Intersection Type Systems for Explicit Substitution Calculi

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Talk's Plan

The λ-calculus

Proposed by Church in 1932. [Church32]

$$\textit{Terms} \qquad M := x \,|\, (M\ M) \,|\, \lambda_x.M$$

Computations (reductions) are made by a unique rule:

$$(\lambda_x.M\ N) \longrightarrow M\{x := N\} \qquad (\beta)$$

Some renaming may be necessary:

$$\lambda_x.M \longrightarrow \lambda_y.M\{x := y\} \qquad (\alpha)$$

Foundation for the Lisp and functional programming languages in general.

The λ-calculus with de Bruijn indices (λ_{dB})

Invented by N.G. de Bruijn[dB72].

Terms $\quad M ::= \underline{n} \mid (M\ M) \mid \lambda.M \quad$ for $n \in \mathbb{N}_* = \mathbb{N} \setminus \{0\}$

Examples

$\lambda.(\lambda.(\underline{1}\ \underline{4}\ \underline{2})\ \underline{1})$

$\lambda.\underline{1} \simeq \lambda x.x \simeq \lambda y.y$

Definition (β-contraction)

$$(\lambda.M\ N) \rhd_\beta \{\underline{1}/N\}M$$

Typing Systems

Simply typed λ-calculus proposed by Church.[Church40]

Classify objects (terms) in the formal system.

$\lambda_{x:int}.x : int \rightarrow int \quad \lambda_{x:bol}.x : bool \rightarrow bool \quad$ (*à la Church*)

$\lambda_{x}.x : int \rightarrow int \quad \lambda_{x}.x : bool \rightarrow bool \quad$ (*à la Curry*)

STLC is related to IPL: Curry-Howard(-de Bruijn) Isomorphism.

If $M : \langle \Gamma \vdash \tau \rangle$ then $\langle \Gamma \vdash \tau \rangle$ is called a typing of M.

Simple types for λ_{dB}

Definition (Simple Types and Contexts)

Types $\sigma, \tau \in \mathcal{S} ::= \mathcal{A} \,|\, \mathcal{S} \to \mathcal{S}$ **Contexts** $\Gamma ::= nil \,|\, \sigma.\Gamma$

System $\lambda_{dB}^{\to}$

$$\underline{1} : \langle \tau.\Gamma \vdash \tau \rangle \ (\text{var}) \qquad \frac{M : \langle \sigma.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash \sigma \to \tau \rangle} \to_i$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \sigma.\Gamma \vdash \tau \rangle} \ (\text{varn}) \qquad \frac{M : \langle \Gamma \vdash \sigma \to \tau \rangle \qquad N : \langle \Gamma \vdash \sigma \rangle}{(M\ N) : \langle \Gamma \vdash \tau \rangle} \to_e$$

Typing Systems Properties

- Subject Reduction (SR)

  If $M : \langle \Gamma \vdash \tau \rangle$ and $M \rightarrow_\beta M'$, then $M' : \langle \Gamma \vdash \tau \rangle$

- Strong or Weak Normalisation (SN and WN) for typeable term.
- Type Inference (M : ?)
- Principal Typing (PT)
- Inhabitation Problem (? : $\langle \Gamma \vdash \tau \rangle$)

Intersection type discipline

- Introduced by Coppo and Dezani-Ciancaglini. [CDC78, CDC80]
- Characterisation of the SN terms of the λ-calculus. [Pottinger80]
- It incorporates type polymorphism in a finitary way:

$$\lambda_x.x : (int \to int) \wedge (bool \to bool)$$

- PT has been verified in IT systems. [Bakel95, SM96a, KW04]
- Exists IT systems for explicit substitution (ES) calculi, e.g. λx [LLDDvB2004]
- There is no IT system for calculi with *composition of substitutions*. (e.g. $\lambda\sigma$ and λs_e)

Explicit Substitution

- Substitution is a *meta level* operation in the λ-calculus

$$(\lambda_x.M\ N) \longrightarrow_\beta M\{x := N\}$$

- Calculi with Explicit Substitutions extends it:

$$(\lambda_x.M\ N) \longrightarrow_{(Beta)} M\langle x := N\rangle$$

▸ JumpES

- There are different approaches of expliciting the substitutions.

