

Intersection Type Systems for Explicit Substitution Calculi

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Talk's Plan

Motivation

The λ -calculus and type systems

Intersection types

Explicit Substitution

Intersection Types for λ_{dB}

The system λ_{dB}^{sm}

Intersection Types for $\lambda\sigma$

The $\lambda\sigma$ -calculus

The system $\lambda\sigma^\wedge$

Intersection Types for λs_e

The system λs_e^{sm}

The system λs_e^\wedge

Conclusion

The λ -calculus

Proposed by Church in 1932. [Church32]

Terms $M := x \mid (M M) \mid \lambda_x.M$

Computations (reductions) are made by a unique rule:

$$(\lambda_x.M N) \longrightarrow M\{x := N\} \quad (\beta)$$

Some renaming may be necessary:

$$\lambda_x.M \longrightarrow \lambda_y.M\{x := y\} \quad (\alpha)$$

Foundation for the Lisp and functional programming languages in general.

The λ -calculus with de Bruijn indices (λ_{dB})

Invented by N.G. de Bruijn[dB72].

Terms $M ::= \underline{n} \mid (M M) \mid \lambda.M$ for $n \in \mathbb{N}_* = \mathbb{N} \setminus \{0\}$

Examples

$\lambda.(\lambda.(\underline{1} \underline{4} \underline{2}) \underline{1})$

$\lambda.\underline{1} \simeq \lambda x.x \simeq \lambda y.y$

Definition (β -contraction)

$$(\lambda.M N) \triangleright_{\beta} \{\underline{1}/N\}M$$

Simply typed λ -calculus proposed by Church.[Church40]

Classify objects (terms) in the formal system.

$\lambda_{x:int}.x : int \rightarrow int$ $\lambda_{x:bool}.x : bool \rightarrow bool$ (*à la Church*)

$\lambda_x.x : int \rightarrow int$ $\lambda_x.x : bool \rightarrow bool$ (*à la Curry*)

STLC is related to IPL: Curry-Howard(-de Bruijn) Isomorphism.

If $M : \langle \Gamma \vdash \tau \rangle$ then $\langle \Gamma \vdash \tau \rangle$ is called a typing of M .

Definition (Simple Types and Contexts)

Types $\sigma, \tau \in \mathcal{S} ::= \mathcal{A} \mid \mathcal{S} \rightarrow \mathcal{S}$

Contexts $\Gamma ::= nil \mid \sigma.\Gamma$

System $\lambda_{dB}^{\rightarrow}$

$\underline{1} : \langle \tau.\Gamma \vdash \tau \rangle$ (var)

$$\frac{M : \langle \sigma.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle} \rightarrow_i$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \sigma.\Gamma \vdash \tau \rangle} \text{ (varn)}$$

$$\frac{M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle \quad N : \langle \Gamma \vdash \sigma \rangle}{(M N) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e$$

Typing Systems Properties

- ▶ **Subject Reduction (SR)**

If $M : \langle \Gamma \vdash \tau \rangle$ and $M \rightarrow_{\beta} M'$, then $M' : \langle \Gamma \vdash \tau \rangle$

- ▶ Strong or Weak Normalisation (SN and WN) for typeable term.

- ▶ Type Inference ($M : ?$)

- ▶ Principal Typing (PT)

- ▶ Inhabitation Problem ($? : \langle \Gamma \vdash \tau \rangle$)

Intersection type discipline

- ▶ Introduced by Coppo and Dezani-Ciancaglini. [CDC78, CDC80]
- ▶ Characterisation of the SN terms of the λ -calculus. [Pottinger80]
- ▶ It incorporates type polymorphism in a finitary way:

$$\lambda_x.x : (int \rightarrow int) \wedge (bool \rightarrow bool)$$

- ▶ PT has been verified in IT systems. [Bakel95, SM96a, KW04]
- ▶ Exists IT systems for explicit substitution (ES) calculi, e.g. λx [LLDDvB2004]
- ▶ There is no IT system for calculi with *composition of substitutions*. (e.g. $\lambda\sigma$ and λs_e)

Explicit Substitution

- ▶ Substitution is a *meta level* operation in the λ -calculus

$$(\lambda_x.M N) \longrightarrow_{\beta} M\{x := N\}$$

- ▶ Calculi with Explicit Substitutions extends it:

$$(\lambda_x.M N) \longrightarrow_{(Beta)} M\langle x := N \rangle$$

▶ JumpES

- ▶ There are different approaches of expliciting the substitutions.

