.16) with $\lambda \searrow 0$.

Approximating argument. Let $\lambda > 0$ and let $f \in \mathcal{V}'$. By the surjectivity property (C.1.16), there exists $x_\lambda \in \mathcal{V}$ such that (with $J = \mathcal{A}$):

$$\lambda J x_\lambda + (I - \mathcal{A}_F)x_\lambda = \lambda J x_\lambda + x_\lambda + i\mathcal{A}x_\lambda + \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda) \ni f \in \mathcal{V}'. \tag{C.1.17}$$

Next we take the duality pairing $\mathcal{V}\times\mathcal{V}'$ of (C.1.17), where $J=\mathcal{A}$, with any $\psi\in\mathcal{V}$,

$$\begin{aligned}
&(\lambda J x_\lambda + (I-\mathcal{A}_F)x_\lambda,\psi)_{\mathcal{H}}\\
&\qquad = (\lambda J x_\lambda + x_\lambda,\psi)_{\mathcal{H}} + i(\mathcal{A}x_\lambda,\psi)_{\mathcal{H}} + (\partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda),\psi)_{\mathcal{H}}\\
&\qquad = (f,\psi)_{\mathcal{H}}, \qquad \psi\in\mathcal{V}. \qquad\qquad \text{(C.1.18)}
\end{aligned}$$

Setting, for now, $\psi = x_\lambda\in\mathcal{V}$, taking the Im part of (C.1.18), and invoking (a_1)–(a_6) and (C.1.14), we obtain

$$(\mathcal{A}x_\lambda,x_\lambda)_{\mathcal{H}} = \|\mathcal{A}^{\frac{1}{2}}x_\lambda\|^2_{\mathcal{H}} = \mathrm{Im}(f,x_\lambda)_{\mathcal{H}} \le \|f\|_{\mathcal{V}'}\|x_\lambda\|_{\mathcal{V}}. \qquad \text{(C.1.19)}$$

Since $\mathcal{V}\equiv\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, $\|x_\lambda\|_{\mathcal{V}} = \|\mathcal{A}^{\frac{1}{2}}x_\lambda\|_{\mathcal{H}}$, then (C.1.19) yields:

$$\begin{cases}
\|x_\lambda\|_{\mathcal{V}} \le \|f\|_{\mathcal{V}'}, \quad \forall\ \lambda>0, \text{ hence} & \text{(C.1.20a)}\\
x_\lambda \to \text{ some } x\in\mathcal{V} \text{ weakly in } \mathcal{V}, \text{ as } \lambda\searrow 0, & \text{(C.1.20b)}
\end{cases}$$

for a subsequence, still denoted by x_λ. Next, define

$$x^*_\lambda \in \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda)\in\mathcal{V}' \text{ continuously in } x_\lambda\in\mathcal{V}, \qquad \text{(C.1.21)}$$

by (C.1.14). Hence, (C.1.21) implies via (C.1.20a)

$$\begin{cases}
\|x^*_\lambda\|_{\mathcal{V}'} \le c\|x_\lambda\|_{\mathcal{V}} \le c\|f\|_{\mathcal{V}'}, \quad \lambda>0, \text{ hence} & \text{(C.1.22a)}\\
x^*_\lambda \to \text{ some } x^*\in\mathcal{V}' \text{ weakly in } \mathcal{V}', \text{ as } \lambda\searrow 0, & \text{(C.1.22b)}
\end{cases}$$

for a subsequence, still denoted by x^*_λ.