The $\lambda\sigma$-calculus [ACCL91]

- Well studied and important application on HOU. [DHK95]
- Two-sorted calculus: Terms and Substitutions.
- Defined with n-ary substitutions.
- Allows *composition* (unrestricted composition).
- $\lambda\sigma$ uses de Bruijn indices.

$$(\lambda.M\ N) \longrightarrow M[N.id] \qquad (Beta)$$

The λs_e-calculus [KR97]

- Natural extension for the λ-calculus à la de Bruijn.
- Has application on HOU [AK01].
- Extension of λs [KR95], allowing composition.
- One-sorted calculus: Terms.
- Introduces operators to realise substitutions and updatings.

$$(\lambda.M\ N) \longrightarrow M\sigma^1 N \qquad (\sigma\text{-generation})$$

Restricted Intersection types in λ_{dB}

Definition (Restricted intersection types and contexts)

1. The **restricted intersection types** are defined by:

$$\tau, \sigma \in \mathcal{T} ::= \mathcal{A} \,|\, \mathcal{U} \to \mathcal{T}$$
$$u, v \in \mathcal{U} ::= \omega \,|\, \mathcal{U} \wedge \mathcal{U} \,|\, \mathcal{T}$$

$\wedge$ is commutative, associative and has ω as neutral element.

2. **Contexts**: $\Gamma ::= nil \,|\, u.\Gamma$ s.t. $u \in \mathcal{U}$

$$nil \wedge \Gamma = \Gamma \wedge nil = \Gamma \quad (u_1.\Gamma) \wedge (u_2.\Delta) = (u_1 \wedge u_2).(\Gamma \wedge \Delta)$$

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The system λ^{SM}_{dB} [VAK2010]

System λ^{SM}_{dB}

$$\frac{\tau \in \mathcal{T}}{\underline{1} : \langle \tau.nil \vdash \tau \rangle}\ \text{var} \qquad \frac{M : \langle nil \vdash \tau \rangle}{\lambda.M : \langle nil \vdash \omega \to \tau \rangle}\ \to_i'$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \omega.\Gamma \vdash \tau \rangle}\ \text{varn} \qquad \frac{M : \langle u.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash u \to \tau \rangle}\ \to_i$$

$$\frac{M_1 : \langle \Gamma \vdash \omega \to \tau \rangle \qquad M_2 : \langle \Delta \vdash \sigma \rangle}{(M_1\ M_2) : \langle \Gamma \wedge \Delta \vdash \tau \rangle}\ \to_e'$$

$$\frac{M_1 : \langle \Gamma \vdash \wedge_{i=1}^{n} \sigma_i \to \tau \rangle \quad M_2 : \langle \Delta^1 \vdash \sigma_1 \rangle \ \ldots \ M_2 : \langle \Delta^n \vdash \sigma_n \rangle}{(M_1\ M_2) : \langle \Gamma \wedge \Delta^1 \wedge \cdots \wedge \Delta^n \vdash \tau \rangle}\ \to_e$$

Examples

$$\frac{\dfrac{\underline{1} : \langle \alpha \to \beta.\mathit{nil} \vdash \alpha \to \beta \rangle \qquad \underline{1} : \langle \alpha.\mathit{nil} \vdash \alpha \rangle}{(\,\underline{1}\ \underline{1}\,) : \langle (\alpha \to \beta) \wedge \alpha.\mathit{nil} \vdash \beta \rangle}}{\lambda.(\,\underline{1}\ \underline{1}\,) : \langle \mathit{nil} \vdash (\alpha \to \beta) \wedge \alpha \to \beta \rangle}$$

$\Omega \equiv (\lambda.(\,\underline{1}\ \underline{1}\,)\ \lambda.(\,\underline{1}\ \underline{1}\,))$ is not typeable in λ_{dB}^{SM}:

$$(\alpha \to \beta) \wedge \alpha \ =^{?}\ (\alpha' \to \beta') \wedge \alpha' \to \beta'$$

System λ_{dB}^{SM} properties

Lemma (Relevance [VAK2010])
If $M : \langle \Gamma \vdash \tau \rangle$*, then* $|\Gamma| = sup(M)$ *and* $\forall i$, $\Gamma_i \neq \omega$ *iff* $\underline{i} \in FI(M)$.

- Every β-nf is typeable.
- A restricted version has PT for all β-nf.[VAK2010]
- SR for β-contraction (for relevant systems).