The $\lambda\sigma$ -calculus [ACCL91]

- ▶ Well studied and important application on HOU. [DHK95]
- ▶ Two-sorted calculus: Terms and Substitutions.
- ▶ Defined with n -ary substitutions.
- ▶ Allows *composition* (unrestricted composition).
- ▶ $\lambda\sigma$ uses de Bruijn indices.

$$(\lambda.M N) \longrightarrow M[N.id] \quad (\text{Beta})$$

The λ_{s_e} -calculus [KR97]

- ▶ Natural extension for the λ -calculus à la de Bruijn.
- ▶ Has application on HOU [AK01].
- ▶ Extension of λ_s [KR95], allowing composition.
- ▶ One-sorted calculus: Terms.
- ▶ Introduces operators to realise substitutions and updatings.

$$(\lambda.M N) \longrightarrow M\sigma^1 N \quad (\sigma\text{-generation})$$

Definition (Restricted intersection types and contexts)

1. The **restricted intersection types** are defined by:

$$\begin{aligned}\tau, \sigma \in \mathcal{T} &::= \mathcal{A} \mid \mathcal{U} \rightarrow \mathcal{T} \\ u, v \in \mathcal{U} &::= \omega \mid \mathcal{U} \wedge \mathcal{U} \mid \mathcal{T}\end{aligned}$$

\wedge is commutative, associative and has ω as neutral element.

2. **Contexts:** $\Gamma ::= nil \mid u.\Gamma$ s.t. $u \in \mathcal{U}$

$$nil \wedge \Gamma = \Gamma \wedge nil = \Gamma \quad (u_1.\Gamma) \wedge (u_2.\Delta) = (u_1 \wedge u_2).(\Gamma \wedge \Delta)$$

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The system λ_{dB}^{SM} [VAK2010]

System λ_{dB}^{SM}

$$\frac{\tau \in \mathcal{T}}{\underline{1} : \langle \tau, nil \vdash \tau \rangle} \text{ var} \quad \frac{M : \langle nil \vdash \tau \rangle}{\lambda.M : \langle nil \vdash \omega \rightarrow \tau \rangle} \rightarrow'_i$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \omega.\Gamma \vdash \tau \rangle} \text{ varn} \quad \frac{M : \langle u.\Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash u \rightarrow \tau \rangle} \rightarrow_i$$

$$\frac{M_1 : \langle \Gamma \vdash \omega \rightarrow \tau \rangle \quad M_2 : \langle \Delta \vdash \sigma \rangle}{(M_1 \ M_2) : \langle \Gamma \wedge \Delta \vdash \tau \rangle} \rightarrow'_e$$

$$\frac{M_1 : \langle \Gamma \vdash \bigwedge_{i=1}^n \sigma_i \rightarrow \tau \rangle \quad M_2 : \langle \Delta^1 \vdash \sigma_1 \rangle \dots M_n : \langle \Delta^n \vdash \sigma_n \rangle}{(M_1 \ M_2) : \langle \Gamma \wedge \Delta^1 \wedge \dots \wedge \Delta^n \vdash \tau \rangle} \rightarrow_e$$

$$\frac{\frac{\underline{1}:\langle\alpha\rightarrow\beta.nil\vdash\alpha\rightarrow\beta\rangle\quad \underline{1}:\langle\alpha.nil\vdash\alpha\rangle}{(\underline{1}\ \underline{1}):\langle(\alpha\rightarrow\beta)\wedge\alpha.nil\vdash\beta\rangle}}{\lambda.(\underline{1}\ \underline{1}):\langle nil\vdash(\alpha\rightarrow\beta)\wedge\alpha\rightarrow\beta\rangle}$$

$\Omega \equiv (\lambda.(\underline{1}\ \underline{1})\ \lambda.(\underline{1}\ \underline{1}))$ is not typeable in λ_{dB}^{SM} :

$$(\alpha\rightarrow\beta)\wedge\alpha \stackrel{?}{=} (\alpha'\rightarrow\beta')\wedge\alpha'\rightarrow\beta'$$

Lemma (Relevance [VAK2010])

If $M : \langle \Gamma \vdash \tau \rangle$, then $|\Gamma| = \text{sup}(M)$ and $\forall i, \Gamma_i \neq \omega$ iff $\underline{i} \in \text{FI}(M)$.

- Every β -nf is typeable.
- A restricted version has PT for all β -nf.[VAK2010]
- SR for β -contraction (for relevant systems).

Subject Reduction (SR)

If $M : \langle \Gamma \vdash \tau \rangle$ and $M \rightarrow_{\beta} M'$, then $M' : \langle \Gamma \vdash \tau \rangle$

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Subject Reduction (SR)

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Definition (FI restriction)

Let $\Gamma \downarrow_M$ be a $\Gamma' \sqsubseteq \Gamma$ s.t. $|\Gamma'| = \text{sup}(M)$ and $\forall i, \Gamma'_i \neq \omega$ iff $i \in FI(M)$.

Theorem (SR for β -contraction in λ_{dB}^{SM})

If $(\lambda.M N) : \langle \Gamma \vdash_{\lambda_{dB}^{SM}} \tau \rangle$ then $\{\underline{1}/N\}M : \langle \Gamma \downarrow_{\{\underline{1}/N\}M} \vdash_{\lambda_{dB}^{SM}} \tau \rangle$.