The limit process. Using (C.1.20a–b), we obtain with $J=\mathcal{A}$ and $\psi\in\mathcal{V}=\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, as $\lambda\searrow 0$:

$$|(\lambda J x_\lambda,\psi)_{\mathcal{H}}| \le \lambda\|x_\lambda\|_{\mathcal{V}}\|\psi\|_{\mathcal{V}} \le \lambda\|f\|_{\mathcal{V}'}\|\psi\|_{\mathcal{V}} \to 0; \qquad \text{(C.1.23)}$$

$$(x_\lambda,\psi)_{\mathcal{H}} + i(\mathcal{A}x_\lambda,\psi)_{\mathcal{H}} = (x_\lambda,\mathcal{A}^{-1}\psi)_{\mathcal{V}} + i(x_\lambda,\psi)_{\mathcal{V}} \to (x,\psi)_{\mathcal{H}} + i(\mathcal{A}x,\psi)_{\mathcal{H}}. \qquad \text{(C.1.24)}$$

Moreover, using (C.1.21) and (C.1.22b), we obtain with $\psi\in\mathcal{V}$, hence $\mathcal{A}\psi\in\mathcal{V}'$, as $\lambda\searrow 0$:

$$(\partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda),\psi)_{\mathcal{H}} = (x^*_\lambda,\psi)_{\mathcal{H}} = (x^*_\lambda,\mathcal{A}\psi)_{\mathcal{V}'} \to (x^*,\psi)_{\mathcal{H}}. \qquad \text{(C.1.25)}$$

Thus, letting $\lambda \searrow 0$ in (C.1.18) and invoking (C.1.23)–(C.1.25), we obtain

$$(Ix + i\mathcal{A}x, \psi)_{\mathcal{H}} + (x^*, \psi)_{\mathcal{H}} = (f, \psi), \quad \forall\, \psi \in \mathcal{V}, \tag{C.1.26}$$

where $x \in \mathcal{V}$ and $x^* \in \mathcal{V}'$ are defined in (C.1.20b) and (C.1.22b), respectively.

Identification $x^* \in \partial(\phi(\mathcal{N}^*\mathcal{A}))(x)$**.** What remains to be done to obtain (C.1.12) from (C.1.26) is to establish the identification that the limit $x^* \in \mathcal{V}'$ obtained in (C.1.22b) and the limit $x \in \mathcal{V}$ obtained in (C.1.20b) are linked to each other by the relationship

$$x^* \in \partial(\phi(\mathcal{N}^*\mathcal{A}))(x) = \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}x), \tag{C.1.27}$$

recalling (C.1.14). To establish (C.1.27), we seek to apply [B.1, Lem. 1.3, p. 42], [B.2, Lem. 1.3, p. 49]. To this end, it suffices to establish that for $x_\lambda, x_\nu \in \mathcal{V}$, $x_\lambda^*, x_\nu^* \in \mathcal{V}'$, we have:

$$\lim_{\lambda,\nu\searrow 0}(x_\lambda - x_\nu, x_\lambda^* - x_\nu^*)_{\mathcal{H}} = 0,\ \ x_\lambda^* \in \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda),\ \ x_\nu^* \in \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\nu), \tag{C.1.28}$$

after which, then, (C.1.27) is proved. Here, by definition of x_λ and x_ν, we have:

$$\lambda J x_\lambda + x_\lambda + i\mathcal{A}x_\lambda + \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda) \quad \ni \quad f \in \mathcal{V}',\ \ \lambda > 0 \tag{C.1.29}$$

$$\nu J x_\nu + x_\nu + i\mathcal{A}x_\nu + \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\nu) \quad \ni \quad f \in \mathcal{V}',\ \ \nu > 0. \tag{C.1.30}$$

Subtracting the second relation from the first, taking the duality pairing $\mathcal{V}\times\mathcal{V}'$ with $(x_\lambda - x_\nu)$, and finally taking the Re part of the resulting expression yields

$$\begin{aligned}(\lambda J x_\lambda - \nu J x_\nu, x_\lambda - x_\nu)_{\mathcal{H}} + \|x_\lambda - x_\nu\|^2_{\mathcal{H}} + \mathrm{Re}(\partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda)& \\ - \ \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\nu), x_\lambda - x_\nu)_{\mathcal{H}} = 0.&\end{aligned} \tag{C.1.31}$$