Subject Reduction (SR)
If $M : \langle \Gamma \vdash \tau \rangle$ and $M \to_\beta M'$, then $M' : \langle \Gamma \vdash \tau \rangle$

System λ_{dB}^{SM} properties

Lemma (Relevance [VAK2010])

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- Every β-nf is typeable.
- A restricted version has PT for all β-nf.[VAK2010]
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Subject Reduction (SR)

If $M : \langle \Gamma \vdash \tau \rangle$ and $M \to_\beta M'$, then $M' : \langle \Gamma \vdash \tau \rangle$

Definition (FI restriction)

Let $\Gamma\lfloor_M$ be a $\Gamma' \sqsubseteq \Gamma$ s.t. $|\Gamma'| = sup(M)$ and $\forall i$, $\Gamma'_i \neq \omega$ iff $\underline{i} \in FI(M)$.

Theorem (SR for β-contraction in λ_{dB}^{SM})

If $(\lambda.M\ N) : \langle \Gamma \vdash_{\lambda_{dB}^{SM}} \tau \rangle$ *then* $\{\underline{1}/N\}M : \langle \Gamma\lfloor_{\{\underline{1}/N\}M} \vdash_{\lambda_{dB}^{SM}} \tau \rangle$.

The system $\lambda^{\wedge}_{dB}$

System $\lambda^{\wedge}_{dB}$

$$\frac{\tau \in \mathcal{T}}{\underline{1} : \langle \tau.nil \vdash \tau \rangle}\ \text{var} \qquad \frac{M : \langle nil \vdash \tau \rangle}{\lambda.M : \langle nil \vdash \omega \to \tau \rangle}\ \to_i'$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \omega.\Gamma \vdash \tau \rangle}\ \text{varn} \qquad \frac{M : \langle u.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash u \to \tau \rangle}\ \to_i$$

$$\frac{M_1 : \langle \Gamma \vdash \omega \to \tau \rangle}{(M_1\ M_2) : \langle \Gamma \vdash \tau \rangle}\ \to_e^{\omega}$$

$$\frac{M_1 : \langle \Gamma \vdash \wedge_{i=1}^{n} \sigma_i \to \tau \rangle \quad M_2 : \langle \Delta^1 \vdash \sigma_1 \rangle \ \ldots \ M_2 : \langle \Delta^n \vdash \sigma_n \rangle}{(M_1\ M_2) : \langle \Gamma \wedge \Delta^1 \wedge \cdots \wedge \Delta^n \vdash \tau \rangle}\ \to_e$$

Lemma

If $M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ *and* $|\Gamma| = m > 0$ *then* $\Gamma_m \neq \omega$ *and* $\forall 1 \leq i \leq |\Gamma|,\ \Gamma_i \neq \omega$ *implies* $\underline{i} \in FI(M)$.

$\lambda\sigma$-calculus [ACCL91]

Syntax

Terms	$M ::= \underline{1} \mid (M\ M) \mid \lambda.M \mid M[S]$
Substitutions	$S ::= id \mid \uparrow \mid M.S \mid S \circ S$

- $\underline{n+1} \cong \underline{1}\,[\uparrow^n]$
- The β-reduction is simulated by the term $M[S]$(*closure*)

$$(\lambda.M\ N) \to M[N.id] \quad (Beta)$$

$\lambda\sigma$ rewriting rules

$(\lambda.M\ N)$	$\longrightarrow$	$M[N.id]$	$(Beta)$
$(M\ N)[S]$	$\longrightarrow$	$(M[S]\ N[S])$	(App)
$\underline{1}\,[M.S]$	$\longrightarrow$	M	$(VarCons)$
$M[id]$	$\longrightarrow$	M	(Id)
$(\lambda.M)[S]$	$\longrightarrow$	$\lambda.(M[\underline{1}.(S\circ\uparrow)])$	(Abs)
$(M[S])[S']$	$\longrightarrow$	$M[S \circ S']$	$(Clos)$
$id \circ S$	$\longrightarrow$	S	(IdL)
$\uparrow\circ(M.S)$	$\longrightarrow$	S	$(ShiftCons)$
$(S_1 \circ S_2) \circ S_3$	$\longrightarrow$	$S_1 \circ (S_2 \circ S_3)$	$(AssEnv)$
$(M.S) \circ S'$	$\longrightarrow$	$M[S'].(S \circ S')$	$(MapEnv)$
$S \circ id$	$\longrightarrow$	S	(IdR)
$\underline{1}\,.\uparrow$	$\longrightarrow$	id	$(VarShift)$
$\underline{1}\,[S].(\uparrow\circ S)$	$\longrightarrow$	S	$(Scons)$