The system λ_{dB}^{\wedge}

System λ_{dB}^{\wedge}

$$\frac{\tau \in \mathcal{T}}{\underline{1} : \langle \tau, nil \vdash \tau \rangle} \text{ var} \quad \frac{M : \langle nil \vdash \tau \rangle}{\lambda.M : \langle nil \vdash \omega \rightarrow \tau \rangle} \rightarrow'_i$$
$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \omega, \Gamma \vdash \tau \rangle} \text{ varn} \quad \frac{M : \langle u, \Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash u \rightarrow \tau \rangle} \rightarrow_i$$
$$\frac{M_1 : \langle \Gamma \vdash \omega \rightarrow \tau \rangle}{(M_1 M_2) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e^{\omega}$$
$$\frac{M_1 : \langle \Gamma \vdash \bigwedge_{i=1}^n \sigma_i \rightarrow \tau \rangle \quad M_2 : \langle \Delta^1 \vdash \sigma_1 \rangle \dots M_2 : \langle \Delta^n \vdash \sigma_n \rangle}{(M_1 M_2) : \langle \Gamma \wedge \Delta^1 \wedge \dots \wedge \Delta^n \vdash \tau \rangle} \rightarrow_e$$

Lemma

If $M : \langle \Gamma \vdash_{\lambda_{dB}^{\wedge}} \tau \rangle$ and $|\Gamma| = m > 0$ then $\Gamma_m \neq \omega$ and $\forall 1 \leq i \leq |\Gamma|, \Gamma_i \neq \omega$ implies $\underline{i} \in FI(M)$.

Syntax

Terms $M ::= \underline{1} \mid (M M) \mid \lambda.M \mid M[S]$

Substitutions $S ::= id \mid \uparrow \mid M.S \mid S \circ S$

- ▶ $\underline{n+1} \cong \underline{1}[\uparrow^n]$
- ▶ The β -reduction is simulated by the term $M[S](closure)$

$$(\lambda.M N) \rightarrow M[N.id] \quad (Beta)$$

$\lambda\sigma$ rewriting rules

$(\lambda.M N)$	\longrightarrow	$M[N.id]$	(Beta)
$(M N)[S]$	\longrightarrow	$(M[S] N[S])$	(App)
$\underline{1}[M.S]$	\longrightarrow	M	(VarCons)
$M[id]$	\longrightarrow	M	(Id)
$(\lambda.M)[S]$	\longrightarrow	$\lambda.(M[\underline{1}.(S \circ \uparrow)])$	(Abs)
$(M[S])[S']$	\longrightarrow	$M[S \circ S']$	(Clos)
$id \circ S$	\longrightarrow	S	(IdL)
$\uparrow \circ (M.S)$	\longrightarrow	S	(ShiftCons)
$(S_1 \circ S_2) \circ S_3$	\longrightarrow	$S_1 \circ (S_2 \circ S_3)$	(AssEnv)
$(M.S) \circ S'$	\longrightarrow	$M[S'].(S \circ S')$	(MapEnv)
$S \circ id$	\longrightarrow	S	(IdR)
$\underline{1}.\uparrow$	\longrightarrow	id	(VarShift)
$\underline{1}[S].(\uparrow \circ S)$	\longrightarrow	S	(Scons)

Simple types for $\lambda\sigma$

The System $\lambda\sigma \rightarrow$

Terms

$$\begin{array}{ll} \text{(var)} & \underline{x} : \langle \tau, \Gamma \vdash \tau \rangle \\ \text{(lambda)} & \frac{M : \langle \sigma, \Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle} \\ \text{(app)} & \frac{M_1 : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle \quad M_2 : \langle \Gamma \vdash \sigma \rangle}{(M_1 M_2) : \langle \Gamma \vdash \tau \rangle} \\ \text{(clos)} & \frac{S : \langle \Gamma \triangleright \Gamma' \rangle \quad M : \langle \Gamma' \vdash \tau \rangle}{M[S] : \langle \Gamma \vdash \tau \rangle} \end{array}$$

Substitutions

$$\begin{array}{ll} \text{(id)} & id : \langle \Gamma \triangleright \Gamma \rangle \\ \text{(cons)} & \frac{M : \langle \Gamma \vdash \tau \rangle \quad S : \langle \Gamma \triangleright \Gamma' \rangle}{M.S : \langle \Gamma \triangleright \tau, \Gamma' \rangle} \\ \text{(shift)} & \uparrow : \langle \tau, \Gamma \triangleright \Gamma \rangle \\ \text{(comp)} & \frac{S : \langle \Gamma \triangleright \Gamma' \rangle \quad S' : \langle \Gamma' \triangleright \Gamma'' \rangle}{S' \circ S : \langle \Gamma \triangleright \Gamma'' \rangle} \end{array}$$