Arguing as in (C.1.23), we readily obtain via (C.1.20a),

$$|(\lambda J x_\lambda - \nu J x_\nu, x_\lambda - x_\nu)_{\mathcal{H}}| \le (\lambda + \nu)\|f\|_{\mathcal{V}'}[2\|f\|_{\mathcal{V}'}] \to 0, \tag{C.1.32}$$

as $\lambda, \nu \searrow 0$. Thus, returning to (C.1.31), we obtain

$$\lim_{\lambda,\nu\searrow 0}\{\|x_\lambda - x_\nu\|^2_{\mathcal{H}} + \mathrm{Re}(\partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\lambda) - \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\nu), x_\lambda - x_\nu)_{\mathcal{H}}\} = 0. \tag{C.1.33}$$

But, by (C.1.14) and monotonicity (C.1.4), each term in (C.1.33) is non-negative and hence each of the two terms in (C.1.33) has limit equal to zero. In particular, we obtain by (a_6),

$$\begin{aligned} 0 &= \lim_{\lambda,\nu\searrow 0} \mathrm{Re}(\partial(\phi(\mathcal{N}^*\mathcal{A})(x_\lambda) - \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\nu), x_\lambda - x_\nu)_H && \text{(C.1.34)} \\ &= \lim_{\lambda,\nu\searrow 0} (\partial(\phi(\mathcal{N}^*\mathcal{A})(x_\lambda) - \partial(\phi(\mathcal{N}^*\mathcal{A}))(x_\nu), x_\lambda - x_\nu)_H && \text{(C.1.35)} \\ &= \lim_{\lambda,\nu\searrow 0} (x^*_\lambda - x^*_\nu, x_\lambda - x_\nu), && \text{(C.1.36)} \end{aligned}$$

and (C.1.28) is proved via (C.1.36). Thus, (C.1.27) is proved. Using now (C.1.27) in (C.1.26), we rewrite the latter via (C.1.14) as

$$(Ix + i\mathcal{A}x, \psi)_{\mathcal{H}} + (\partial(\phi(\mathcal{N}^*\mathcal{A}))(x), \psi)_{\mathcal{H}} = ((I - \mathcal{A}_F)x, \psi)_{\mathcal{H}} = (f, \psi)_{\mathcal{H}}, \ \forall\ \psi \in \mathcal{V}, \tag{C.1.37}$$

so that, then, the surjectivity property of $[I - \mathcal{A}_F]$: $\mathcal{V}$ onto $\mathcal{V}'$ in (C.1.12) is established, as desired, by virtue of (C.1.37).

Step 2(ii). (From (C.1.12) to (C.1.11)). Let now $f = F \in \mathcal{H} \subset \mathcal{V}'$. By Step 2(i), there exists $x \in \mathcal{V} = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ such that (C.1.12) holds true. It remains to show that such $x \in \mathcal{D}(\mathcal{A}_F)$, as defined in (C.1.5b). By (C.1.12) we have that

$$i\mathcal{A}x + \mathcal{A}\mathcal{N}\partial\phi(\mathcal{A}\mathcal{N}x) = \mathcal{A}[ix + \mathcal{N}\partial\phi(\mathcal{A}\mathcal{N}x)] \ni F - x \in \mathcal{H}, \tag{C.1.38}$$

hence $[ix + \mathcal{N}\partial\phi(\mathcal{A}\mathcal{N}x)] \in \mathcal{D}(\mathcal{A})$ with $x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. Hence, invoking (C.1.5b), we see that $x \in \mathcal{D}(\mathcal{A}_F)$, as desired. Thus, (C.1.12) implies (C.1.11), i.e., the surjectivity $(I - \mathcal{A}_F) :\ \mathcal{D}(\mathcal{A}_F)$ onto $\mathcal{H}$. Moreover, (C.1.33) likewise implies that $\{x_\lambda\}$ is a Cauchy sequence in $\mathcal{H}$ and hence, by (C.1.20b) and uniqueness of the limit, $x_\lambda \to x$ (strongly) in $\mathcal{H}$. Returning to (C.1.20a) with $f = F \in \mathcal{H}$, we take the limit and obtain

$$\|x\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}. \tag{C.1.39}$$

Theorem C.1.1 is proved.