Simple types for $\lambda\sigma$

The System $\lambda\sigma^{\rightarrow}$

Terms

$$\text{(var)} \quad \underline{1} : \langle \tau.\Gamma \vdash \tau \rangle \qquad \text{(app)} \quad \frac{M_1 : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle \qquad M_2 : \langle \Gamma \vdash \sigma \rangle}{(M_1 \; M_2) : \langle \Gamma \vdash \tau \rangle}$$

$$\text{(lambda)} \quad \frac{M : \langle \sigma.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle} \qquad \text{(clos)} \quad \frac{S : \langle \Gamma \rhd \Gamma' \rangle \qquad M : \langle \Gamma' \vdash \tau \rangle}{M[S] : \langle \Gamma \vdash \tau \rangle}$$

Substitutions

$$\text{(id)} \quad id : \langle \Gamma \rhd \Gamma \rangle \qquad \text{(cons)} \quad \frac{M : \langle \Gamma \vdash \tau \rangle \qquad S : \langle \Gamma \rhd \Gamma' \rangle}{M.S : \langle \Gamma \rhd \tau.\Gamma' \rangle}$$

$$\text{(shift)} \quad \uparrow : \langle \tau.\Gamma \rhd \Gamma \rangle \qquad \text{(comp)} \quad \frac{S : \langle \Gamma \rhd \Gamma' \rangle \qquad S' : \langle \Gamma' \rhd \Gamma'' \rangle}{S' \circ S : \langle \Gamma \rhd \Gamma'' \rangle}$$

$\lambda\sigma^{\rightarrow}$ Properties

- Has SR property.[ACCL91]
- Typeable terms are WN.[Munoz97]
- $\lambda\sigma^{\rightarrow}$ has PT. [VAK2008]
  (equivalence with Wells' PT)
- System $\lambda\sigma$ is not PSN:

  "If M is SN in λ then M is SN in $\lambda\sigma$"
  (Melliès' example [Mellies95])
- Unrestricted composition is the one to blame.[Ritter99]

The system $\lambda\sigma^{\wedge}$

Terms

$$\frac{}{\underline{1} : \langle \tau.nil \vdash \tau \rangle}\ (\text{var}) \qquad \frac{M : \langle u.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash u \to \tau \rangle}\ \to_i$$

$$\frac{M_1 : \langle \Gamma \vdash \omega \to \tau \rangle}{(M_1\ M_2) : \langle \Gamma \vdash \tau \rangle}\ \to_e^{\omega} \qquad \frac{M : \langle nil \vdash \tau \rangle}{\lambda.M : \langle nil \vdash \omega \to \tau \rangle}\ \to_i'$$

$$\frac{M_1 : \langle \Gamma \vdash \wedge_{i=1}^{n} \sigma_i \to \tau \rangle \qquad M_2 : \langle \Delta^1 \vdash \sigma_1 \rangle \ \ldots\ M_2 : \langle \Delta^n \vdash \sigma_n \rangle}{(M_1\ M_2) : \langle \Gamma \wedge \Delta^1 \wedge \cdots \wedge \Delta^n \vdash \tau \rangle}\ \to_e$$

$$\frac{S : \langle \Gamma \rhd \Gamma' \rangle \qquad M : \langle \Gamma' \vdash \tau \rangle}{M[S] : \langle \Gamma \vdash \tau \rangle}\ (\text{clos})$$

The system $\lambda\sigma^{\wedge}$

Substitutions

$$(\wedge\text{-cons})\ \frac{M:\langle\Delta^1 \vdash \sigma_1\rangle \ \ldots \ M:\langle\Delta^n \vdash \sigma_n\rangle \qquad S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Delta^1 \wedge \cdots \wedge \Delta^n \wedge \Delta \rhd (\wedge_{i=1}^n \sigma_i).\Delta'\rangle}$$

$$(\mathrm{id})\ \frac{\Gamma \neq \Delta.\omega^{\underline{m}}}{id:\langle\Gamma \rhd \Gamma\rangle} \qquad (\mathrm{comp})\ \frac{S:\langle\Gamma \rhd \Gamma''\rangle \qquad S':\langle\Gamma'' \rhd \Gamma'\rangle}{S' \circ S:\langle\Gamma \rhd \Gamma'\rangle}$$

$$(nil\text{-shift})\ \uparrow:\langle nil \rhd nil\rangle \qquad (nil\text{-cons})\ \frac{S:\langle\Delta \rhd nil\rangle}{M.S:\langle\Delta \rhd nil\rangle}$$

$$(\mathrm{shift})\ \frac{\Gamma \neq \Delta.\omega^{\underline{n}}}{\uparrow:\langle\omega.\Gamma \rhd \Gamma\rangle} \qquad (\omega\text{-cons})\ \frac{S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Delta \rhd \omega.\Delta'\rangle}, \Delta' \neq \omega^{\underline{n}}$$