- ▶ Has SR property.[ACCL91]
- ▶ Typeable terms are WN.[Munoz97]
- ▶ $\lambda\sigma^{\rightarrow}$ has PT. [VAK2008]
(equivalence with Wells' PT)
- ▶ System $\lambda\sigma$ is not PSN:
"If M is SN in λ then M is SN in $\lambda\sigma$ "
(Melliès' example [Mellies95])
- ▶ Unrestricted composition is the one to blame.[Ritter99]

Terms

$$\frac{}{\underline{1} : \langle \tau, nil \vdash \tau \rangle} \text{ (var)} \qquad \frac{M : \langle u, \Gamma \vdash \tau \rangle}{\lambda.M : \langle \Gamma \vdash u \rightarrow \tau \rangle} \rightarrow_i$$

$$\frac{M_1 : \langle \Gamma \vdash \omega \rightarrow \tau \rangle}{(M_1 M_2) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e^\omega \qquad \frac{M : \langle nil \vdash \tau \rangle}{\lambda.M : \langle nil \vdash \omega \rightarrow \tau \rangle} \rightarrow'_i$$

$$\frac{M_1 : \langle \Gamma \vdash \bigwedge_{i=1}^n \sigma_i \rightarrow \tau \rangle \quad M_2 : \langle \Delta^1 \vdash \sigma_1 \rangle \dots M_2 : \langle \Delta^n \vdash \sigma_n \rangle}{(M_1 M_2) : \langle \Gamma \wedge \Delta^1 \wedge \dots \wedge \Delta^n \vdash \tau \rangle} \rightarrow_e$$

$$\frac{S : \langle \Gamma \triangleright \Gamma' \rangle \quad M : \langle \Gamma' \vdash \tau \rangle}{M[S] : \langle \Gamma \vdash \tau \rangle} \text{ (clos)}$$

Substitutions

$$(\wedge\text{-cons}) \frac{M:\langle\Delta^1 \vdash \sigma_1\rangle \dots M:\langle\Delta^n \vdash \sigma_n\rangle \quad S:\langle\Delta \triangleright \Delta'\rangle}{M.S:\langle\Delta^1 \wedge \dots \wedge \Delta^n \wedge \Delta \triangleright (\wedge_{i=1}^n \sigma_i).\Delta'\rangle}$$

$$(\text{id}) \frac{\Gamma \neq \Delta.\omega^m}{\text{id}:\langle\Gamma \triangleright \Gamma\rangle}$$

$$(\text{comp}) \frac{S:\langle\Gamma \triangleright \Gamma''\rangle \quad S':\langle\Gamma'' \triangleright \Gamma'\rangle}{S' \circ S:\langle\Gamma \triangleright \Gamma'\rangle}$$

$$(\text{nil-shift}) \uparrow:\langle\text{nil} \triangleright \text{nil}\rangle$$

$$(\text{nil-cons}) \frac{S:\langle\Delta \triangleright \text{nil}\rangle}{M.S:\langle\Delta \triangleright \text{nil}\rangle}$$

$$(\text{shift}) \frac{\Gamma \neq \Delta.\omega^n}{\uparrow:\langle\omega.\Gamma \triangleright \Gamma\rangle}$$

$$(\omega\text{-cons}) \frac{S:\langle\Delta \triangleright \Delta'\rangle}{M.S:\langle\Delta \triangleright \omega.\Delta'\rangle}, \Delta' \neq \omega^n$$

Lemma

If $M: \langle \Gamma \vdash \tau \rangle$ and $|\Gamma| = m > 0$, then $\Gamma_m \neq \omega$. In particular, if $S: \langle \Gamma \triangleright \Gamma' \rangle$ and $|\Gamma| = m > 0$ then $\Gamma_m \neq \omega$ and if $|\Gamma'| = m' > 0$ then $\Gamma'_{m'} \neq \omega$.

Lemma (Generation for substitutions in $\lambda\sigma^\wedge$)

1. $S: \langle \text{nil} \triangleright \text{nil} \rangle$ for any substitution S .
2. If $M.S: \langle \Gamma \triangleright \text{nil} \rangle$ then $S: \langle \Gamma \triangleright \text{nil} \rangle$.
3. If $M.S: \langle \Gamma \triangleright \omega.\Gamma' \rangle$ then $S: \langle \Gamma \triangleright \Gamma' \rangle$ and $\Gamma' \neq \omega^n$.
4. If $M.S: \langle \Gamma \triangleright \Gamma' \rangle$ for $\Gamma' = \bigwedge_{i=1}^m \sigma_i.\Gamma''$ then $S: \langle \Gamma''' \triangleright \Gamma'' \rangle$ and $\forall i$, $M: \langle \Gamma^i \vdash \sigma_i \rangle$ s.t. $\Gamma = \Gamma''' \wedge \Gamma^1 \wedge \dots \wedge \Gamma^m$.