C.2 The Nonlinear Schrödinger Equation

In this subsection, we show that the setting of Section C.1—in particular, Theorem C.1.1—applies to the following Schrödinger problem, with nonlinearities both in the interior and on the boundary of Ω. As in Section 1, let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be an open bounded domain with, say, C^2-boundary $\partial\Omega \equiv \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. In Ω, we consider the Schrödinger problem

$$\begin{cases} y_t = i\Delta y - f(y) & \text{in } Q \equiv (0,T] \times \Omega; & \text{(C.2.1a)} \\ y(0,\,\cdot\,) = y_0 \in L_2(\Omega) & & \text{(C.2.1b)} \\ y|_{\Sigma_0} \equiv 0, \ \dfrac{\partial y}{\partial \nu} = ig(y) & \text{in } \Sigma_k \equiv (0,T] \times \Gamma_k, \ k = 0, 1, & \text{(C.2.1c)} \end{cases}$$

which reduces to (2.1a–c) for $f \equiv 0$. Here, $f, g : \mathbb{C} \to \mathbb{C}$ are complex, single-valued continuous functions with $f(0) = g(0) = 0$. Moreover, $f, g : \ \mathbb{C} \to \mathbb{C}$ are assumed to be sub-differentials $\partial j_k : \mathbb{C} \to \mathbb{C}$:

$$g(z) \quad = \quad \partial j_1(z) = \{z^* \in \mathbb{C} : j_1(z) - j_1(v) \leq (z^*, z - v)_{\mathbb{C}} = \mathrm{Re}\{z^*(\bar{z} - \bar{v})\}\}; \tag{C.2.2}$$

$$f(z) \quad = \quad \partial j_2(z) = \{z^* \in \mathbb{C} : j_2(z) - j_2(v) \leq (z^*, z - v)_{\mathbb{C}} = \mathrm{Re}\{z^*(\bar{z} - \bar{v})\}\}; \tag{C.2.3}$$

of functions $j_k : \ \mathbb{C} \to \overline{\mathbb{R}} =]-\infty, \infty]$ which are lower semi-continuous, convex, proper, so that f and g are monotone functions in the following sense

$$\mathrm{Im}\{f(z)\bar{z}\} \equiv \mathrm{Im}\{g(z)\bar{z}\} \equiv 0; \tag{C.2.4a}$$

$$\mathrm{Re}\{(f(z) - f(v))(\bar{z} - \bar{v})\} > 0; \ \mathrm{Re}\{(g(z) - g(v))(\bar{z} - \bar{v})\} \geq 0, \quad \forall \ z, v \in \mathbb{C}; \tag{C.2.4b}$$

in particular, for $v = 0$, we obtain

$$\mathrm{Re}\{f(z)\bar{z}\} = f(z)\bar{z} = |f(z)\bar{z}| \geq 0; \text{ and } \mathrm{Re}\{g(z)\bar{z}\} = g(z)\bar{z} = |g(z)\bar{z}| \geq 0 \tag{C.2.5}$$

(same as (2.2a–b) for g). Under the present setting, Theorem C.1.1 specializes then to the following (semigroup) well-posedness and regularity result for the Schrödinger problem (C.2.1a–c).

Theorem C.2.1. *Assume the above setting, in particular assumption (C.2.4a–b). Then, the following results hold true for the y-problem (C.2.1):*

(a) (well-posedness) For any initial condition $y_0 \in L_2(\Omega)$*, problem (C.2.1) defines a unique (nonlinear semigroup) solution* $y(t; y_0)$ *satisfying*

$$y_0 \in L_2(\Omega) \Rightarrow y(\,\cdot\,; y_0) \in C([0, \infty); L_2(\Omega)) \tag{C.2.6}$$

continuously. The generator $\mathcal{A}_F$ *of the corresponding nonlinear (feedback) semigroup is given explicitly in (C.2.24) below.*