System $\lambda\sigma^\wedge$ properties

Lemma

If $M : \langle \Gamma \vdash \tau \rangle$ *and* $|\Gamma| = m > 0$*, then* $\Gamma_m \neq \omega$*. In particular, if* $S : \langle \Gamma \rhd \Gamma' \rangle$ *and* $|\Gamma| = m > 0$ *then* $\Gamma_m \neq \omega$ *and if* $|\Gamma'| = m' > 0$ *then* $\Gamma'_{m'} \neq \omega$*.*

Lemma (Generation for substitutions in $\lambda\sigma^\wedge$)

1. $S : \langle nil \rhd nil \rangle$ *for any substitution* S*.*
2. *If* $M.S : \langle \Gamma \rhd nil \rangle$ *then* $S : \langle \Gamma \rhd nil \rangle$*.*
3. *If* $M.S : \langle \Gamma \rhd \omega.\Gamma' \rangle$ *then* $S : \langle \Gamma \rhd \Gamma' \rangle$ *and* $\Gamma' \neq \omega^{\underline{n}}$*.*
4. *If* $M.S : \langle \Gamma \rhd \Gamma' \rangle$ *for* $\Gamma' = \wedge_{i=1}^{m} \sigma_i.\Gamma''$ *then* $S : \langle \Gamma''' \rhd \Gamma'' \rangle$ *and* $\forall i$*,* $M : \langle \Gamma^i \vdash \sigma_i \rangle$ *s.t.* $\Gamma = \Gamma''' \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m$*.*

Lemma (Generation for substitutions in $\lambda\sigma^\wedge$(cont.))

5. *If* $S : \langle \Gamma \rhd nil \rangle$ *then* $\Gamma = nil$.
6. $\uparrow^m : \langle \Gamma \rhd \Gamma' \rangle$ *iff either* $\Gamma = \Gamma' = nil$ *or* $\Gamma = \omega^{\underline{m}}.\Gamma'$, *where* $\Gamma' \neq \Delta.\omega^{\underline{n}}$.
7. *If* $S : \langle \Gamma \rhd \Gamma' \rangle$ *and* $S : \langle \Delta \rhd \Delta' \rangle$ *then* $S : \langle \Gamma \wedge \Delta \rhd \Gamma' \wedge \Delta' \rangle$.
8. *If* $S : \langle \Gamma \rhd \Delta^1 \wedge \Delta^2 \rangle$ *for* $\Delta^1 \neq \Delta'.\omega^{\underline{m}}$ *and* $\Delta^2 \neq \Delta''.\omega^{\underline{m}}$, *then* $\Gamma = \Gamma^1 \wedge \Gamma^2$ *s.t.* $S : \langle \Gamma^1 \rhd \Delta^1 \rangle$ *e* $S : \langle \Gamma^2 \rhd \Delta^2 \rangle$.

Lemma (SR for $\lambda\sigma^\wedge$)

If $M : \langle \Gamma \vdash \tau \rangle$ *and* $M \longrightarrow_{\lambda\sigma} M'$ *then* $M' : \langle \Gamma \vdash \tau \rangle$. *Particularly, if* $S : \langle \Gamma \rhd \Gamma' \rangle$ *and* $S \longrightarrow_{\lambda\sigma} S'$ *then* $S' : \langle \Gamma \rhd \Gamma' \rangle$.

jumpEnd

λs-calculus [KR95]

Syntax

Terms $\quad M ::= \underline{n} \,|\, (M\ M) \,|\, \lambda.M \,|\, M\sigma^i M \,|\, \varphi^j_k M,$