Lemma (Generation for substitutions in $\lambda\sigma^\wedge$ (cont.))

5. If $S:\langle\Gamma \triangleright nil\rangle$ then $\Gamma = nil$.
6. $\uparrow^m:\langle\Gamma \triangleright \Gamma'\rangle$ iff either $\Gamma = \Gamma' = nil$ or $\Gamma = \omega^m.\Gamma'$, where $\Gamma' \neq \Delta.\omega^n$.
7. If $S:\langle\Gamma \triangleright \Gamma'\rangle$ and $S:\langle\Delta \triangleright \Delta'\rangle$ then $S:\langle\Gamma \wedge \Delta \triangleright \Gamma' \wedge \Delta'\rangle$.
8. If $S:\langle\Gamma \triangleright \Delta^1 \wedge \Delta^2\rangle$ for $\Delta^1 \neq \Delta'.\omega^m$ and $\Delta^2 \neq \Delta''.\omega^m$, then $\Gamma = \Gamma^1 \wedge \Gamma^2$ s.t. $S:\langle\Gamma^1 \triangleright \Delta^1\rangle$ e $S:\langle\Gamma^2 \triangleright \Delta^2\rangle$.

Lemma (SR for $\lambda\sigma^\wedge$)

If $M:\langle\Gamma \vdash \tau\rangle$ and $M \rightarrow_{\lambda\sigma} M'$ then $M':\langle\Gamma \vdash \tau\rangle$. Particularly, if $S:\langle\Gamma \triangleright \Gamma'\rangle$ and $S \rightarrow_{\lambda\sigma} S'$ then $S':\langle\Gamma \triangleright \Gamma'\rangle$.

▶ jumpEnd

Syntax

Terms $M ::= \underline{n} \mid (M M) \mid \lambda.M \mid M\sigma^i M \mid \varphi_k^j M,$

where $n, i, j \in \mathbb{N}^*$ and $k \in \mathbb{N}$

- ▶ The β -substitution is simulated by the term $M\sigma^i N$

$$(\lambda.M N) \rightarrow M\sigma^1 N \quad (\sigma\text{-generation})$$

$(\lambda.M N)$	\longrightarrow	$M\sigma^1 N$	$(\sigma\text{-generation})$
$(\lambda.M)\sigma^i N$	\longrightarrow	$\lambda.(M\sigma^{i+1} N)$	$(\sigma\text{-}\lambda\text{-transition})$
$(M_1 M_2)\sigma^i N$	\longrightarrow	$((M_1\sigma^i N) (M_2\sigma^i N))$	$(\sigma\text{-app-transition})$
$\underline{n}\sigma^i N$	\longrightarrow	$\begin{cases} \underline{n-1} & \text{if } n > i \\ \varphi_0^i N & \text{if } n = i \\ \underline{n} & \text{if } n < i \end{cases}$	$(\sigma\text{-destruction})$
$\varphi_k^i(\lambda.M)$	\longrightarrow	$\lambda.(\varphi_{k+1}^i M)$	$(\varphi\text{-}\lambda\text{-transition})$
$\varphi_k^i(M_1 M_2)$	\longrightarrow	$((\varphi_k^i M_1) (\varphi_k^i M_2))$	$(\varphi\text{-app-transition})$
$\varphi_k^i \underline{n}$	\longrightarrow	$\begin{cases} \underline{n+i-1} & \text{if } n > k \\ \underline{n} & \text{if } n \leq k \end{cases}$	$(\varphi\text{-destruction})$

λ_{s_e} rewriting rules [KR97]

$$\begin{array}{lll} (M_1 \sigma^i M_2) \sigma^j N & \longrightarrow & (M_1 \sigma^{j+1} N) \sigma^i (M_2 \sigma^{j-i+1} N) \quad \text{if } i \leq j \quad (\sigma\text{-}\sigma\text{-transition}) \\ (\varphi_k^i M) \sigma^j N & \longrightarrow & \varphi_k^{i-1} M \quad \text{if } k < j < k + 1 \quad (\sigma\text{-}\varphi\text{-transition 1}) \\ (\varphi_k^i M) \sigma^j N & \longrightarrow & \varphi_k^i (M \sigma^{j-i+1} N) \quad \text{if } k + i \leq j \quad (\sigma\text{-}\varphi\text{-transition 2}) \\ \varphi_k^i (M \sigma^j N) & \longrightarrow & (\varphi_{k+1}^i M) \sigma^j (\varphi_{k+1-j}^i N) \quad \text{if } j \leq k + 1 \quad (\varphi\text{-}\sigma\text{-transition}) \\ \varphi_k^i (\varphi_l^j M) & \longrightarrow & \varphi_l^j (\varphi_{k+1-j}^i M) \quad \text{if } l + j \leq k \quad (\varphi\text{-}\varphi\text{-transition 1}) \\ \varphi_k^i (\varphi_l^j M) & \longrightarrow & \varphi_l^{j+i-1} M \quad \text{if } l \leq k < l + j \quad (\varphi\text{-}\varphi\text{-transition 2}) \end{array}$$