(b) (regularity) Let, in particular, $y_0 \in H^2(\Omega)$ *satisfy the following compatibility conditions:*

$$y_0 \in H^2(\Omega): \; y_0|_{\Gamma_0} = 0, \; \left.\frac{\partial y_0}{\partial \nu}\right|_{\Gamma_1} = ig(y_0), \; \textit{so that } y_0 \in \mathcal{D}(\mathcal{A}_F), \tag{C.2.7}$$

the domain of the generator $\mathcal{A}_F$*. Moreover, let* f *be bounded from* $H^1(\Omega)$ *to* $L_2(\Omega)$*. Then, the corresponding unique solution* $y(t; y_0)$ *guaranteed by part (a) satisfies*

$$y(\,\cdot\,; y_0) \in C([0, \infty); \mathcal{D}(\mathcal{A}_F)), \quad \mathcal{D}(\mathcal{A}_F) \subset H^1_{\Gamma_0}(\Omega). \tag{C.2.8}$$

Remark C.2.1. This specialized Theorem C.2.1 greatly extends the well-posedness result in [Li.1, Thm. 10.1, p. 133] where $g \equiv 0$ and f has the special form: $f(z) = |z|^p z$, $p > 0$. Recall (2.25) and Example 3.1.

Proof. We need to put the Schrödinger problem (C.2.1) into the abstract model (C.1.0) covered by Theorem C.1.1.

Step 1. To this end, we define the following spaces and operators:

(i)

$$\mathcal{H} \equiv L_2(\Omega); \; \mathcal{U} \equiv L_2(\Gamma) \times L_2(\Omega); \; \mathcal{U}_0 \equiv H^{\frac{1}{2}}(\Gamma) \times H^1; \tag{C.2.9}$$

(ii)

$$\mathcal{A}y = -\Delta y; \; \mathcal{D}(\mathcal{A}) = \left\{ y \in H^2(\Omega): \; y|_{\Gamma_0} = 0, \; \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_1} = 0 \right\} \tag{C.2.10}$$

(same as the operator A in (1.8));

(iii) the Neumann map N in (1.9), so that

$$N^*\mathcal{A}\chi = \begin{cases} 0 & \text{on } \Gamma_0, \\ \chi|_{\Gamma_1} & \text{on } \Gamma_1, \end{cases} \quad \text{for } \chi \in H^1_{\Gamma_0}(\Omega) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}). \tag{C.2.11}$$

Next, we introduce Hilbert space-valued functions generated by j_1, j_2:

$$\phi_1 : L_2(\Gamma) \to \overline{R},\ \phi_1(u) = \begin{cases} \int_\Gamma j_1(u(x))dx & \text{if } j_1(u) \in L_1(\Gamma) \\ \infty & \text{otherwise;} \end{cases} \tag{C.2.12}$$

$$\phi_2 : L_2(\Gamma) \to \overline{R},\ \phi_2(u) = \begin{cases} \int_\Gamma j_2(u(x))dx & \text{if } j_2(u) \in L_2(\Gamma) \\ \infty & \text{otherwise;} \end{cases} \tag{C.2.13}$$

It is well known [B.1, Prop. 2.8, p. 61] that the functions ϕ_1, ϕ_2 are convex, lower semi-continuous, and proper, due to the assumed properties of j_1, j_2. Hence [B.1, Thm. 2.1, p. 54] the corresponding sub-differentials satisfy (on complex spaces):

$$\partial\varphi_1 : \quad L_2(\Gamma) \supset \mathcal{D}(\partial\varphi_1) \to L_2(\Gamma) \text{ is maximal monotone;} \tag{C.2.14}$$

$$\partial\varphi_2 : \quad L_2(\Omega) \supset \mathcal{D}(\partial\varphi_2) \to L_2(\Omega) \text{ is maximal monotone;} \tag{C.2.15}$$

$$\mathcal{D}(\partial\varphi_1) = \{u \in L_2(\Gamma) : j_1(u(x)) \in L_1(\Gamma);\ \partial j_1(u) \in L_2(\Gamma)\}; \tag{C.2.16}$$

$$\mathcal{D}(\partial\varphi_2) = \{u \in L_2(\Omega) : j_2(u(x)) \in L_1(\Omega);\ \partial j_2(u) \in L_2(\Omega)\}. \tag{C.2.17}$$