where $n, i, j \in \mathbb{N}^*$ and $k \in \mathbb{N}$

- The β-substitution is simulated by the term $M\sigma^i N$

$$(\lambda.M\ N) \to M\sigma^1 N \quad (\sigma\text{-generation})$$

λs rewriting rules

$$
\begin{array}{llll}
(\lambda.M\ N) & \longrightarrow & M\sigma^1 N & (\sigma\text{-generation}) \\
(\lambda.M)\,\sigma^i N & \longrightarrow & \lambda.(M\sigma^{i+1}N) & (\sigma\text{-}\lambda\text{-transition}) \\
(M_1\ M_2)\sigma^i N & \longrightarrow & ((M_1\sigma^i N)\ (M_2\sigma^i N)) & (\sigma\text{-app-transition}) \\
\underline{n}\,\sigma^i N & \longrightarrow & \begin{cases} \underline{n-1} & \text{if } n > i \\ \varphi^i_0 N & \text{if } n = i \\ \underline{n} & \text{if } n < i \end{cases} & (\sigma\text{-destruction}) \\
\varphi^i_k(\lambda.M) & \longrightarrow & \lambda.(\varphi^i_{k+1} M) & (\varphi\text{-}\lambda\text{-transition}) \\
\varphi^i_k(M_1\ M_2) & \longrightarrow & ((\varphi^i_k M_1)\ (\varphi^i_k M_2)) & (\varphi\text{-app-transition}) \\
\varphi^i_k\,\underline{n} & \longrightarrow & \begin{cases} \underline{n+i-1} & \text{if } n > k \\ \underline{n} & \text{if } n \leq k \end{cases} & (\varphi\text{-destruction})
\end{array}
$$

λs_e rewriting rules [KR97]

$(M_1 \sigma^i M_2) \sigma^j N$	$\longrightarrow$	$(M_1 \sigma^{j+1} N) \sigma^i (M_2 \sigma^{j-i+1} N)$	if $i \leq j$	(σ-σ-transition)
$(\varphi^i_k M) \sigma^j N$	$\longrightarrow$	$\varphi^{i-1}_k M$	if $k < j < k+i$	(σ-φ-transition 1)
$(\varphi^i_k M)\, \sigma^j N$	$\longrightarrow$	$\varphi^i_k (M \sigma^{j-i+1} N)$	if $k + i \leq j$	(σ-φ-transition 2)
$\varphi^i_k (M \sigma^j N)$	$\longrightarrow$	$(\varphi^i_{k+1} M) \sigma^j (\varphi^i_{k+1-j} N)$	if $j \leq k+1$	(φ-σ-transition)
$\varphi^i_k (\varphi^j_l M)$	$\longrightarrow$	$\varphi^j_l (\varphi^i_{k+1-j} M)$	if $l + j \leq k$	(φ-φ-transition 1)
$\varphi^i_k (\varphi^j_l M)$	$\longrightarrow$	$\varphi^{j+i-1}_l M$	if $l \leq k < l + j$	(φ-φ-transition 2)

Simple types for λs_e

The System $\lambda s_e{}^{\rightarrow}$

$$\underline{1} : \langle \tau.\Gamma \vdash \tau \rangle \ (\mathrm{var}) \qquad \frac{M : \langle \sigma.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle} \rightarrow_i$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \sigma.\Gamma \vdash \tau \rangle} \ (\mathrm{varn}) \qquad \frac{M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle \qquad N : \langle \Gamma \vdash \sigma \rangle}{(M\ N) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e$$

$$(\varphi)\ \frac{M : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau \rangle}{\varphi^i_k M : \langle \Gamma \vdash \tau \rangle},\ \text{where } |\Gamma| \geq k + i - 1$$

$$(\sigma)\ \frac{N : \langle \Gamma_{\geq i} \vdash \rho \rangle \qquad M : \langle \Gamma_{<i}.\rho.\Gamma_{\geq i} \vdash \tau \rangle}{M \sigma^i N : \langle \Gamma \vdash \tau \rangle},\ \text{where } |\Gamma| \geq i - 1$$

$\lambda s_e{}^{\to}$ Properties [KR2000]

- Has SR property for both λs and λs_e.
- Typeable terms are SN in λs.
- Typeable terms are WN in λs_e.
- $\lambda s_e{}^{\to}$ has PT. [VAK2008]
  (equivalence with Wells' PT)
- System λs_e is not PSN.
  (Guillaume's example [Guillaume2000])