The System $\lambda s_e \rightarrow$

$$\underline{1} : \langle \tau. \Gamma \vdash \tau \rangle \text{ (var)} \qquad \frac{M : \langle \sigma. \Gamma \vdash \tau \rangle}{\lambda. M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle} \rightarrow_i$$

$$\frac{\underline{n} : \langle \Gamma \vdash \tau \rangle}{\underline{n+1} : \langle \sigma. \Gamma \vdash \tau \rangle} \text{ (varn)} \qquad \frac{M : \langle \Gamma \vdash \sigma \rightarrow \tau \rangle \quad N : \langle \Gamma \vdash \sigma \rangle}{(M N) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e$$

$$(\varphi) \frac{M : \langle \Gamma_{\leq k}. \Gamma_{\geq k+i} \vdash \tau \rangle}{\varphi_k^i M : \langle \Gamma \vdash \tau \rangle}, \text{ where } |\Gamma| \geq k + i - 1$$

$$(\sigma) \frac{N : \langle \Gamma_{\geq i} \vdash \rho \rangle \quad M : \langle \Gamma_{< i}. \rho. \Gamma_{\geq i} \vdash \tau \rangle}{M \sigma^i N : \langle \Gamma \vdash \tau \rangle}, \text{ where } |\Gamma| \geq i - 1$$

- ▶ Has SR property for both λ_s and λ_{S_e} .
- ▶ Typeable terms are SN in λ_s .
- ▶ Typeable terms are WN in λ_{S_e} .
- ▶ $\lambda_{S_e} \rightarrow$ has PT. [VAK2008]
(equivalence with Wells' PT)
- ▶ System λ_{S_e} is not PSN.
(Guillaume's example [Guillaume2000])

Intersection types for λ_S

The System λ_S^{SM}

$$(\omega\text{-}\varphi) \frac{M : \langle \Gamma \vdash \tau \rangle}{\varphi_k^i M : \langle \Gamma_{\leq k} \cdot \omega^{\underline{i-1}} \cdot \Gamma_{>k} \vdash \tau \rangle}, |\Gamma| > k \quad (\omega\text{-}\sigma) \frac{N : \langle \Delta \vdash \rho \rangle \quad M : \langle \Gamma \vdash \tau \rangle}{M\sigma^i N : \langle \Gamma_{<i} \cdot \Gamma_{>i} \vdash \tau \rangle}, \Gamma_i = \omega$$

$$(\text{nil}\text{-}\varphi) \frac{M : \langle \Gamma \vdash \tau \rangle}{\varphi_k^i M : \langle \Gamma \vdash \tau \rangle}, |\Gamma| \leq k \quad (\text{nil}\text{-}\sigma) \frac{N : \langle \Delta \vdash \rho \rangle \quad M : \langle \Gamma \vdash \tau \rangle}{M\sigma^i N : \langle \Gamma \vdash \tau \rangle}, |\Gamma| < i$$

$$(\wedge\text{-}\text{nil}\text{-}\sigma) \frac{N : \langle \text{nil} \vdash \sigma_1 \rangle \dots N : \langle \text{nil} \vdash \sigma_m \rangle \quad M : \langle \omega^{\underline{i-1}} \cdot \wedge_{j=1}^m \sigma_j \cdot \text{nil} \vdash \tau \rangle}{M\sigma^i N : \langle \text{nil} \vdash \tau \rangle}$$

$$(\wedge\text{-}\omega\text{-}\sigma) \frac{N : \langle \text{nil} \vdash \sigma_1 \rangle \dots N : \langle \text{nil} \vdash \sigma_m \rangle \quad M : \langle \Gamma \vdash \tau \rangle}{M\sigma^i N : \langle \Gamma_{<(i-k)} \cdot \text{nil} \vdash \tau \rangle}, \Gamma_i = \wedge_{j=1}^m \sigma_j \text{ (*)}$$

$$(\wedge\text{-}\sigma) \frac{N : \langle \Delta^1 \vdash \sigma_1 \rangle \dots N : \langle \Delta^m \vdash \sigma_m \rangle \quad M : \langle \Gamma \vdash \tau \rangle}{M\sigma^i N : \langle (\Gamma_{<i} \cdot \Gamma_{>i}) \wedge \omega^{\underline{i-1}} \cdot (\Delta^1 \wedge \dots \wedge \Delta^m) \vdash \tau \rangle}, \Gamma_i = \wedge_{j=1}^m \sigma_j \text{ (**)}$$

(*) $\Gamma = \Gamma_{<(i-k)} \cdot \omega^{\underline{k}} \cdot \wedge_{j=1}^m \sigma_j \cdot \text{nil}$ and $\Gamma_{(i-k-1)} \neq \omega$

(**) $\Delta^k \neq \text{nil}$, for some $1 \leq k \leq m$, or $\Gamma_{>i} \neq \text{nil}$

Lemma (Relevance for λs^{SM})

If $M : \langle \Gamma \vdash \tau \rangle$, then $|\Gamma| = \text{sav}(M)$ and $\forall i, \Gamma_i \neq \omega$ iff $i \in AI(M)$.