Moreover, for $u \in \mathcal{D}(\partial\varphi_k)$, we have

$$\partial\varphi_k(x) = \partial j_k(u(x)),\ \text{a.e. } k = 1, 2. \tag{C.2.18}$$

Step 2. In view of Step 1, the y-Schrödinger problem (C.2.1) can be rewritten abstractly as

$$y_t = -i\mathcal{A}y - \mathcal{A}N\partial\phi_1(N^*\mathcal{A}y) - \partial\phi_2(y). \tag{C.2.19}$$

Step 3. To put (C.2.19) into the definitive abstract form (C.1.0), we next introduce

(i)

$$\phi : \mathcal{U} \to \overline{\mathbb{R}}, \text{ defined by } \phi \begin{bmatrix} g \\ y \end{bmatrix} = \phi_1(g) + \phi_2(y); \tag{C.2.20}$$

(ii) $\mathcal{N} : \mathcal{U} \to \mathcal{H} \equiv L_2(\Omega)$ defined by

$$\mathcal{N} \begin{bmatrix} g \\ y \end{bmatrix} = Ng + \mathcal{A}^{-1}y, \; g \in L_2(\Gamma), \; y \in L_2(\Omega). \tag{C.2.21}$$

The function $\phi : \mathcal{U} \to \overline{\mathbb{R}}$ is convex, lower semicontinuous, and proper. Moreover, one readily verifies that

$$\mathcal{N}^*\mathcal{A}y = \begin{bmatrix} N^*\mathcal{A}y \\ y \end{bmatrix}; \quad \partial\phi \begin{bmatrix} g \\ y \end{bmatrix} = \begin{bmatrix} \partial\phi_1(g) \\ \partial\phi_2(y) \end{bmatrix}. \tag{C.2.22}$$

In the above notation, we further have

$$\begin{aligned} \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}y) &= \mathcal{A}N\partial\phi_1(N^*\mathcal{A}y) + \mathcal{A}\mathcal{A}^{-1}\partial\phi_2(\mathcal{A}\mathcal{A}^{-1}y) \\ &= \mathcal{A}N\partial\phi_1(N^*\mathcal{A}y) + \partial\varphi_2(y). \end{aligned} \tag{C.2.23}$$

Using (C.2.23) in (C.2.19) yields the following abstract model of the y-Schrödinger problem (C.2.1):

$$y_t = -i\mathcal{A}y - \mathcal{A}\mathcal{N}\partial\phi(\mathcal{N}^*\mathcal{A}y) = (-A_F)y, \tag{C.2.24}$$

which is precisely the required abstract model (C.1.0).

Step 4. In order to invoke Theorem C.1.1, we need to verify the surjectivity condition (a_4)

$$\mathcal{N}^*\mathcal{A}^{\frac{1}{2}} : \quad \text{bounded } \mathcal{H} \text{ onto } \mathcal{U}_0, \text{ equivalently,} \tag{C.2.25}$$

$$\mathcal{N}^*\mathcal{A} : \quad \text{bounded } H^1(\Omega) \text{ onto } H^{\frac{1}{2}}(\Gamma) \times H^1(\Omega), \tag{C.2.26}$$

recalling (C.2.9). But by (C.2.22), we have

$$\mathcal{N}^*\mathcal{A}y = [N^*\mathcal{A}y, y] = [y|_{\Gamma_1}, y], \quad y \in H^1_{\Gamma_0}(\Omega), \tag{C.2.27}$$

so that properties (C.2.25) hold true, since the trace $y \in H^1_{\Gamma_0}(\Omega) \to y|_{\Gamma_1} \in H^{\frac{1}{2}}(\Gamma_1)$ is bounded and surjective.