Intersection types for λs

The System λs^{SM}

$$(\omega\text{-}\varphi)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{\varphi^i_k M:\langle\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{>k}\vdash\tau\rangle},\ |\Gamma|>k \qquad (\omega\text{-}\sigma)\ \frac{N:\langle\Delta\vdash\rho\rangle \qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle\Gamma_{<i}.\Gamma_{>i}\vdash\tau\rangle},\ \Gamma_i=\omega$$

$$(nil\text{-}\varphi)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{\varphi^i_k M:\langle\Gamma\vdash\tau\rangle},\ |\Gamma|\leq k \qquad (nil\text{-}\sigma)\ \frac{N:\langle\Delta\vdash\rho\rangle \qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle\Gamma\vdash\tau\rangle},\ |\Gamma|<i$$

$$(\wedge\text{-}nil\text{-}\sigma)\ \frac{N:\langle nil\vdash\sigma_1\rangle\dots N:\langle nil\vdash\sigma_m\rangle \qquad M:\langle\omega^{\underline{i-1}}.\wedge_{j=1}^m\sigma_j.nil\vdash\tau\rangle}{M\sigma^i N:\langle nil\vdash\tau\rangle}$$

$$(\wedge\text{-}\omega\text{-}\sigma)\ \frac{N:\langle nil\vdash\sigma_1\rangle\dots N:\langle nil\vdash\sigma_m\rangle \qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle\Gamma_{<(i-k)}.nil\vdash\tau\rangle},\ \Gamma_i=\wedge_{j=1}^m\sigma_j\ (*)$$

$$(\wedge\text{-}\sigma)\ \frac{N:\langle\Delta^1\vdash\sigma_1\rangle\dots N:\langle\Delta^m\vdash\sigma_m\rangle \qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle(\Gamma_{<i}.\Gamma_{>i})\wedge\omega^{\underline{i-1}}.(\Delta^1\wedge\cdots\wedge\Delta^m)\vdash\tau\rangle},\ \Gamma_i=\wedge_{j=1}^m\sigma_j\ (**)$$

(*) $\Gamma=\Gamma_{<(i-k)}.\omega^{\underline{k}}.\wedge_{j=1}^m\sigma_j.nil$ and $\Gamma_{(i-k-1)}\neq\omega$
(**) $\Delta^k\neq nil$, for some $1\leq k\leq m$, or $\Gamma_{>i}\neq nil$

System λs^{SM} properties

Lemma (Relevance for λs^{SM})

If $M : \langle \Gamma \vdash \tau \rangle$, *then* $|\Gamma| = sav(M)$ *and* $\forall i$, $\Gamma_i \neq \omega$ *iff* $\underline{i} \in AI(M)$.

Theorem (SR for s in λs^{SM})

Let $M : \langle \Gamma \vdash \tau \rangle$. *If* $M \to_s M'$, *then* $M' : \langle \Gamma \vdash \tau \rangle$.

Definition (AI restriction)

Let $\Gamma\lfloor_M$ be a $\Gamma' \sqsubseteq \Gamma$ s.t. $|\Gamma'| = sav(M)$ and $\forall i$, $\Gamma'_i \neq \omega$ iff $\underline{i} \in AI(M)$.

Theorem (SR to λs^{SM} for the simulation of β-contraction)

Let $(\lambda.M\ M') \in \Lambda_{dB}$. *If* $(\lambda.M\ M') : \langle \Gamma \vdash \tau \rangle$, *then* $\{\underline{1}/M'\}M : \langle \Gamma\lfloor_{\{\underline{1}/M'\}M} \vdash \tau \rangle$.

Intersection types for λs_e

Let $A \equiv (\underline{1}\ \underline{1})$:

- $M \equiv (\underline{3}\,\sigma^1 A)\,\sigma^1\lambda.A$ is typeable in λs^{SM}
- $M \to_{\lambda s_e} M'$, for $M' \equiv (\underline{3}\,\sigma^2\lambda.A)\,\sigma^1(A\,\sigma^1\lambda.A)$.
- M' is not typeable in λs^{SM}.

We change three typing rules:

$$\frac{M : \langle \Gamma \vdash \omega \to \tau \rangle}{(M\ N) : \langle \Gamma \vdash \tau \rangle} \to_e^{\omega}$$

$$(nil\text{-}\sigma)\ \frac{M : \langle \Gamma \vdash \tau \rangle}{M\sigma^i N : \langle \Gamma \vdash \tau \rangle},\ |\Gamma| \leq i \qquad (\omega\text{-}\sigma)\ \frac{M : \langle \Gamma \vdash \tau \rangle}{M\sigma^i N : \langle \Gamma_{<i}.\Gamma_{>i} \vdash \tau \rangle},\ \Gamma_i = \omega$$