Theorem (SR for s in λs^{SM})

Let $M : \langle \Gamma \vdash \tau \rangle$. If $M \rightarrow_s M'$, then $M' : \langle \Gamma \vdash \tau \rangle$.

Definition (AI restriction)

Let $\Gamma \downarrow_M$ be a $\Gamma' \sqsubseteq \Gamma$ s.t. $|\Gamma'| = \text{sav}(M)$ and $\forall i, \Gamma'_i \neq \omega$ iff $i \in \text{AI}(M)$.

Theorem (SR to λs^{SM} for the simulation of β -contraction)

Let $(\lambda.M M') \in \Lambda_{dB}$. If $(\lambda.M M') : \langle \Gamma \vdash \tau \rangle$, then
 $\{\underline{1}/M'\}M : \langle \Gamma \downarrow_{\{\underline{1}/M'\}M} \vdash \tau \rangle$.

Intersection types for λs_e

Let $A \equiv (\underline{1} \ \underline{1})$:

- $M \equiv (\underline{\exists} \sigma^1 A) \sigma^1 \lambda.A$ is typeable in λs^{SM}
- $M \rightarrow_{\lambda s_e} M'$, for $M' \equiv (\underline{\exists} \sigma^2 \lambda.A) \sigma^1 (A \sigma^1 \lambda.A)$.
- M' is not typeable in λs^{SM} .

We change three typing rules:

$$\begin{array}{c} \frac{M : \langle \Gamma \vdash \omega \rightarrow \tau \rangle}{(M \ N) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e^\omega \\ \\ (nil-\sigma) \frac{M : \langle \Gamma \vdash \tau \rangle}{M \sigma^i N : \langle \Gamma \vdash \tau \rangle}, |\Gamma| \leq i \qquad (\omega-\sigma) \frac{M : \langle \Gamma \vdash \tau \rangle}{M \sigma^i N : \langle \Gamma_{<i} . \Gamma_{>i} \vdash \tau \rangle}, \Gamma_i = \omega \end{array}$$

Intersection types for λs_e

Let $A \equiv (\underline{1} \ \underline{1})$:

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We change three typing rules:

$$\begin{array}{c} M: \langle \Gamma \vdash \omega \rightarrow \tau \rangle \\ \hline (M \ N): \langle \Gamma \vdash \tau \rangle \quad \rightarrow_e^\omega \end{array}$$
$$(nil-\sigma) \frac{M: \langle \Gamma \vdash \tau \rangle}{M \sigma^i N: \langle \Gamma \vdash \tau \rangle}, |\Gamma| \leq i \qquad (\omega-\sigma) \frac{M: \langle \Gamma \vdash \tau \rangle}{M \sigma^i N: \langle \Gamma_{<i} . \Gamma_{>i} \vdash \tau \rangle}, \Gamma_i = \omega$$

Intersection types for λs_e

Let $A \equiv (\underline{1} \ \underline{1})$:

- $M \equiv (\underline{\exists} \sigma^1 A) \sigma^1 \lambda.A$ is typeable in λs^{SM}
- $M \rightarrow_{\lambda s_e} M'$, for $M' \equiv (\underline{\exists} \sigma^2 \lambda.A) \sigma^1 (A \sigma^1 \lambda.A)$.
- M' is not typeable in λs^{SM} .

We change three typing rules:

$$\begin{array}{c} \frac{M : \langle \Gamma \vdash \omega \rightarrow \tau \rangle}{(M \ N) : \langle \Gamma \vdash \tau \rangle} \rightarrow_e^\omega \\ (nil-\sigma) \frac{M : \langle \Gamma \vdash \tau \rangle}{M \sigma^i N : \langle \Gamma \vdash \tau \rangle}, |\Gamma| \leq i \quad (\omega-\sigma) \frac{M : \langle \Gamma \vdash \tau \rangle}{M \sigma^i N : \langle \Gamma_{<i} . \Gamma_{>i} \vdash \tau \rangle}, \Gamma_i = \omega \end{array}$$