Step 5. All the hypotheses (a_1)–(a_6) of Theorem C.1.1 hold true for the y-problem (C.2.1). Applying Theorem C.1.1, we obtain Theorem C.2.1. □

Appendix D

This Appendix D collects material on the L_p-elliptic theory that is needed in Section 4—the key result is (D.20)—and which is not readily available in the literature. It is beyond [M-S.1] and [T.1]. We refer to [Tay.1, Chapter 3]. We warmly thank Marius Mitrea (Math Department, U. of Missouri, Columbia) and Irina Mitrea (Math Department, U. of Virginia) for the present appendix.

Notation, preliminaries. Let

$$E(x) = \begin{cases} \dfrac{c_n}{|x|^{n-2}}, & n = 3, 4, \dots \qquad \text{(D.1a)} \\ \dfrac{1}{4\pi} \ln |x|, & n = 2. \qquad \text{(D.1b)} \end{cases}$$

(1) We introduce the *boundary* $\to$ *domain operators* (single and double layer potentials)

$$(\mathcal{S}f)(x) = \int_{\partial\Omega} E(x-y) f(y) d\sigma_y, \quad x \in \Omega; \tag{D.2}$$

$$(\mathcal{D}f)(x) = \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)] f(y) d\sigma_y, \quad x \in \Omega. \tag{D.3}$$

(2) We introduce the *boundary* $\to$ *boundary operators*

$$(Sf)(x) = \int_{\partial\Omega} E(x-y) f(y) d\sigma_y, \quad x \in \partial\Omega; \tag{D.4}$$

$$(Kf)(x) = \text{p.v.} \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)] f(y) d\sigma_y, \quad x \in \partial\Omega. \tag{D.5}$$

(3) The *jump relations* are

$$\mathcal{S}f|_{\partial\Omega} = Sf, \qquad \mathcal{D}f|_{\partial\Omega} = \left(\frac{1}{2} I + K\right) f; \tag{D.6}$$

$$\partial_\nu \mathcal{S} f|_{\partial\Omega} = \Big(-\frac{1}{2}\, I + K^* \Big) f. \tag{D.7}$$

(4) The solution of the Dirichlet Boundary Value Problem (BVP):

$$\begin{cases} \Delta v = 0 & \text{in } \Omega; \quad \text{(D.8a)} \\ v|_{\partial\Omega} = f & \text{in } \partial\Omega, \quad \text{(D.8b)} \end{cases}$$

is

$$v = \mathcal{D}\Big[\Big(\frac{1}{2}\, I + K^* \Big)^{-1} f \Big], \tag{D.9}$$

with corresponding Dirichlet-Neumann map

$$\partial_\nu v|_{\partial\Omega} \equiv \Lambda_{DN}(f) = \Big(-\frac{1}{2}\, I + K^* \Big)(S^{-1} f) \tag{D.10}$$

(denoted $\mathcal{N}$ in [Tay.1, Eqn. (4.94), p. 171].

(5) The solution of the Neumann BVP:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \quad \text{(D.11a)} \\ \partial_\nu u = g & \text{in } \partial\Omega, \quad \text{(D.11b)} \end{cases}$$

is

$$u = \mathcal{S}\Big[\Big(-\frac{1}{2}\, I + K^* \Big)^{-1} g \Big], \tag{D.12}$$

with corresponding Neumann-Dirichlet map

$$u|_{\partial\Omega} \equiv \Lambda_{ND}(g) = S\Big[\Big(-\frac{1}{2}\, I + K^* \Big)^{-1} g \Big]. \tag{D.13}$$

(6) Boundedness of K, K^*, S (boundary $\to$ boundary):

(a)

$$K, K^* : \text{ bounded } L^p(\partial\Omega) \to L^p(\partial\Omega),\ 1 < p < \infty,\ \partial\Omega = \text{Lipschitz}; \tag{D.14}$$

(b)

$$K : \text{ bounded } W^{1,p}(\partial\Omega) \to W^{1,p}(\partial\Omega),\ 1 < p < \infty,\ \partial\Omega = \text{Lipschitz}; \tag{D.15}$$