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Intersection types for λs_e

System $\lambda s_e{}^{\wedge}$

$$(nil\text{-}\sigma)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle\Gamma\vdash\tau\rangle},\ |\Gamma|<i \qquad (\omega\text{-}\sigma)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle\Gamma_{<i}.\Gamma_{>i}\vdash\tau\rangle},\ \Gamma_i=\omega$$

$$(\wedge\text{-}nil\text{-}\sigma)\ \frac{N:\langle nil\vdash\sigma_1\rangle\ \dots\ N:\langle nil\vdash\sigma_m\rangle \qquad M:\langle\omega\,\underline{^{i-1}}.\ \wedge_{j=1}^m\sigma_j.nil\vdash\tau\rangle}{M\sigma^i N:\langle nil\vdash\tau\rangle}$$

$$(\wedge\text{-}\omega\text{-}\sigma)\ \frac{N:\langle nil\vdash\sigma_1\rangle\ \dots\ N:\langle nil\vdash\sigma_m\rangle \qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle\Gamma_{<(i-k)}.nil\vdash\tau\rangle},\ \Gamma_i=\wedge_{j=1}^m\sigma_j\ (*)$$

$$(\wedge\text{-}\sigma)\ \frac{N:\langle\Delta^1\vdash\sigma_1\rangle\ \dots\ N:\langle\Delta^m\vdash\sigma_m\rangle \qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^i N:\langle(\Gamma_{<i}.\Gamma_{>i})\wedge\omega\,\underline{^{i-1}}.(\Delta^1\wedge\cdots\wedge\Delta^m)\vdash\tau\rangle},\ \Gamma_i=\wedge_{j=1}^m\sigma_j\ (**)$$

$$(\omega\text{-}\varphi)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{\varphi_k^i M:\langle\Gamma_{\le k}.\omega\,\underline{^{i-1}}.\Gamma_{>k}\vdash\tau\rangle},\ |\Gamma|>k \qquad (nil\text{-}\varphi)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{\varphi_k^i M:\langle\Gamma\vdash\tau\rangle},\ |\Gamma|\le k$$

(*) $\Gamma=\Gamma_{<(i-k)}.\omega\,\underline{^{k}}.\ \wedge_{j=1}^m\sigma_j.nil$ and $\Gamma_{(i-k-1)}\neq\omega$

(**) $\Delta^k\neq nil$, for some $1\le k\le m$, or $\Gamma_{>i}\neq nil$

System $\lambda s_e{}^{\wedge}$ properties

Lemma

If $M\!:\!\langle\Gamma \vdash \tau\rangle$ *for* $|\Gamma| = m > 0$, *then* $\Gamma_m \neq \omega$.

Theorem (SR for $\lambda s_e{}^{\wedge}$)

If $M\!:\!\langle\Gamma \vdash \tau\rangle$ *and* $M \rightarrow_{\lambda s_e} M'$, *then* $M'\!:\!\langle\Gamma \vdash \tau\rangle$.

Conclusions

- The λ-calculus is important for theoretical and practical purposes in Computer Science.

- Explicit substitution fills a gap between the theory and real implementations.

- Intersection types allows one to add polymorphism in a computation type system in a "machine friendly" way.

Contributions

- IT systems was introduced for two ES calculi: $\lambda\sigma$ and λs_e.
- Both systems satisfy the property of SR.
- In order to obtain those IT systems, we presented an IT system for λ_{dB} and introduced an IT system for λs.
- We introduced the syntactic notion of available indices in order to present a proper notion of relevance for λs.
- We introduced a proper notion of SR for both relevant systems.
- The SR for relevant system is satisfied in each calculus for the (simulation in the case of λs) β-contraction.

Open questions

- Verify whether $\lambda\sigma^{\wedge}$ and $\lambda s_e{}^{\wedge}$ characterise WN.
- Verify whether λ_{dB}^{SM} and λs^{SM} characterise SN.
- Investigate the notion for PT on $\lambda_{dB}^{\wedge}$, $\lambda\sigma^{\wedge}$ and $\lambda s_e{}^{\wedge}$.

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The λx-calculus rewriting rules[Lins86, BR95]

$$x\langle x := N\rangle \longrightarrow N \qquad \text{(xv)}$$

$$x\langle y := N\rangle \longrightarrow x \quad \text{if } x \not\equiv y \qquad \text{(xvgc)}$$

$$(\lambda_x.M)\langle y := N\rangle \longrightarrow \lambda_x.M\langle y := N\rangle \qquad \text{(xab)}$$

$$(M_1\ M_2)\langle y := N\rangle \longrightarrow (M_1\langle y := N\rangle\ M_2\langle y := N\rangle) \qquad \text{(xap)}$$

where $x \not\equiv y$ and $x \notin FV(N)$ in (xab)