Intersection types for λ_{S_e}

System $\lambda_{S_e}^\wedge$

$$(nil-\sigma) \frac{M:\langle \Gamma \vdash \tau \rangle}{M\sigma^i N:\langle \Gamma \vdash \tau \rangle}, |\Gamma| < i \quad (\omega-\sigma) \frac{M:\langle \Gamma \vdash \tau \rangle}{M\sigma^i N:\langle \Gamma_{<i}.\Gamma_{>i} \vdash \tau \rangle}, \Gamma_i = \omega$$

$$(\wedge-nil-\sigma) \frac{N:\langle nil \vdash \sigma_1 \rangle \dots N:\langle nil \vdash \sigma_m \rangle \quad M:\langle \omega \stackrel{i-1}{\cdot} \wedge_{j=1}^m \sigma_j \cdot nil \vdash \tau \rangle}{M\sigma^i N:\langle nil \vdash \tau \rangle}$$

$$(\wedge-\omega-\sigma) \frac{N:\langle nil \vdash \sigma_1 \rangle \dots N:\langle nil \vdash \sigma_m \rangle \quad M:\langle \Gamma \vdash \tau \rangle}{M\sigma^i N:\langle \Gamma_{<(i-k)} \cdot nil \vdash \tau \rangle}, \Gamma_i = \wedge_{j=1}^m \sigma_j \quad (*)$$

$$(\wedge-\sigma) \frac{N:\langle \Delta^1 \vdash \sigma_1 \rangle \dots N:\langle \Delta^m \vdash \sigma_m \rangle \quad M:\langle \Gamma \vdash \tau \rangle}{M\sigma^i N:\langle (\Gamma_{<i}.\Gamma_{>i}) \wedge \omega \stackrel{i-1}{\cdot} (\Delta^1 \wedge \dots \wedge \Delta^m) \vdash \tau \rangle}, \Gamma_i = \wedge_{j=1}^m \sigma_j \quad (**)$$

$$(\omega-\varphi) \frac{M:\langle \Gamma \vdash \tau \rangle}{\varphi_k^i M:\langle \Gamma_{\leq k} \cdot \omega \stackrel{i-1}{\cdot} \Gamma_{>k} \vdash \tau \rangle}, |\Gamma| > k \quad (nil-\varphi) \frac{M:\langle \Gamma \vdash \tau \rangle}{\varphi_k^i M:\langle \Gamma \vdash \tau \rangle}, |\Gamma| \leq k$$

(*) $\Gamma = \Gamma_{<(i-k)} \cdot \omega \stackrel{k}{\cdot} \wedge_{j=1}^m \sigma_j \cdot nil$ and $\Gamma_{(i-k-1)} \neq \omega$

(**) $\Delta^k \neq nil$, for some $1 \leq k \leq m$, or $\Gamma_{>i} \neq nil$

Lemma

If $M : \langle \Gamma \vdash \tau \rangle$ for $|\Gamma| = m > 0$, then $\Gamma_m \neq \omega$.

Theorem (SR for $\lambda_{S_e}^\wedge$)

If $M : \langle \Gamma \vdash \tau \rangle$ and $M \rightarrow_{\lambda_{S_e}} M'$, then $M' : \langle \Gamma \vdash \tau \rangle$.

Conclusions

- The λ -calculus is important for theoretical and practical purposes in Computer Science.
- Explicit substitution fills a gap between the theory and real implementations.
- Intersection types allows one to add polymorphism in a computation type system in a “machine friendly” way.

- IT systems was introduced for two ES calculi: $\lambda\sigma$ and λs_e .
- Both systems satisfy the property of SR.
- In order to obtain those IT systems, we presented an IT system for λ_{dB} and introduced an IT system for λs .
- We introduced the syntactic notion of available indices in order to present a proper notion of relevance for λs .
- We introduced a proper notion of SR for both relevant systems.
- The SR for relevant system is satisfied in each calculus for the (simulation in the case of λs) β -contraction.

Open questions

- Verify whether $\lambda_{\sigma}^{\wedge}$ and $\lambda_{s_e}^{\wedge}$ characterise WN.
- Verify whether λ_{dB}^{SM} and λ_s^{SM} characterise SN.
- Investigate the notion for PT on λ_{dB}^{\wedge} , $\lambda_{\sigma}^{\wedge}$ and $\lambda_{s_e}^{\wedge}$.

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$$x \langle x := N \rangle \longrightarrow N \quad (\text{xv})$$

$$x \langle y := N \rangle \longrightarrow x \quad \text{if } x \not\equiv y \quad (\text{xvgc})$$

$$(\lambda_x.M) \langle y := N \rangle \longrightarrow \lambda_x.M \langle y := N \rangle \quad (\text{xab})$$

$$(M_1 M_2) \langle y := N \rangle \longrightarrow (M_1 \langle y := N \rangle M_2 \langle y := N \rangle) \quad (\text{xap})$$

where $x \not\equiv y$ and $x \notin FV(N)$ in (xab)