(c)

$$K^* : \text{ bounded } W^{-1,p}(\partial\Omega) \to W^{-1,p}(\partial\Omega),\ 1 < p < \infty,\ \partial\Omega = \text{Lipschitz}; \tag{D.16}$$

(d)

$$S : \text{ bounded } L^p(\partial\Omega) \to W^{1,p}(\partial\Omega),\ 1 < p < \infty,\ \partial\Omega = \text{Lipschitz}. \tag{D.17}$$

(7) Isomorphism (bounded and boundedly invertible) from boundary to boundary:

(a)

$$-\frac{1}{2}I + K^* : \text{ isomorphic } L^p_{\text{zero}}(\partial\Omega) \simeq L^p(\partial\Omega), \tag{D.18}$$

$$\begin{aligned} \text{if: } \partial\Omega = \text{Lipschitz} \quad &\text{and} \quad 1 < p \leq 2; \\ \partial\Omega \in C^1 \quad &\text{and} \quad 1 < p < \infty, \end{aligned}$$

where the subindex "zero" in $f \in L^p_{\text{zero}}(\partial\Omega)$ means $\int_{\partial\Omega} f = 0$.

(b)

$$S : \text{ isomorphic } L^p_{\text{zero}}(\partial\Omega) \simeq W^{1,p}(\partial\Omega)/\mathbb{R}, \tag{D.19}$$

$$\begin{aligned} \text{if: } \partial\Omega = \text{Lipschitz} \quad &\text{and} \quad 1 < p \leq 2; \\ \partial\Omega \in C^1 \quad &\text{and} \quad 1 < p < \infty. \end{aligned}$$

(c)

$$\Lambda_{ND} : \text{ isomorphic } L^p_{\text{zero}}(\partial\Omega) \simeq W^{1,p}(\partial\Omega)/\mathbb{R}, \tag{D.20}$$

$$\begin{aligned} \text{if: } \partial\Omega = \text{Lipschitz} \quad &\text{and} \quad 1 < p \leq 2; \\ \partial\Omega \in C^1 \quad &\text{and} \quad 1 < p < \infty. \end{aligned}$$

For $1 > p = \frac{4}{r} = \frac{4}{3-\epsilon} > \frac{4}{3}$, (C.20) yields the result needed and quoted in (4.19) and (4.26).

(8) $\mathcal{S}$ is smoothing of order $1 + \frac{1}{p}$:

$$\mathcal{S} : \qquad \text{bounded } W^{-1,p}(\partial\Omega) \to B^{p,p^*}_{\frac{1}{p}}(\Omega); \tag{D.21}$$

$$\mathcal{S} : \qquad \text{bounded } L^p(\partial\Omega) \to B^{p,p^*}_{1+\frac{1}{p}}(\Omega); \tag{D.22}$$

$$p^* = \max\{p, 2\}, \quad 1 < p < \infty.$$

(9) $\mathcal{D}$ is smoothing of order $\frac{1}{p}$, $1 < p < \infty$:

$$\mathcal{D}: \qquad L^p(\partial\Omega) \to B^{p,p^*}_{\frac{1}{p}}(\Omega); \tag{D.23}$$

$$\mathcal{D}: \qquad W^{1,p}(\partial\Omega) \to B^{p,p^*}_{1+\frac{1}{p}}(\Omega). \tag{D.24}$$

(10) In particular, with reference to problem (D.11a-b) as needed in (4.19), where $2 > p = \frac{4}{r} = \frac{4}{3-\epsilon} > \frac{4}{3}$, we have $p^* = 2$ and thus formula (D.12) along with (D.18) and (D.22) yields

$$g \in L^p(\partial\Omega), \quad p = \frac{4}{r} = \frac{4}{3-\epsilon} \to u \in B^{p,2}_{1+\frac{1}{p}} \tag{D.25}$$

for the solution u of problem (D.11a-b).

This *optimal* interior regularity result (D.25) is not quite enough, however, to produce *just by trace theory* the result needed in (4.19)—i.e., (4.20).

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